

Towards the Mass Shift and Discrete Chiral Symmetry in the Lattice QED Hamiltonian

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Work in Progress



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Contents

Introduction

Discrete Chiral Transformation and Mass Shift

Summary

Recently Progress in Schwinger Model

Schwinger Model: $(1 + 1)$ D Fermion system coupled to $U(1)$ gauge

[Schwinger, 1962]

- Exactly Solvable! [Coleman et al., 1975]
- Playground for confinement, mass gap, and bosonization
- Consistency check between numerical simulations [Byrnes et al., 2002]

Recently, [Dempsey et al., 2022] pointed out that the fermion mass slightly shifts on the lattice space.

————→ **Mass Shift**

- The Hamiltonian has an exact chiral symmetry.
- It exhibits a chiral anomaly.
- Adding mass shift improves numerical simulations.

Can we extend the mass shift to a $3 + 1$ D case?

Schwinger Model

We first review a $(1 + 1)$ D case, namely Schwinger model.

$$H(\theta) = \frac{e^2 a}{2} \left(\Pi_n + \frac{\theta}{2\pi} \right)^2 + \frac{1}{i2a} \left(\chi_n^\dagger e^{iA_n} \chi_{n+1} - h.c. \right) + m(-1)^n \chi_n^\dagger \chi_n$$

Theta Term Staggered Fermion [Kogut and Susskind, 1975]

$$[A_n, \Pi_m] = i\delta_{nm}, \quad \left\{ \chi_n^\dagger, \chi_m \right\} = \delta_{nm}$$

- e : coupling constant
- θ : theta angle
- a : lattice spacing
- m : mass parameter

A Dirac fermion is described by $\psi_x = \begin{pmatrix} \chi_{2x} \\ \chi_{2x+1} \end{pmatrix}$.

The spinless fermion χ can be interpreted as the component of ψ .

Discrete Chiral Symmetry

We define a one-site shift operator $\mathcal{V}$ by

$$\mathcal{V}\chi_n\mathcal{V}^{-1} = \chi_{n+1}, \quad \mathcal{V}A_n\mathcal{V}^{-1} = A_{n+1}.$$

This operator flips the mass term,

$$m \sum_n (-1)^n \chi_n^\dagger \chi_n \rightarrow -m \sum_n (-1)^n \chi_n^\dagger \chi_n,$$

and acts on the Dirac fermion as

$$\psi_x = \begin{pmatrix} \chi_{2x} \\ \chi_{2x+1} \end{pmatrix} \rightarrow \begin{pmatrix} \chi_{2x+1} \\ \chi_{2x+2} \end{pmatrix} = \sigma_1 \psi_x + \mathcal{O}(a).$$

—→ $\mathcal{V}$ can be interpreted as an axial $U(1)$ rotation by $\frac{\pi}{2}$!
 $\mathcal{V}$ is a discrete chiral transformation!

Gauss Law

In the large mass limit, the ground state $|\text{gs}\rangle$ satisfy

$$\chi_n |\text{gs}\rangle = 0 \quad (n: \text{ even}).$$

We assume that $|\text{gs}\rangle$ is neutral under

$$\chi_n \rightarrow e^{i\lambda_n} \chi_n, \quad A_n \rightarrow A_n - \lambda_{n+1} + \lambda_n.$$

Then, we get a Gauss law constraint

$$\Pi_n - \Pi_{n-1} = q_n, \quad q_n = \chi_n^\dagger \chi_n - \frac{1}{2}(1 - (-1)^n).$$

Note that q_n changes to

$$\mathcal{V} q_n \mathcal{V}^{-1} = \chi_{n+1}^\dagger \chi_{n+1} - \frac{1}{2}(1 - (-1)^n) = q_{n+1} + (-1)^n.$$

Mass Shift

We assume that the Gauss law is invariant under $\mathcal{V}$.

To cancel out $(-1)^n$, Π_n transforms

$$\Pi_n \rightarrow \Pi_{n+1} + \frac{1}{2}(-1)^n + l,$$

where $l \in \mathbb{Z} + \frac{1}{2}$ since Π_n takes an integer value.

