43rd International Symposium on Lattice Field Theory
College Park, 29 July 2026

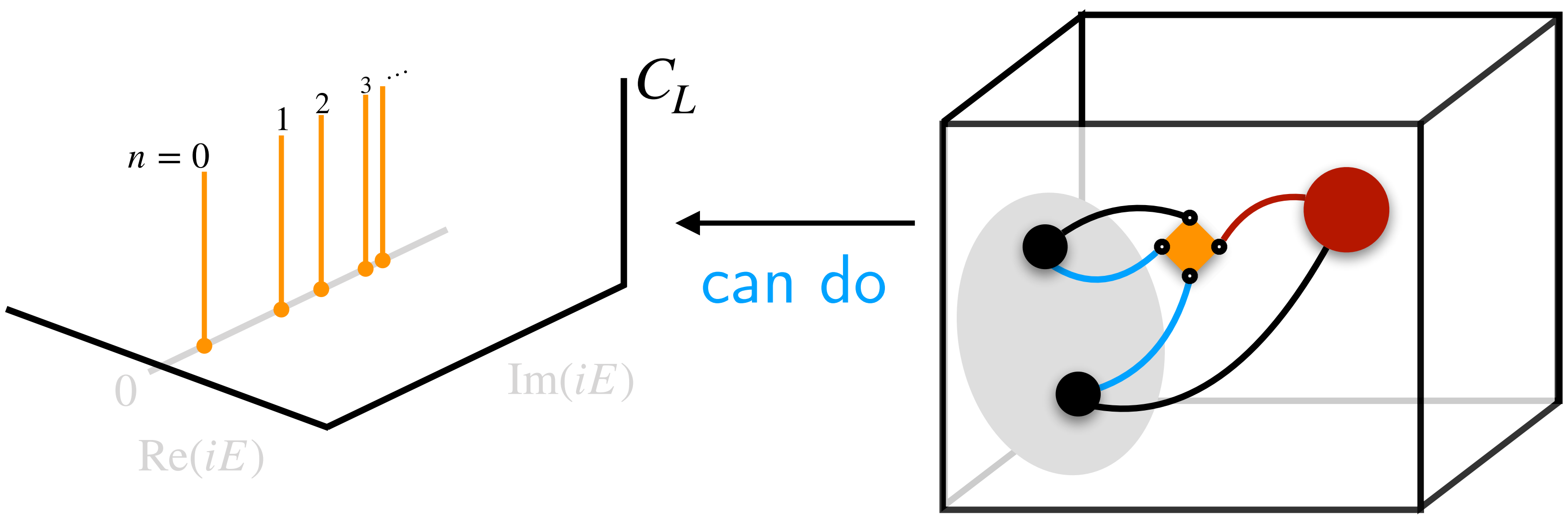
A perturbative N-particle finite-volume formalism

