Tensor renormalization group approach to entanglement negativity

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Work in collaboration with

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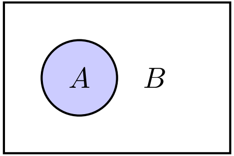
Entanglement negativity [Vidal-Werner, 2002]

- A measure of quantum entanglement between two subsystems A and B
- Defined by the density matrix ρ of $A \cup B$:

$$\mathcal{N} = \frac{\|\rho^{T_B}\| - 1}{2}$$

cf. $S_A = -\mathrm{tr}\rho_A \log \rho_A, \;\; \rho_A = \mathrm{tr}_B \rho$



T_B: partial transpose with respect to subsystem B

$$\rho_{(I_A \textcolor{red}{I_B})(J_A \textcolor{red}{J_B})} = (\rho^{T_B})_{(I_A \textcolor{red}{J_B})(J_A \textcolor{red}{I_B})}$$

$\|M\|$: trace norm of matrix M (sum of absolute values of eigenvalues)

$$\|M\| = \sum_i |\lambda_i|$$

Details of negativity

- Peres separability criterion:
  Subsystems A and B are separable $\Rightarrow \rho^{T_B}$ has no negative eigenvalues

- Negativity $\mathcal{N}$ is defined such that it quantifies the degree of violation of the Peres separability criterion:

$$\mathcal{N} \equiv \frac{||\rho^{T_B}|| - 1}{2} = \frac{1}{2}\sum_i (|\lambda_i| - \lambda_i) = \sum_{\text{negative eigenvalues}} |\lambda_i|,$$

  No entanglement between A and $B \Rightarrow \mathcal{N} = 0$

- In this work, we calculate logarithmic negativity $E_{\mathcal{N}}$:

$$E_{\mathcal{N}} \equiv \log ||\rho^{T_B}|| = \log(2\mathcal{N} + 1)$$

Entanglement negativity vs. entanglement entropy

Negativity can extract only quantum correlations in mixed states

E.g., consider entanglement between two qubits A and B in the following cases:

1. Pure state with entanglement: $\rho = |\psi\rangle\langle\psi|, |\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$S_A = S_B = \log 2, \quad \mathcal{N} = \frac{1}{2}$$

   Both correctly detect entanglement

2. Mixed state without entanglement: $\rho = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$

$$S_A = S_B = \log 2, \quad \mathcal{N} = 0$$

   EE detects classical correlations, while negativity correctly detects no entanglement

Advantages of entanglement negativity

- Entanglement negativity can extract quantum correlations in mixed states
- It can be a useful probe for analyzing phase transitions, especially in finite-temperature systems, such as the QGP phase transition
- Entanglement negativity is important quantity in the context of the quantum information theory, so it may be a key to exploring the quantum information aspects of quantum field theories
- It is computable by various methods
  cf. analytical calculations in CFT[Calabrese et al., 2013], linked-cluster expansion[Sherman et al., 2015], and quantum Monte Carlo[Chun et al., 2013]

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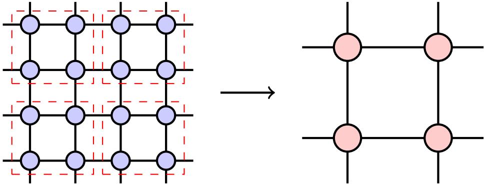
We use the tensor renormalization group (TRG) approach
to compute negativity

Tensor renormalization group (TRG) [Levin-Nave, 2006]

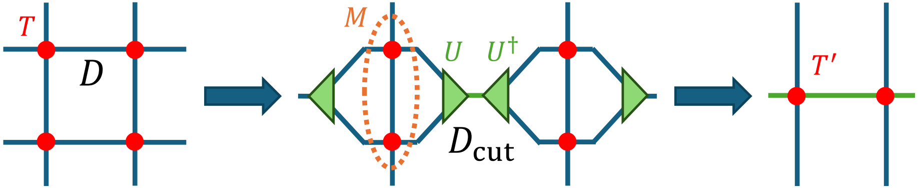
- Partition functions and path integrals can be expressed as a product of local tensors, which is called a tensor network
- Contraction of tensors are computed by iterative coarse-graining
- Various coarse-graining algorithms have been proposed

$$Z = \sum_{a,b,c,d,\dots} T_{abcd} T_{defg} \cdots =$$



Higher-Order TRG algorithm [Xie et al., 2012]

- Two neighboring tensors are contracted to form a new tensor:



- Isometry matrix U diagonalizes the matrix $MM^\dagger$, and contains the $D_{\mathrm{cut}}(< D^2)$ largest singular values of $MM^\dagger$.
- It can be applied to higher-dimensional systems

Advantages of TRG method

- Sign problem free
  Applicable to finite-density systems, real-time evolution, systems with a θ term, etc.

