Towards the Mass Shift and Discrete Chiral Symmetry in the Lattice QED Hamiltonian

Shoto Aoki (Riken iTHEMS, UC Berkeley)
with Yoshio Kikukawa and Toshinari Takemoto (U. Tokyo)

July 30, 2026 @ Maryland

Arxiv:2511.06198

Work in Progress

RIKEN Center for Interdisciplinary Theoretical and Mathematical Sciences

Contents

Recently Progress in Schwinger Model

Schwinger Model: $(1+1)$D Fermion system coupled to $U(1)$ gauge

[Schwinger, 1962]

- Exactly Solvable! [Coleman et al., 1975]
- Playground for confinement, mass gap, and bosonization
- Consistency check between numerical simulations [Byrnes et al., 2002]

Recently, [Dempsey et al., 2022] pointed out
that the fermion mass slightly shifts on the lattice space.
$\longrightarrow$ Mass Shift

- The Hamiltonian has an exact chiral symmetry.
- It exhibits a chiral anomaly.
- Adding mass shift improves numerical simulations.

Can we extend the mass shift to a $3+1$D case?

Schwinger Model

We first review a $(1+1)$D case, namely Schwinger model.

$$H(\theta) = \frac{e^2 a}{2}\left(\Pi_n + \underbrace{\frac{\theta}{2\pi}}_{\text{Theta Term}}\right)^2 + \underbrace{\frac{1}{i2a}\left(\chi_n^\dagger e^{iA_n}\chi_{n+1} - h.c.\right) + m(-1)^n \chi_n^\dagger \chi_n}_{\text{Staggered Fermion [Kogut and Susskind, 1975]}}$$

$$[A_n, \Pi_m] = i\delta_{nm}, \ \left\{\chi_n^\dagger, \chi_m\right\} = \delta_{nm}$$

- e: coupling constant
- a: lattice spacing
- θ: theta angle
- m: mass parameter

A Dirac fermion is described by $\psi_x = \begin{pmatrix} \chi_{2x} \\ \chi_{2x+1} \end{pmatrix}$.

The spinless fermion χ can be interpreted as the component of ψ.

Discrete Chiral Symmetry

We define a one-sift shift operator $\mathcal{V}$ by

$$\mathcal{V}\chi_n\mathcal{V}^{-1} = \chi_{n+1},\ \mathcal{V}A_n\mathcal{V}^{-1} = A_{n+1}.$$

This oeprator flips the mass term,

$$m\sum_n(-1)^n\chi_n^\dagger\chi_n \to -m\sum_n(-1)^n\chi_n^\dagger\chi_n,$$

and acts on the Dirac fermion as

$$\psi_x = \begin{pmatrix}\chi_{2x}\\ \chi_{2x+1}\end{pmatrix} \to \begin{pmatrix}\chi_{2x+1}\\ \chi_{2x+2}\end{pmatrix} = \sigma_1\psi_x + \mathcal{O}(a).$$

$\longrightarrow$ $\mathcal{V}$ can be interpreted as an axial $U(1)$ rotation by $\frac{\pi}{2}$!
$\mathcal{V}$ is a discrete chiral transformation!

In the large mass limit, the ground state $|\mathrm{gs}\rangle$ satisfy

$$\chi_n |\mathrm{gs}\rangle = 0 \quad (n\text{: even}).$$

We assume that $|\mathrm{gs}\rangle$ is neutral under

$$\chi_n \to e^{i\lambda_n}\chi_n,\ A_n \to A_n - \lambda_{n+1} + \lambda_n.$$

Then, we get a Gauss law constraint

$$\Pi_n - \Pi_{n-1} = q_n,\ q_n = \chi_n^\dagger \chi_n - \frac{1}{2}(1-(-1)^n).$$

Note that q_n changes to

$$\mathcal{V} q_n \mathcal{V}^{-1} = \chi_{n+1}^\dagger \chi_{n+1} - \frac{1}{2}(1-(-1)^n) = q_{n+1} + {\color{red}(-1)^n}.$$

Mass Shift

We assume that the Gauss law is invariant under $\mathcal{V}$.

To cancel out $(-1)^n$, Π_n transforms

$$\Pi_n \to \Pi_{n+1} + \frac{1}{2}(-1)^n + l,$$

where $l \in \mathbb{Z} + \frac{1}{2}$ since Π_n takes an integer value.