Rajnandini Mukherjee

based on work with Maxwell T. Hansen

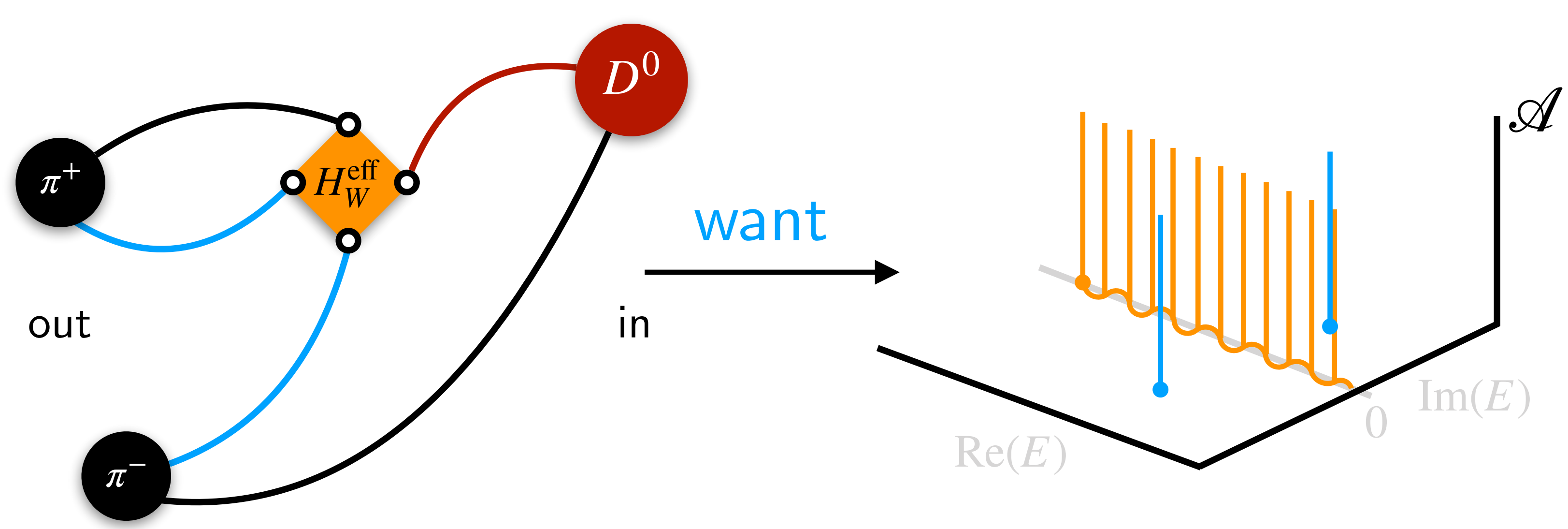
THE UNIVERSITY of EDINBURGH

Motivation



goal: **multi-hadron processes**

hadronic decays, scattering, resonances



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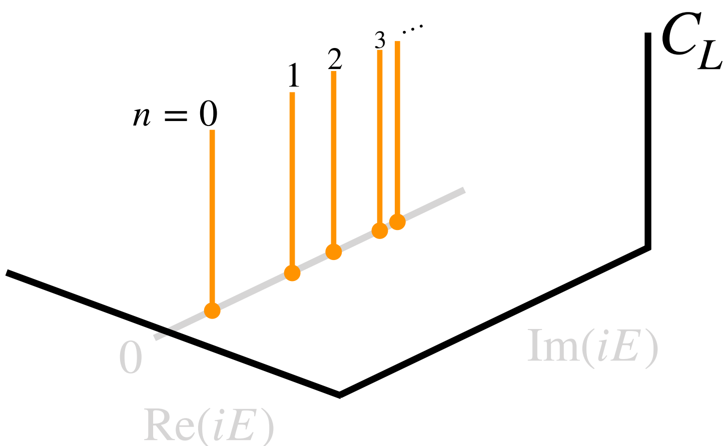
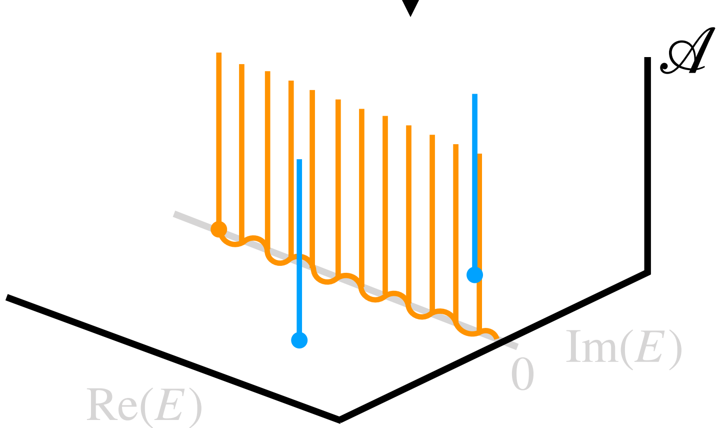


$$E_n(L) = E_n^{\text{free}}(L) + \delta_n(L)$$

spectrum $\xleftrightarrow[\text{formalism}]{\text{FV}}$ **quantisation condition** (QC) $\xrightarrow{\text{fit}}$ **K-matrices** $\xrightarrow[\text{integral eqns}]{\text{unitarity}}$ **scattering amplitudes**