Then, we get

$$\mathcal{V}H(\theta)\mathcal{V}^{-1} = H(\theta + \pi) - 2\left(m + \frac{e^2 a}{8}\right) \sum_n (-1)^n \chi_n^\dagger \chi_n.$$

→ $m = -\frac{e^2 a}{8}$: Mass Shift (from Gauss law)

$\theta \rightarrow \theta + \pi$: Chiral anomaly (from integrerness of Π_n)

The mass shift improves numerical simulations! [Dempsey et al., 2022]

We employ the QED Hamiltonian.

- Chiral transformation is equivalent to one-site shift in the diagonal direction [Kogut and Susskind, 1975].
- We derive a mass shift from the Gauss law.
- We predict a chiral anomaly under a magnetic field.

Overview: Hersh Singh(Fri, 9:30)

Numerical Simulation: Simone Romiti(Thu, 16:30), Emil Otis Rosanowski(Thu, 16:30)

Non-Abelian: Lena Funcke(Thu, 16:10), David Ward(Wed), Joshua Lin(Thu, 16:50)

2D case: Sriram Bharadwaj(Thu, 16:50), Etsuko Itou(Tue), Hiromasa Watanabe(Fri, 15:20)

Contents

Introduction

Discrete Chiral Transformation and Mass Shift

Summary

Contents

Introduction

Discrete Chiral Transformation and Mass Shift

Summary

QED in Hamiltonian Formulation

We consider an N^3 lattice space with a lattice spacing a and employ a QED Hamiltonian in Villain formulation [Villain, 1975].

$$H(\theta) = H_g(\theta) + H_f, \quad (B = dA + 2\pi n, \quad (ijk) = (123), (231), (312))$$

$$H_g(\theta) = \sum_x \sum_i \left[\frac{e^2}{2a} \left(\Pi_{x,i} + \frac{\theta}{8\pi^2} (B_{x+\hat{i},jk} + B_{x-\hat{j}-\hat{k},jk}) \right)^2 + \frac{1}{2ae_g^2} B_{x,jk}^2 \right],$$

$$H_f = \sum_x \left[\frac{1}{i2a} \sum_i (\eta_i(x) \chi_x^\dagger e^{iA_{x,i}} \chi_{x+\hat{i}} - h.c.) + m\epsilon(x) \chi_x^\dagger \chi_x \right]$$

$$\eta_1(x) = 1, \quad \eta_2(x) = (-1)^{x_1}, \quad \eta_3(x) = (-1)^{x_1+x_2}, \quad \epsilon(x) = (-1)^{x_1+x_2+x_3}.$$

- e_g : magnetic coupling constant
- $n_{x,jk} \in \mathbb{Z}$: magnetic flux through the plaquette (x, jk)

Villain Formulation: [Evan Berkowitz(Mon)]

Discrete Chiral Transformation

In the 3D lattice, the shift operator in the diagonal direction is related to the chiral transformation.

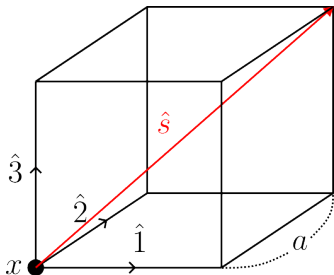
$$\begin{aligned}\mathcal{V}\chi_x\mathcal{V}^{-1} &= i(-1)^{x_2}\chi_{x+\hat{s}}, \\ \mathcal{V}A_{x,i}\mathcal{V}^{-1} &= A_{x+\hat{s},i}, \\ \mathcal{V}n_{x,ij}\mathcal{V}^{-1} &= n_{x+\hat{s},ij}\end{aligned}$$

A Gauss law is also given by

$$\sum_i (\Pi_{x,i} - \Pi_{x-\hat{i},i}) = q_x, \quad q_x = \chi_x^\dagger \chi_x - \frac{1}{2}(1 - \epsilon(x)),$$

and q_x changes to

$$\mathcal{V}q_x\mathcal{V}^{-1} = q_{x+\hat{s}} + \epsilon(x).$$



Decomposition: $\Pi = \Pi^L + \Pi^T$

As same as the Schwinger model, the mass shift is derived from the Gauss law. We decompose Π_x into

- the longitudinal part

$$\Pi_{x,i}^L = \sum_{p \neq 0} \frac{e^{ixp}}{\sqrt{N^3}} \frac{f_i f_j^*}{|f|^2} \Pi_j(p) = \sum_{p \neq 0} \frac{e^{ixp}}{\sqrt{N^3}} \frac{-f_i}{|f|^2} q(p), \quad (f_i(p) = e^{ip_i} - 1)$$

- the transverse part $\Pi^T = \Pi - \Pi^L$.