Then, we get

$$\mathcal{V}H(\theta)\mathcal{V}^{-1} = H(\theta+\pi) - 2\left(m + \frac{e^2 a}{8}\right)\sum_n (-1)^n \chi_n^\dagger \chi_n.$$

$\longrightarrow$ $m = -\frac{e^2 a}{8}$: Mass Shift (from Gauss law)

$\theta \to \theta + \pi$: Chiral anomaly (from integerness of Π_n)

The mass shift improves numerical simulations! [Dempsey et al., 2022]

Our Work

We employ the QED Hamiltonian.

- Chiral transformation is equivalent to one-site shift in the diagonal direction [Kogut and Susskind, 1975].
- We derive a mass shift from the Gauss law.
- We predict a chiral anomaly under a magnetic field.

Overview: Hersh Singh(Fri, 9:30)
Numerical Simulation: Simone Romiti(Thu, 16:30), Emil Otis Rosanowski(Thu, 16:30)
Non-Abelian: Lena Funcke(Thu, 16:10), David Ward(Wed), Joshua Lin(Thu, 16:50)

2D case: Sriram Bharadwaj(Thu, 16:50), Etsuko Itou(Tue), Hiromasa Watanabe(Fri, 15:20)

Contents

Contents

QED in Hamiltonian Formulation

We consider an N^3 lattice space with a lattice spacing a and employ a QED Hamiltonian in Villain formulation [Villain, 1975].

$$H(\theta) = H_g(\theta) + H_f,\ (B = dA + 2\pi n,\ (ijk) = (123), (231), (312))$$

$$H_g(\theta) = \sum_x \sum_i \left[\frac{e^2}{2a} \left(\Pi_{x,i} + \frac{\theta}{8\pi^2} \left(B_{x+\hat{i},jk} + B_{x-\hat{j}-\hat{k},jk} \right) \right)^2 + \frac{1}{2ae_g^2} B_{x,jk}^2 \right],$$

$$H_f = \sum_x \left[\frac{1}{i2a} \sum_i \left(\eta_i(x) \chi_x^\dagger e^{iA_{x,i}} \chi_{x+\hat{i}} - h.c. \right) + m\epsilon(x) \chi_x^\dagger \chi_x \right]$$

$$\eta_1(x) = 1,\ \eta_2(x) = (-1)^{x_1},\ \eta_3(x) = (-1)^{x_1+x_2},\ \epsilon(x) = (-1)^{x_1+x_2+x_3}.$$

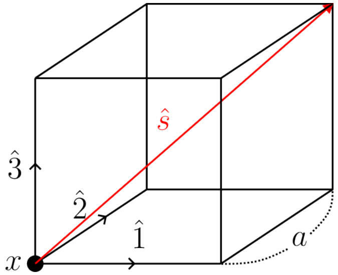
- e_g: magnetic coupling constant
- $n_{x,jk} \in \mathbb{Z}$: magnetic flux through the plaquette (x, jk)

Villain Formulation: [Evan Berkowitz(Mon)]

Discrete Chiral Transformation

In the 3D lattice, the shift operator in the diagonal direction is related to the chiral transformation.

$$\begin{aligned}\mathcal{V}\chi_x\mathcal{V}^{-1} &= i(-1)^{x_2}\chi_{x+\hat{s}},\\ \mathcal{V}A_{x,i}\mathcal{V}^{-1} &= A_{x+\hat{s},i},\\ \mathcal{V}n_{x,ij}\mathcal{V}^{-1} &= n_{x+\hat{s},ij}\end{aligned}$$



A Gauss law is also given by

$$\sum_i(\Pi_{x,i}-\Pi_{x-\hat{i},i}) = q_x,\ \ q_x = \chi_x^\dagger\chi_x - \frac{1}{2}(1-\epsilon(x)),$$

and q_x changes to

$$\mathcal{V}q_x\mathcal{V}^{-1} = q_{x+\hat{s}} + \epsilon(x).$$

Decomposition: $\Pi = \Pi^L + \Pi^T$

As same as the Schwinger model, the mass shift is derived from the Gauss law. We decompose Π_x into

- the longitudinal part

$$\Pi^L_{x,i} = \sum_{p\neq 0} \frac{e^{ixp}}{\sqrt{N^3}} \frac{f_i f_j^*}{|f|^2} \Pi_j(p) = \sum_{p\neq 0} \frac{e^{ixp}}{\sqrt{N^3}} \frac{-f_i}{|f|^2} q(p), \ \ (f_i(p) = e^{ip_i} - 1)$$

- the transverse part $\Pi^T = \Pi - \Pi^L$.