$\downarrow$ analytic continuation



Motivation

States with the same FV quantum numbers mix

$$
\begin{aligned}
\left|E_n, L\right\rangle = \;& C_{\pi\pi}(E_n, L)\,\left|\pi\pi, \text{out}\right\rangle_{\text{KLP*}} \\
&+ C_{4\pi}(E_n, L)\,\left|\pi\pi\pi\pi, \text{out}\right\rangle_{\text{KLP}} \\
&+ C_{K\overline{K}}(E_n, L)\left|K\overline{K}, \text{out}\right\rangle_{\text{KLP}} \\
&+ \;(\eta\eta, 6\pi, \ldots)
\end{aligned}
$$

*K-matrix-like piece

spectrum $\xleftrightarrow[\text{formalism}]{\text{FV}}$ **quantisation condition** (QC) $\xrightarrow{\text{fit}}$ **K-matrices** $\xrightarrow[\text{integral eqns}]{\text{unitarity}}$ **scattering amplitudes**

- QC2 ✓ *Lüscher (1986)*
- QC3 ~✓ but **challenging**! *Hansen, Sharpe, Döring, Mai, Hammer, Pang, Rusetsky,… S Sharpe Lattice2025 talk*

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we want to go beyond this!

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Q: can we neglect higher multi-particle effects?

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- QC4, …, QCN **??**

Q: can we neglect higher multi-particle effects?

Q: how can we quantify higher multi-particle effects without a non-perturbative QCN?

N-particle quantisation condition in PT

Finite-volume two-point function:

$$C_L(E, \mathbf{P}) = \int_0^\infty dt\, e^{iEt - \varepsilon t} \langle 0 \,|\, \hat{\mathcal{O}}_{\mathbf{P}}(t) \hat{\mathcal{O}}^\dagger_{\mathbf{P}}(0) \,|\, 0 \rangle$$

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$$= i\langle 0\,|\,\hat{\mathcal{O}}_{\mathbf{P}}(0)\ \hat{G}(E,\mathbf{P},L)\ \hat{\mathcal{O}}^\dagger_{\mathbf{P}}(0)\,|\,0\rangle$$

$$\boxed{\hat{G}(E,\mathbf{P},L) = \frac{1}{E\,-\,\hat{H}(\mathbf{P},L)\,+\,i\varepsilon}}$$

Green's function

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$$\boxed{\hat{G}(E,\mathbf{P},L) = \frac{1}{E - \hat{H}(\mathbf{P},L) + i\varepsilon}}$$ Green's function

non-interacting

$$\boxed{\hat{G}_0 = \frac{1}{E - \hat{H}_0 + i\varepsilon}}$$

$$\boxed{\hat{H}(\mathbf{P},L) = \hat{H}_0(\mathbf{P},L) + \hat{V}(\mathbf{P},L)}$$ finite volume Hamiltonian

non-interacting Hamiltonian

(PT)

interaction potential

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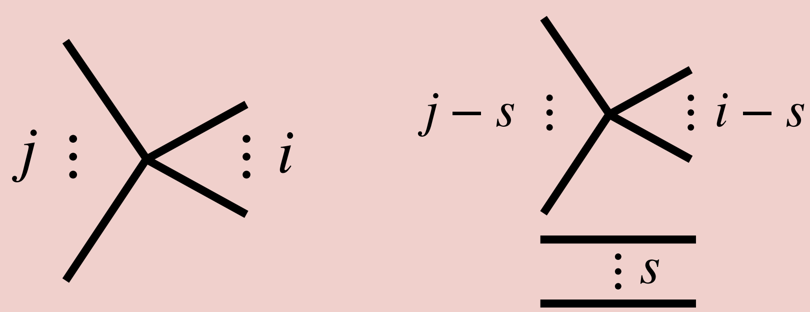
Key steps:

1

$$V_{ij} = V^{\mathsf{c}}_{ij} + V^{\mathsf{d}}_{ij}$$

connected **d**isconnected



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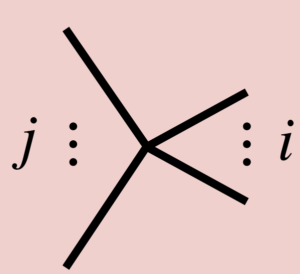
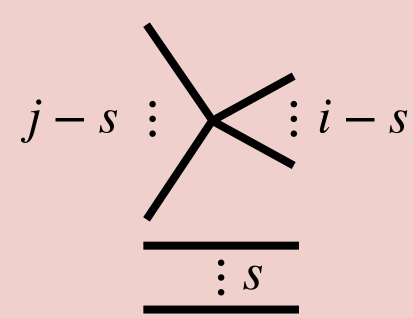
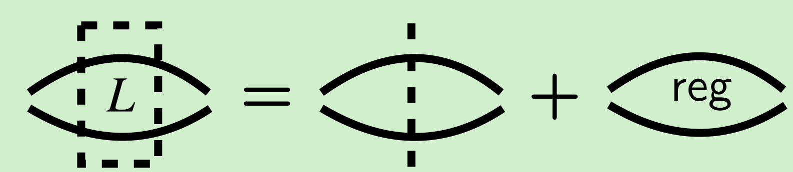




2

$$G_0 = S_0 + R_0$$

singular **r**egular



$$E_{\min} < E < E_{\max}$$

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$$= C_L^{\text{reg}} + i\mathscr{L} \frac{1}{S^{-1} - B^{\mathsf{c}}} \mathscr{R}^\dagger \quad \text{all singular behaviour in } S$$

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Poles of $C_L(E, \mathbf{P})$ satisfy

$$\boxed{\det \left[S(E, \mathbf{P}, L)^{-1} - B^{\mathsf{c}}(E, \mathbf{P}, L) \right] = 0} \qquad \textit{all-orders QC}$$

- $S^{-1} = S_0^{-1} - B^{\mathsf{d}}$
- B^{c}, B^{d}: effective interactions from V^{c}, V^{d} and R_0

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- B^{c}, B^{d}: effective interactions from V^{c}, V^{d} and R_0
- At leading order: $B^{\mathsf{c/d},\text{LO}} = V^{\mathsf{c/d}}$

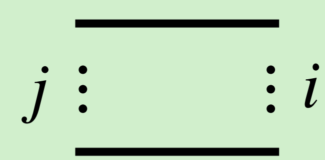
$$\boxed{\det\left[S^{\text{LO}}(E,\mathbf{P},L)^{-1} - B^{\mathsf{c},\text{LO}}(E,\mathbf{P},L)\right] = 0} \quad \textit{LO QC}$$

Scalar field theory

Identical spin-zero particles in a box:

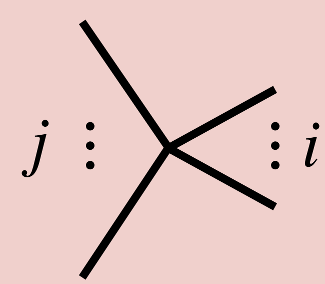
$$\mathscr{L} = \frac{1}{2}\partial_\mu\varphi\partial_\mu\varphi + \frac{m^2}{2}\varphi^2 + \sum_{k>3}\frac{g_k}{k!}\varphi^k$$

- single flavour (m)
- total momentum $\mathbf{P} = \mathbf{0}$
- A_1^+ irrep of O_h