- No replica trick required
  Direct computation of entanglement negativity and entanglement entropy
  [Yang et al., 2016][Bazavov et al., 2017][Luo-Kuramashi, 2023]

- Applicable to higher-dimensional systems
  Computationally expensive, but still feasible

Target model

- We calculate the half-space logarithmic negativity $E_{\mathcal{N}} = \log ||\rho^{T_B}||$ of the one-dimensional transverse-field Ising model at finite temperature

$$\hat{H} = -J \sum_{i=1}^{L} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - h_x \sum_{i=1}^{L} \hat{\sigma}_i^x$$

with periodic boundary conditions $\hat{\sigma}_{L+1}^z = \hat{\sigma}_1^z$

- We consider the density matrix $\hat{\rho}$ of a Gibbs state at finite temperature

$$\hat{\rho} = \frac{e^{-\beta \hat{H}}}{Z}, \quad Z = \mathrm{tr} e^{-\beta \hat{H}},$$

and represent $\hat{\rho}$ as a two-dimensional tensor network

Quantum to classical mapping

- Exponential of the Hamiltonian can be decomposed using the Suzuki-Trotter decomposition:

$$e^{-\beta \hat{H}} = \lim_{N\to\infty} \left(\exp\left[\frac{\beta J}{N} \sum_{i=1}^{L} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \right] \exp\left[\frac{\beta h_x}{N} \sum_{i=1}^{L} \hat{\sigma}_i^x \right] \right)^N$$

- The 1D TFIM is mapped to a 2D classical anisotropic Ising model:

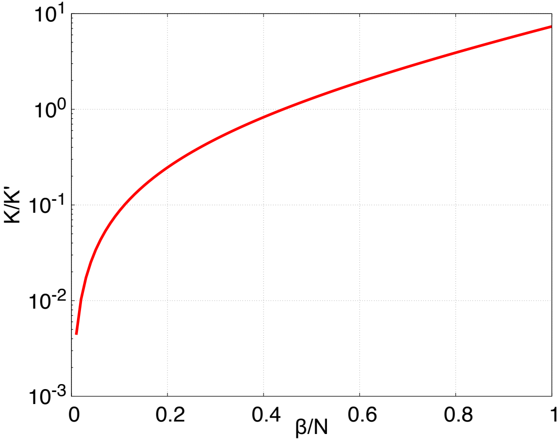
$$\langle \{s\}_\tau | \, e^{-\beta \hat{H}} \, | \{s\}_0 \rangle$$

$$= \lim_{N\to\infty} C^{LN} \sum_{\{s\}_{1,\dots,\{s\}_{N-1}}} \exp\left[K \sum_{i=1}^{L} \sum_{n=1}^{N} s_{i,n} s_{i+1,n} + K' \sum_{i=1}^{L} \sum_{n=0}^{N-1} s_{i,n} s_{i,n+1} \right],$$

where $\{s\}_m = \{s_{1,m}, \dots, s_{L,m}\}$, $|\{s\}_m\rangle \equiv \otimes_{i=1}^{L} |s_{i,m}\rangle$, $\sigma_i^z |s_{i,m}\rangle = s_{i,m} |s_{i,m}\rangle$,

and $C = \sqrt{\frac{1}{2} \sinh\left(\frac{2\beta h_x}{N}\right)}$, $K = \frac{\beta J}{N}$, $K' = \frac{1}{2} \log \coth\left(\frac{\beta h_x}{N}\right)$

β/N dependence of coupling constants K and K'



- Coupling in the spatial direction:

$$K = \frac{\beta J}{N}$$

- Coupling in the imaginary time direction:

$$K' = \frac{1}{2} \log \coth \left(\frac{\beta h_x}{N} \right)$$

- The system becomes highly anisotropic for large N (small β)

Tensor network representation of density matrix

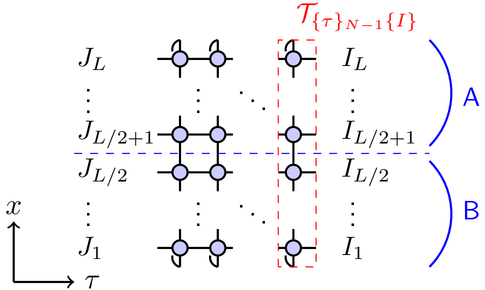
With an appropriate change of basis, we obtain the following tensor network:

$$\langle \{I\} | \, e^{-\beta \hat{H}} \, | \{J\} \rangle = \lim_{N \to \infty} \mathcal{T}_{\{J\}\{\tau\}_1} \mathcal{T}_{\{t\}_1\{\tau\}_2} \times \cdots \times \mathcal{T}_{\{\tau\}_{N-1}\{I\}},$$