Since $q(p) = \sum_x \frac{e^{-ipx}}{\sqrt{N^3}} q_x$ changes to

$$\mathcal{V} q(p) \mathcal{V}^{-1} = e^{i(p_1+p_2+p_3)} q(p) + \frac{1}{\sqrt{N^3}} \delta^3(p - (\pi,\pi,\pi)).$$

Thus, we get $\mathcal{V}\Pi^L_{x,i}\mathcal{V}^{-1} = \Pi^L_{x+\hat{s},i} + \frac{1}{6}\epsilon(x)$

Transverse part: Π^T

Unlike the Schwinger model, Π takes a real value.
$\longrightarrow$ We cannot determine the chiral anomaly from the Gauss law.

We focus on

- no monopole sector $dn = 0 \longrightarrow n = dl + \frac{h}{N^2}\ (h \in \mathbb{Z})$
- $e_g \ll e \ll 1$ region $\longrightarrow \left\langle e^{idA} \right\rangle \simeq e^{i2\pi \frac{h}{N^2}} = e^{id\bar{A}}$

Assuming $A = \bar{A} + \frac{W}{N}$, we get

$$H_{eff}(\theta) = \sum_i \left[\frac{e^2}{2aN} \left(-i \frac{\partial}{\partial W_i} + \frac{\theta}{2\pi} h_{jk} \right)^2 + \frac{1}{2e_g^2 aN} (2\pi h)_{jk}^2 \right] + \frac{e^2}{2a} \sum_{x,i} (\Pi^L_{x,i})^2 + H_f(W; h),$$

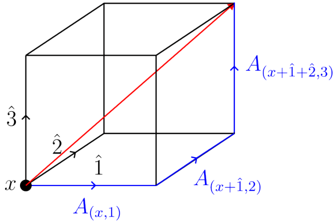
where W_i is a Wilson loop in the i-th direction.

Axial Charge

At $h_{12} = h_{23} = h_{31} = h \in \mathbb{Z}$ sector, we can define an axial charge

$$Q_A = \sum_x \chi_x^\dagger \frac{1}{2}(\Gamma\chi_x + h.c.)$$

$$\Gamma\chi_x = \exp\Big(i(A_{x,1} + A_{x+\hat{1},2} + iA_{x+\hat{1}+\hat{2},3})\Big) \times i(-1)^{x_2} e^{-i\pi h/N^2} \chi(x+\hat{s},t)$$



In the continuum limit, Q_A converges to $\gamma_5 \otimes 1$. Then,

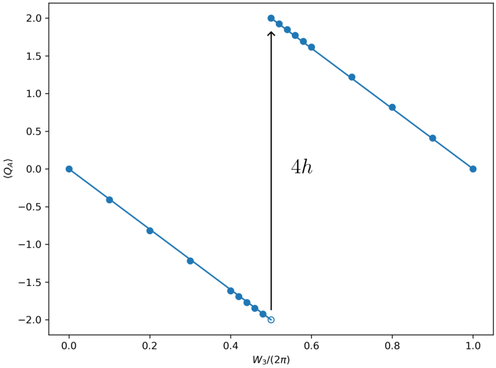
$$\exp\Big(i\frac{\pi}{2}Q_A\Big) \simeq \mathcal{V}$$

Q_A belongs to the Onsager algebra [Catterall et al., 2025; Onogi and Yamaoka, 2025].

Axial Anomaly (Numerical Simulation)

The axial anomaly is detected from the flow of $\langle Q_A \rangle = \mathrm{Tr}_{E<0} Q_A$.

We change W_3, adiabatically $(m = 0,\ N = 8,\ h = 1)$.



$$\mathcal{V}(-i\frac{\partial}{\partial W_3})\mathcal{V}^\dagger \simeq -i\frac{\partial}{\partial W_3} + \frac{\pi}{2}\frac{1}{2\pi}4h = -i\frac{\partial}{\partial W_3} + h$$

Mass Shift

Recalling

$$\mathcal{V}\Pi^L_{x,i}\mathcal{V}^{-1} = \Pi^L_{x+\hat{s},i} + \frac{1}{6}\epsilon(x),\ \mathcal{V}(-i\frac{\partial}{\partial W_3})\mathcal{V}^{-1} \simeq -i\frac{\partial}{\partial W_3} + h,$$

we obtain

$$\mathcal{V}H_{eff}(\theta)\mathcal{V}^{-1} \simeq H_{eff}(\theta + 2\pi) - 2\sum_x \left(m + \frac{e^2}{24a}\right)\epsilon(x)\chi^\dagger_x\chi_x.$$

- The massless point is shifted $m = 0 \to -\frac{e^2}{24a}$.
- $\theta \to \theta + 2\pi$ is consistent with the chiral anomaly with $N_f = 2$.