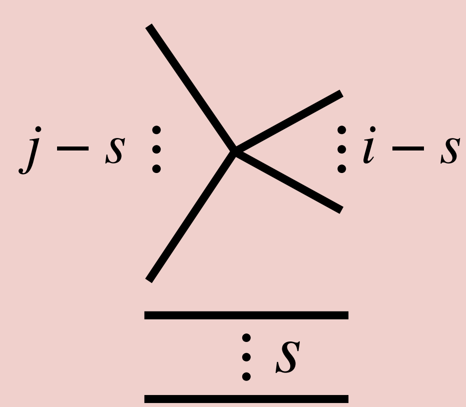


$$G_{0,ij}(E) \sim \frac{\delta_{j,i}}{E - E_n^{(0)} + i\varepsilon}$$

$g_{j\leftarrow i} = g_{i+j}$



$$V^{\mathsf{c}}_{ij} \sim g_{j\leftarrow i}$$



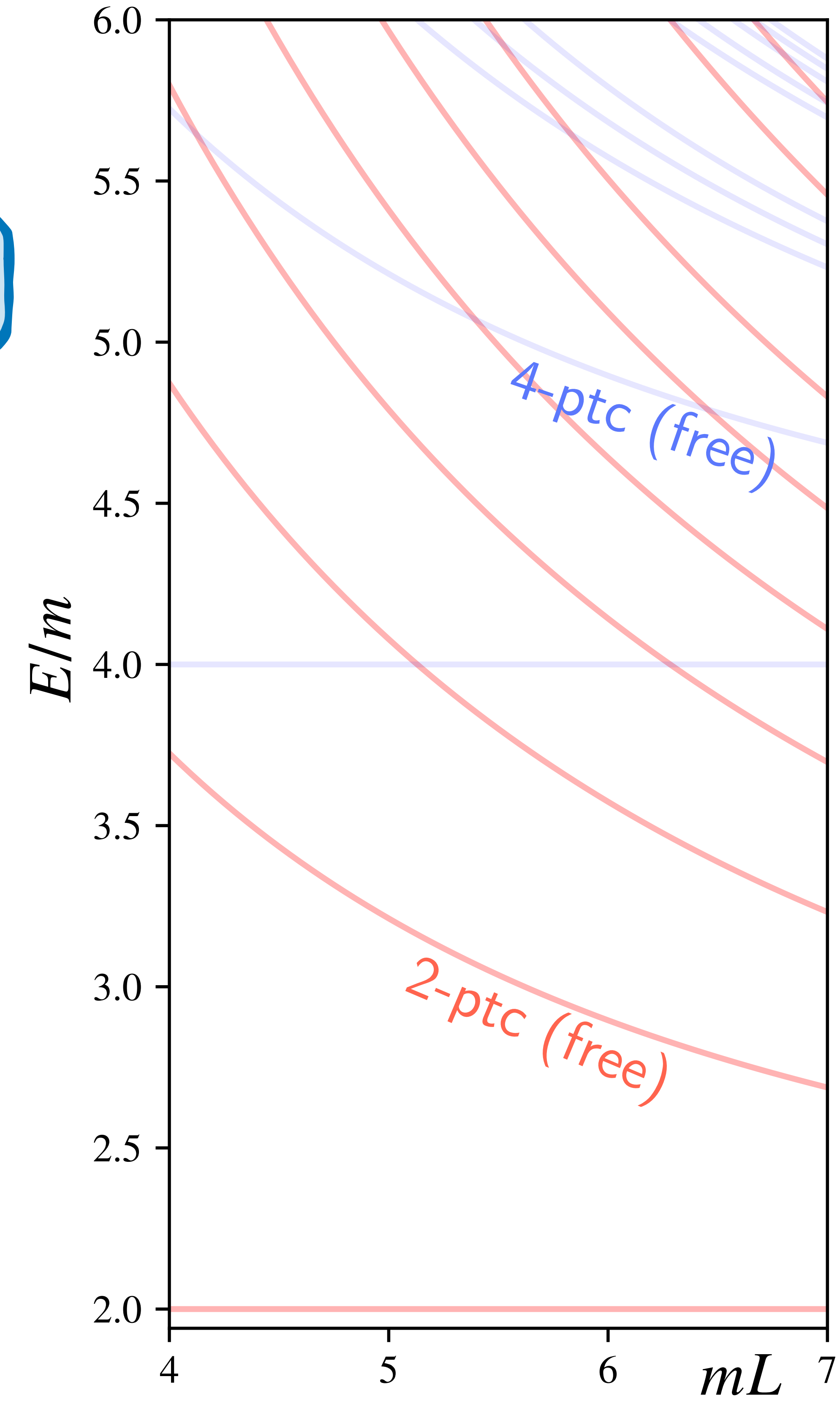
$$V^{\mathsf{d}}_{ij} \sim \sum_s g_{(j-s)\leftarrow(i-s)} \prod^s \delta$$

$N \leq 4$ results

2 and 4-particle coupled channel system

Proceedings from Lattice2025

$$\det\left[S^{\mathrm{LO}}(E,\mathbf{P},L)^{-1} - B^{\mathrm{c,LO}}(E,\mathbf{P},L)\right] = 0$$