Since $q(p) = \sum_x \frac{e^{-ipx}}{\sqrt{N^3}} q_x$ changes to

$$\mathcal{V} q(p) \mathcal{V}^{-1} = e^{i(p_1 + p_2 + p_3)} q(p) + \frac{1}{\sqrt{N^3}} \delta^3(p - (\pi, \pi, \pi)).$$

Thus, we get $\mathcal{V} \Pi_{x,i}^L \mathcal{V}^{-1} = \Pi_{x+\hat{s},i}^L + \frac{1}{6} \epsilon(x)$

Transverse part: Π^T

Unlike the Schwinger model, Π takes a real value.

————→ We cannot determine the chiral anomaly from the Gauss law.

We focus on

- no monopole sector $dn = 0$ —————→ $n = dl + \frac{h}{N^2}$ ($h \in \mathbb{Z}$)
- $e_g \ll e \ll 1$ region —————→ $\langle e^{idA} \rangle \simeq e^{i2\pi \frac{h}{N^2}} = e^{id\bar{A}}$

Assuming $A = \bar{A} + \frac{W}{N}$, we get

$$H_{eff}(\theta) = \sum_i \left[\frac{e^2}{2aN} \left(-i \frac{\partial}{\partial W_i} + \frac{\theta}{2\pi} h_{jk} \right)^2 + \frac{1}{2e_g^2 a N} (2\pi h)_{jk}^2 \right] \\ + \frac{e^2}{2a} \sum_{x,i} (\Pi_{x,i}^L)^2 + H_f(W; h),$$

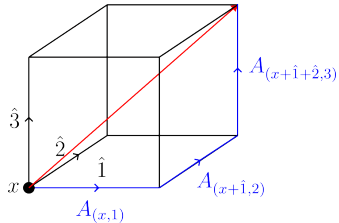
where W_i is a Wilson loop in the i -th direction.

Axial Charge

At $h_{12} = h_{23} = h_{31} = h \in \mathbb{Z}$ sector, we can define an axial charge

$$Q_A = \sum_x \chi_x^\dagger \frac{1}{2} (\Gamma \chi_x + h.c.)$$

$$\Gamma \chi_x = \exp \left(i(A_{x,1} + A_{x+\hat{1},2} + iA_{x+\hat{1}+\hat{2},3}) \right) \\ \times i(-1)^{x_2} e^{-i\pi h/N^2} \chi(x + \hat{s}, t)$$



In the continuum limit, Q_A converges to $\gamma_5 \otimes 1$. Then,

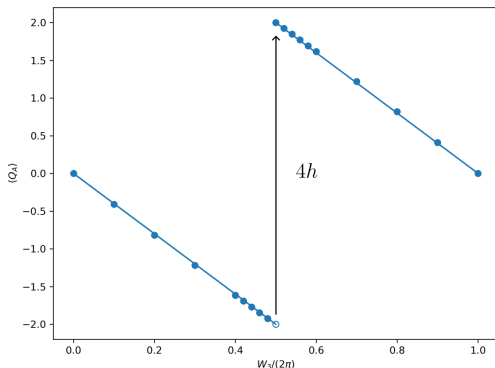
$$\exp \left(i \frac{\pi}{2} Q_A \right) \simeq \mathcal{V}$$

Q_A belongs to the Onsager algebra [Catterall et al., 2025; Onogi and Yamaoka, 2025].

Axial Anomaly (Numerical Simulation)

The axial anomaly is detected from the flow of $\langle Q_A \rangle = \text{Tr}_{E<0} Q_A$.