Contents

Summary

We employ the QED Hamiltonian.

[Mass shift]

- The mass shift is exactly derived from the Gauss law.
- The interaction generates the non-trivial mass term.

[Shifting θ by 2π]

- We need a lot of assumptions to derive the shift,

$$dn = 0,\ e_g \ll e \ll 1,\ h_{12} = h_{23} = h_{31},\ \mathcal{V} \simeq e^{i\frac{\pi}{2}Q_A}.$$

- However, our result suggests

$$\mathcal{V}H(\theta)\mathcal{V}^{-1} \simeq H(\theta + 2\pi) - 2\sum_x \left(m + \frac{e^2}{24a}\right)\epsilon(x)\chi_x^\dagger \chi_x.$$

Reference i

Byrnes, T., Sriganesh, P., Bursill, R. J., and Hamer, C. J. (2002). Density matrix renormalization group approach to the massive Schwinger model. Phys. Rev. D, 66:013002.

Catterall, S., Pradhan, A., and Samlodia, A. (2025). Symmetries and Anomalies of Hamiltonian Staggered Fermions.

Coleman, S. R., Jackiw, R., and Susskind, L. (1975). Charge Shielding and Quark Confinement in the Massive Schwinger Model. Annals Phys., 93:267.

Dempsey, R., Klebanov, I. R., Pufu, S. S., and Zan, B. (2022). Discrete chiral symmetry and mass shift in the lattice Hamiltonian approach to the Schwinger model. Phys. Rev. Res., 4(4):043133.

Kogut, J. B. and Susskind, L. (1975). Hamiltonian Formulation of Wilson's Lattice Gauge Theories. Phys. Rev. D, 11:395–408.

Onogi, T. and Yamaoka, T. (2025). Non-singlet conserved charges and anomalies in 3+1 D staggered fermions.

Schwinger, J. S. (1962). Gauge Invariance and Mass. 2. Phys. Rev., 128:2425–2429.

Reference ii

Villain, J. (1975). Theory of one-dimensional and two-dimensional magnets with an easy magnetization plane. 2. The Planar, classical, two-dimensional magnet. J. Phys. (France), 36:581–590.

Chiral Transformation on $(3+1)$D

In $(3+1)$D, a fermion Hamiltonian [Kogut and Susskind, 1975] is given by

$$H = \frac{1}{2a}\sum_{j=1}^{3}\left(\eta_j(x)\chi^\dagger_{x+\hat{\mu}}e^{iA_{x,\mu}}\chi_x - h.c.\right) + m\epsilon(x)\chi^\dagger_x\chi_x$$

$\eta_1(x) = 1$, $\eta_2(x) = (-1)^{x_1}$, $\eta_3(x) = (-1)^{x_1+x_2}$, $\epsilon(x) = (-1)^{x_1+x_2+x_3}$.

- Chiral transformation:
  $\chi_x \to \Gamma_5\chi_x = i(-1)^{x_2}\chi_{x+\hat{1}+\hat{2}+\hat{3}}$
- Chiral Anomaly:
  $\frac{\mathrm{d}}{\mathrm{d}t}\left\langle\chi^\dagger\Gamma\chi\right\rangle \to -2\frac{1}{2\pi^2}F\wedge F$
  [SA, Kikukawa, and Takemoto, 2025]

$\longrightarrow$ This result impiles the 2π shift of a θ-term.

Towards the Mass Shift and Discrete Chiral Symmetry

To derive the mass shift, we need to construct a θ-term with

$$\theta \to \theta + 2\pi$$

under the chiral transformation.

$\longrightarrow$ It can be interpreted as $\underbrace{\mathcal{T}\text{-transformation}}_{\text{One duality in } (3+1)\text{D Maxwell}}$.

In this talk, we consider a $(3+1)$D Maxwell theory.

- Analyze the duality structure
- Determine the θ-term.