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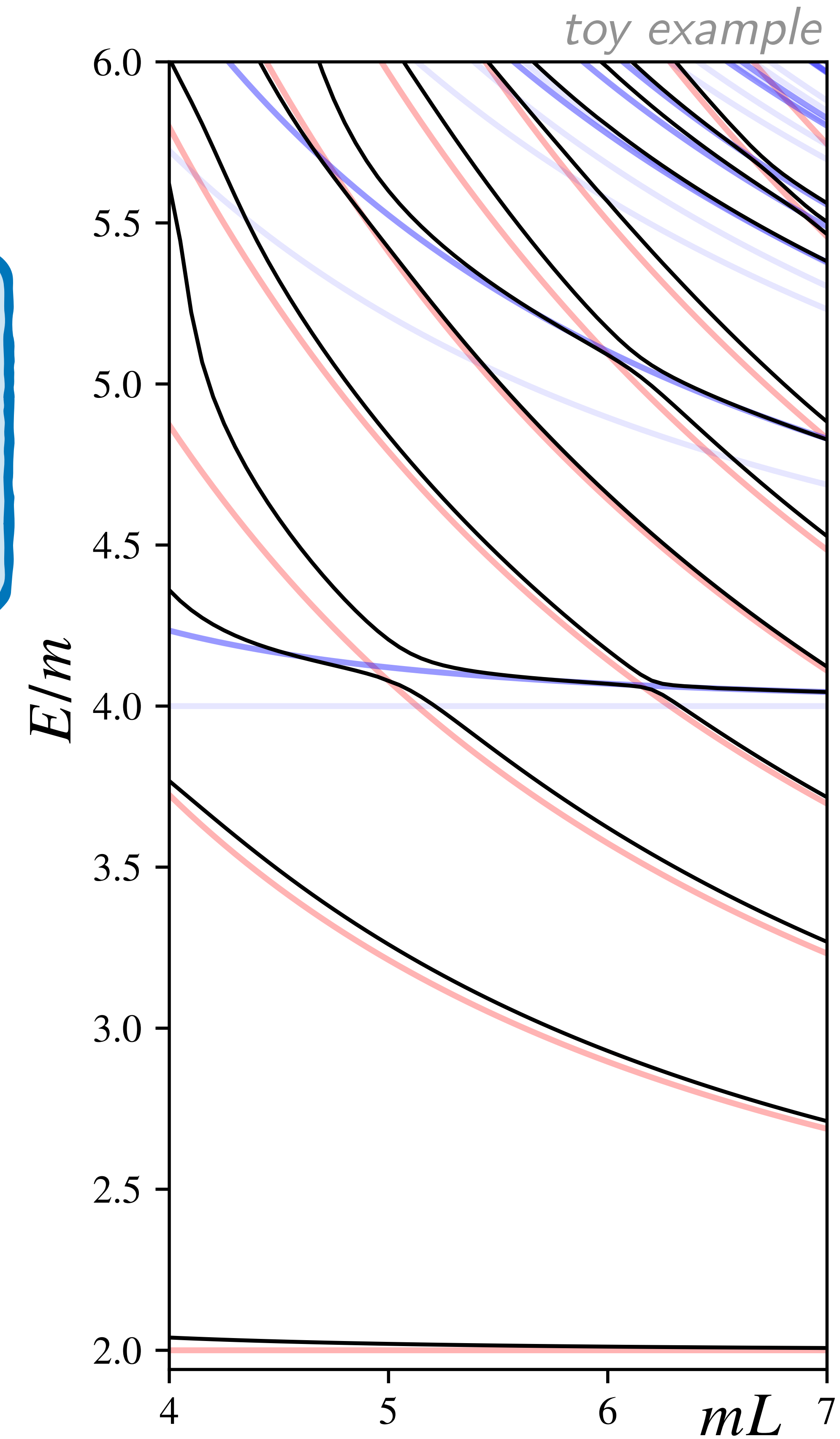
$$\det\left[\begin{pmatrix} S_{0,2}(E,\mathbf{0},L)^{-1} & 0 \\ 0 & S_4(E,\mathbf{0},L)^{-1}\end{pmatrix} - \begin{pmatrix} g_{2\leftarrow 2} & g_{2\leftarrow 4} \\ g_{4\leftarrow 2} & g_{4\leftarrow 4}\end{pmatrix}\right] = 0$$