We change W_3 , adiabatically ($m = 0$, $N = 8$, $h = 1$).



$$\mathcal{V}(-i\frac{\partial}{\partial W_3})\mathcal{V}^\dagger \simeq -i\frac{\partial}{\partial W_3} + \frac{\pi}{2} \frac{1}{2\pi} 4h = -i\frac{\partial}{\partial W_3} + h$$

Recalling

$$\mathcal{V}\Pi_{x,i}^L\mathcal{V}^{-1} = \Pi_{x+\hat{s},i}^L + \frac{1}{6}\epsilon(x), \quad \mathcal{V}\left(-i\frac{\partial}{\partial W_3}\right)\mathcal{V}^{-1} \simeq -i\frac{\partial}{\partial W_3} + h,$$

we obtain

$$\mathcal{V}H_{eff}(\theta)\mathcal{V}^{-1} \simeq H_{eff}(\theta + 2\pi) - 2\sum_x \left(m + \frac{e^2}{24a}\right)\epsilon(x)\chi_x^\dagger\chi_x.$$

- The massless point is shifted $m = 0 \rightarrow -\frac{e^2}{24a}$.
- $\theta \rightarrow \theta + 2\pi$ is consistent with the chiral anomaly with $N_f = 2$.

Contents

Introduction

Discrete Chiral Transformation and Mass Shift

Summary

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We employ the QED Hamiltonian.

[Mass shift]

- The mass shift is exactly derived from the Gauss law.
- The interaction generates the non-trivial mass term.

[Shifting θ by 2π]

- We need a lot of assumptions to derive the shift,

$$dn = 0, \quad e_g \ll e \ll 1, \quad h_{12} = h_{23} = h_{31}, \quad \mathcal{V} \simeq e^{i\frac{\pi}{2}Q_A}.$$

- However, our result suggests

$$\mathcal{V}H(\theta)\mathcal{V}^{-1} \simeq H(\theta + 2\pi) - 2 \sum_x \left(m + \frac{e^2}{24a} \right) \epsilon(x) \chi_x^\dagger \chi_x.$$

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Chiral Transformation on $(3 + 1)\text{D}$

In $(3 + 1)\text{D}$, a fermion Hamiltonian [Kogut and Susskind, 1975] is given by

$$H = \frac{1}{2a} \sum_{j=1}^3 \left(\eta_j(x) \chi_{x+\hat{\mu}}^\dagger e^{iA_{x,\mu}} \chi_x - h.c. \right) + m\epsilon(x) \chi_x^\dagger \chi_x$$

$$\eta_1(x) = 1, \eta_2(x) = (-1)^{x_1}, \eta_3(x) = (-1)^{x_1+x_2}, \epsilon(x) = (-1)^{x_1+x_2+x_3}.$$

- Chiral transformation:

$$\chi_x \rightarrow \Gamma_5 \chi_x = i(-1)^{x_2} \chi_{x+\hat{1}+\hat{2}+\hat{3}}$$

- Chiral Anomaly:

$$\frac{d}{dt} \langle \chi^\dagger \Gamma \chi \rangle \rightarrow -\textcolor{red}{2} \frac{1}{2\pi^2} F \wedge F$$

[SA, Kikukawa, and Takemoto, 2025]

————→ This result implies the $\textcolor{red}{2}\pi$ shift of a θ -term.

Towards the Mass Shift and Discrete Chiral Symmetry

To derive the mass shift, we need to construct a θ -term with

$$\theta \rightarrow \theta + 2\pi$$

under the chiral transformation.

————→ It can be interpreted as $\mathcal{T}$ -transformation.

One duality in $(3+1)$ D Maxwell

In this talk, we consider a $(3+1)$ D Maxwell theory.

- Analyze the duality structure
- Determine the θ -term.

[Shao et al., 2026]: $(3+1)$ D compact scalar theory

[?]: $(2+1)$ D fractonic field theory

Diagonalization of $H_g(\theta)$

We recall

$$H_g(\theta) = \sum_x \sum_i \left[\frac{e^2}{2a} \left(\Pi_{x,i} + \frac{\theta}{8\pi^2} \left(B_{x+\hat{i},jk} + B_{x-\hat{j}-\hat{k},jk} \right) \right)^2 + \frac{1}{2ae_g^2} B_{x,jk}^2 \right]$$

with $B = dA + 2\pi n$. The Hilbert space can be decomposed by

$$\mathcal{H} = \bigoplus_{h,dn} \mathcal{H}_{h,dn},$$

where $h = \frac{1}{N} \sum_x n_{x,ij}$. The lowest energy state in $\mathcal{H}_{h,dn}$ is expressed by

$$|\psi\rangle_{h,dn} = \sum_n \exp\left(-\frac{1}{2} \sqrt{\frac{e}{e_g}} B \frac{1}{\sqrt{d\partial}} B\right)$$