[Shao et al., 2026]: $(3+1)$D compact scalar theory

[?]: $(2+1)$D fractonic field theory

Diagonalization of $H_g(\theta)$

We recall

$$H_g(\theta) = \sum_x \sum_i \left[\frac{e^2}{2a} \left(\Pi_{x,i} + \frac{\theta}{8\pi^2} \left(B_{x+\hat{i},jk} + B_{x-\hat{j}-\hat{k},jk} \right) \right)^2 + \frac{1}{2ae_g^2} B_{x,jk}^2 \right]$$

with $B = dA + 2\pi n$. The Hilbert space can be decomposed by

$$\mathcal{H} = \bigoplus_{h,dn} \mathcal{H}_{h,dn},$$

where $h = \frac{1}{N} \sum_x n_{x,ij}$. The lowest energy state in $\mathcal{H}_{h,dn}$ is expressed by

$$|\psi\rangle_{h,dn} = \sum_n \exp\left(-\frac{1}{2} \sqrt{\frac{e}{e_g}} B \frac{1}{\sqrt{d\partial}} B \right)$$

Diagonalization of $H_g(\theta)$

The Hilbert space can be decomposed by

$$\mathcal{H} = \bigoplus_{h,dn} \mathcal{H}_{h,dn} \ (h = \frac{1}{N}\sum_x n_{x,ij}).$$

The lowest energy state in $\mathcal{H}_{h,dn}$ is expressed by

$$|\psi\rangle_{h,dn} = \sum_{\substack{n \\ h,dn:\text{fixed}}} \exp\left(-\frac{1}{2}\frac{1}{\sqrt{ee_g}}B\frac{P_d^{(2)}}{\sqrt{d\partial}}B - i\frac{\theta}{8\pi^2}\phi^\infty(A,n)\right)$$
$$\exp\left(iA(P_0^{(1)}+P_d^{(1)})m - i2\pi n\frac{1}{d\partial}dm\right)|n\rangle .$$

$$E = \frac{e^2}{2a}\sum_x (m + \frac{\theta}{2\pi}\star n)(P_0^{(1)}+P_d^{(1)})(m+\frac{\theta}{2\pi}\star n)$$
$$+\frac{4\pi^2}{2e_g^2 a}\sum_x\left(\frac{h^2}{N^4} + dn\frac{1}{d\partial}dn\right) + \frac{1}{2}\sqrt{\frac{e}{e_g}}\,\mathrm{tr}\left(\sqrt{d\partial}\right)$$

$e_g \ll e \ll 1$ region

- Photonic excitation modes have $E \sim \sqrt{\frac{e}{e_g}}$.
- If $e_g \ll e \ll 1$, such an excitation is surpressed.

$$\left\langle e^{idA} \right\rangle = e^{-C\sqrt{ee_g}} \exp\left(i2\pi \frac{h}{N^2} \right) = e^{-C\sqrt{ee_g}} e^{id\bar{A}}$$

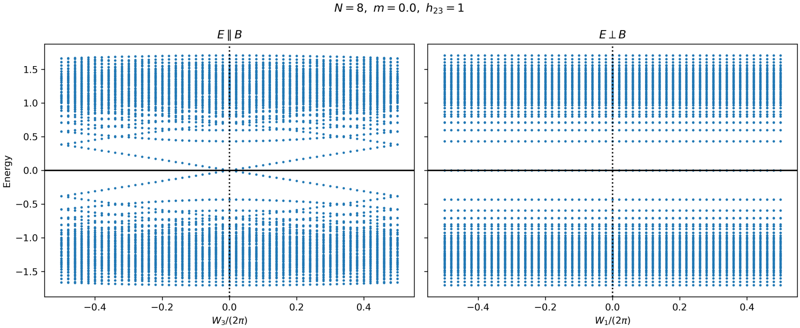
Then, we can write $A = \bar{A} + \frac{W}{N}$ and

$$\mathcal{V} \exp(iW_i) \mathcal{V}^{-1} = \exp\left(iW_i + i2\pi \frac{h_{ij}}{N^2} + i2\pi \frac{h_{ik}}{N^2} \right).$$

If $h_{12} = h_{23} = h_{31}$, $\mathcal{V} \exp(iW_i) \mathcal{V}^{-1} = \exp(iW_i)$

Spectral Flow: $B_1 = B_2 = 0,\ B_3 = \frac{2\pi}{N^2}$

We consider a magnetic field in the x_3 direction with $h = 1$ and add parallel/orthonormal electric fields.



Figures imply the chiral anomaly $\propto E_3 B_3$

Mass Shift with Dynamical Gauge Field

If

$$\mathcal{V}(-i\frac{\partial}{\partial W_3})\mathcal{V}^\dagger = -i\frac{\partial}{\partial W_3} + h_{12}$$

holds rigorously, we can recover the θ term

$$\mathcal{V}(\Pi^T_{x,3})\mathcal{V}^\dagger = \Pi^T_{x+\hat{3},3} + (dA + 2\pi n)_{x+\hat{3},12} + (dA + 2\pi n)_{x-\hat{1}-\hat{2},12}$$

by using a unitary transformation [SA, Kikukawa, Takemoto, 2607.25332]