where $\mathcal{T}$ is a transfer matrix and is defined by

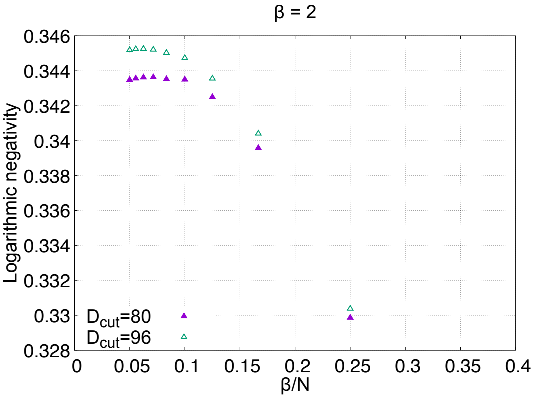
$$\begin{aligned} \mathcal{T}_{\{s\}\{s'\}} \equiv& T_{x_1 x_2 s_1 s'_1} T_{x_2 x_3 s_2 s'_2} \cdots T_{x_L x_1 s_L s'_L}, \\ T_{abcd} \equiv& 2C\delta(\mathrm{mod}(a+b+c+d, 2)) \cosh K \cosh K' (\sqrt{\tanh K})^{a+b} (\sqrt{\tanh K'})^{c+d}, \end{aligned}$$

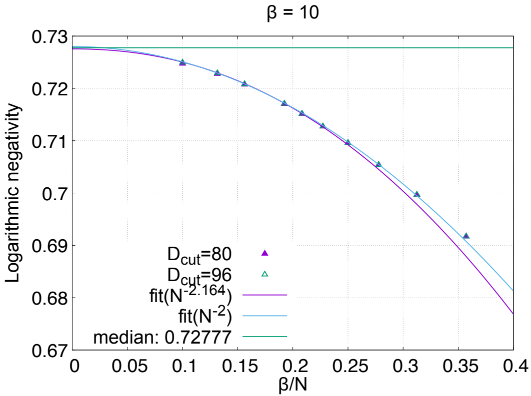
$$\langle \{I\} | \, e^{-\beta \hat{H}} \, | \{J\} \rangle =$$



Dcut dependence of logarithmic negativity

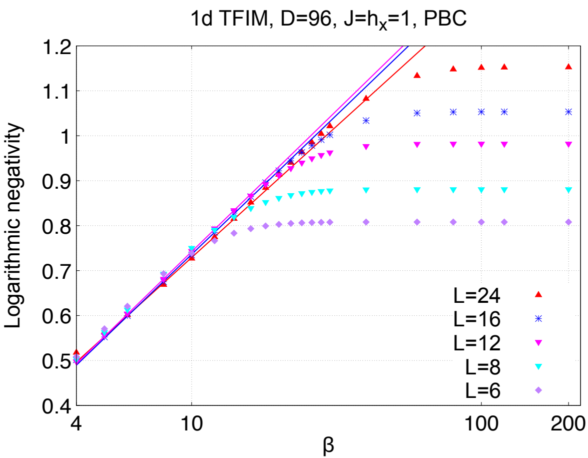
1D TFIM, $J = h_x = 1.0, L = 24$, PBC





At low temperatures, $D_{\mathrm{cut}} = 96$ is sufficient for convergence

Temperature dependence



- It saturates at low temperatures
- Expected scaling for large L:

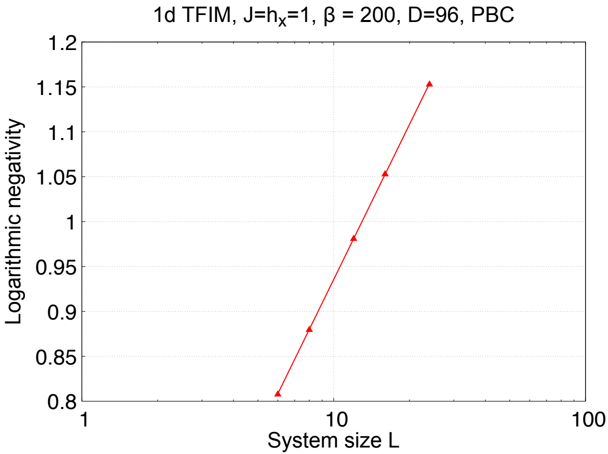
$$E_{\mathcal{N}} = \frac{c}{2} \log \beta + \text{const.},$$

where c is the central charge
$c = 0.5$ in the 1D TFIM

- Central charge

$$c = 0.507 \ (L = 24)$$
$$c = 0.538 \ (L = 16)$$
$$c = 0.545 \ (L = 12)$$

Zero temperature limit



- Analytical result:
  [Calabrese et al., 2013]

$$E_{\mathcal{N}} = \frac{c}{2} \log L + \text{const.}$$

- Central charge

$$c = 0.4983(9)$$

Our result is consistent with the theoretical value $c = 0.5$

Conclusion

Results

- We have computed the logarithmic negativity in 1D TFIM from the density matrix of a Gibbs state using tensor renormalization group methods
- Entanglement entropy can also be computed from the density matrix of quantum systems in the same way
- Our result is consistent with known results, confirming the validity of our calculations

Future directions

Our interest

- Analyzing phase transitions in quantum systems, especially in finite temperature systems
- Exploring quantum information aspects of QFT

Future work

- More general configurations of subsystems

  General sizes of subsystems A, B, entanglement between disjoint intervals, higher dimensions, etc.

- Analysis of finite temperature systems

- Applications to quantum field theories