Diagonalization of $H_g(\theta)$

The Hilbert space can be decomposed by

$$\mathcal{H} = \bigoplus_{h,dn} \mathcal{H}_{h,dn} \quad (h = \frac{1}{N} \sum_x n_{x,ij}).$$

The lowest energy state in $\mathcal{H}_{h,dn}$ is expressed by

$$|\psi\rangle_{h,dn} = \sum_{\substack{n \\ h,dn:\text{fixed}}} \exp\left(-\frac{1}{2}\sqrt{\frac{e}{e_g}}B\frac{P_d^{(2)}}{\sqrt{d\partial}}B - i\frac{\theta}{8\pi^2}\phi^\infty(A,n)\right) \\ \exp\left(iA(P_0^{(1)} + P_d^{(1)})m - i2\pi n\frac{1}{d\partial}dm\right) |n\rangle.$$

$$E = \frac{e^2}{2a} \sum_x \left(m + \frac{\theta}{2\pi} \star n\right) (P_0^{(1)} + P_d^{(1)}) \left(m + \frac{\theta}{2\pi} \star n\right) \\ + \frac{4\pi^2}{2e_g^2 a} \sum_x \left(\frac{h^2}{N^4} + dn\frac{1}{d\partial}dn\right) + \frac{1}{2}\sqrt{\frac{e}{e_g}} \text{tr}\left(\sqrt{d\partial}\right)$$

$e_g \ll e \ll 1$ **region**

- Photonic excitation modes have $E \sim \sqrt{\frac{e}{e_g}}$.
- If $e_g \ll e \ll 1$, such an excitation is suppressed.

$$\left\langle e^{idA} \right\rangle = e^{-C\sqrt{\frac{e_g}{e}}} \exp\left(i2\pi \frac{h}{N^2}\right) = e^{-C\sqrt{\frac{e_g}{e}}} e^{id\bar{A}}$$

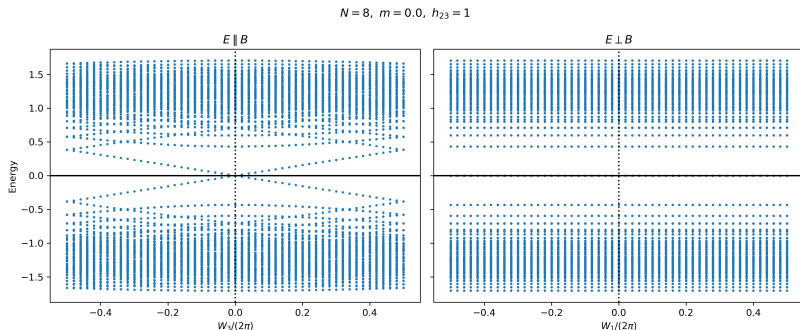
Then, we can write $A = \bar{A} + \frac{W}{N}$ and

$$\mathcal{V} \exp(iW_i) \mathcal{V}^{-1} = \exp\left(iW_i + i2\pi \frac{h_{ij}}{N^2} + i2\pi \frac{h_{ik}}{N^2}\right).$$

If $h_{12} = h_{23} = h_{31}$, $\mathcal{V} \exp(iW_i) \mathcal{V}^{-1} = \exp(iW_i)$

Spectral Flow: $B_1 = B_2 = 0$, $B_3 = \frac{2\pi}{N^2}$

We consider a magnetic field in the x_3 direction with $h = 1$ and add parallel/orthonormal electric fields.



Figures imply the chiral anomaly $\propto E_3 B_3$

Mass Shift with Dynamical Gauge Field

If

$$\mathcal{V}(-i\frac{\partial}{\partial W_3})\mathcal{V}^\dagger = -i\frac{\partial}{\partial W_3} + h_{12}$$

holds rigorously, we can recover the θ term

$$\mathcal{V}(\Pi_{x,3}^T)\mathcal{V}^\dagger = \Pi_{x+\hat{3},3}^T + (dA + 2\pi n)_{x+\hat{3},12} + (dA + 2\pi n)_{x-\hat{1}-\hat{2},12}$$

by using a unitary transformation [SA, Kikukawa, Takemoto, 2607.25332]