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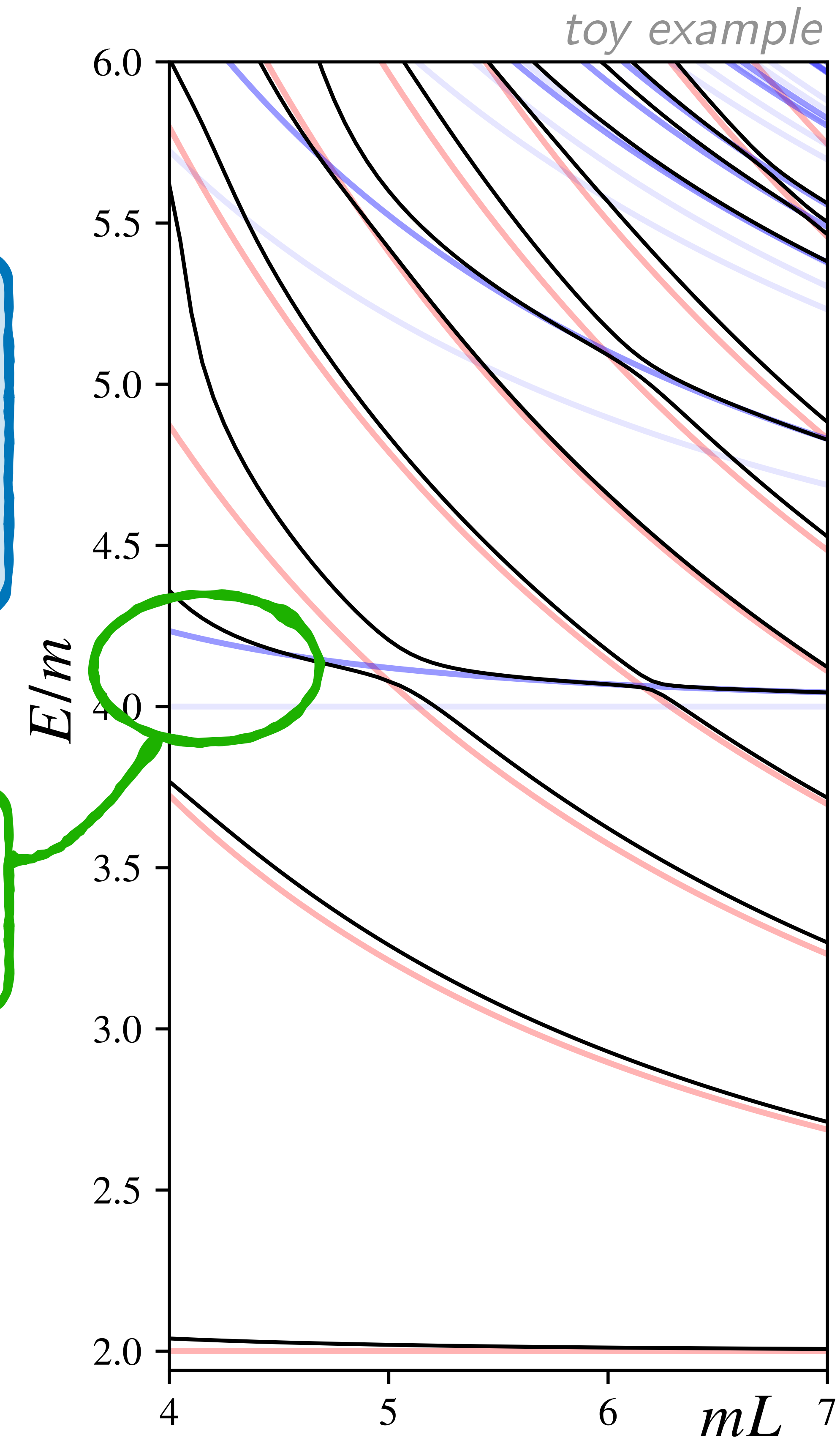
$S_4 \ni$ LO $2 \to 2$ rescattering, using **TOPT**

$$E_4^{(0)}(L) = 4m + \binom{4}{2}\frac{g_{2\leftarrow 2}}{4m^2L^3} + \ldots$$

Huang & Yang (1957)



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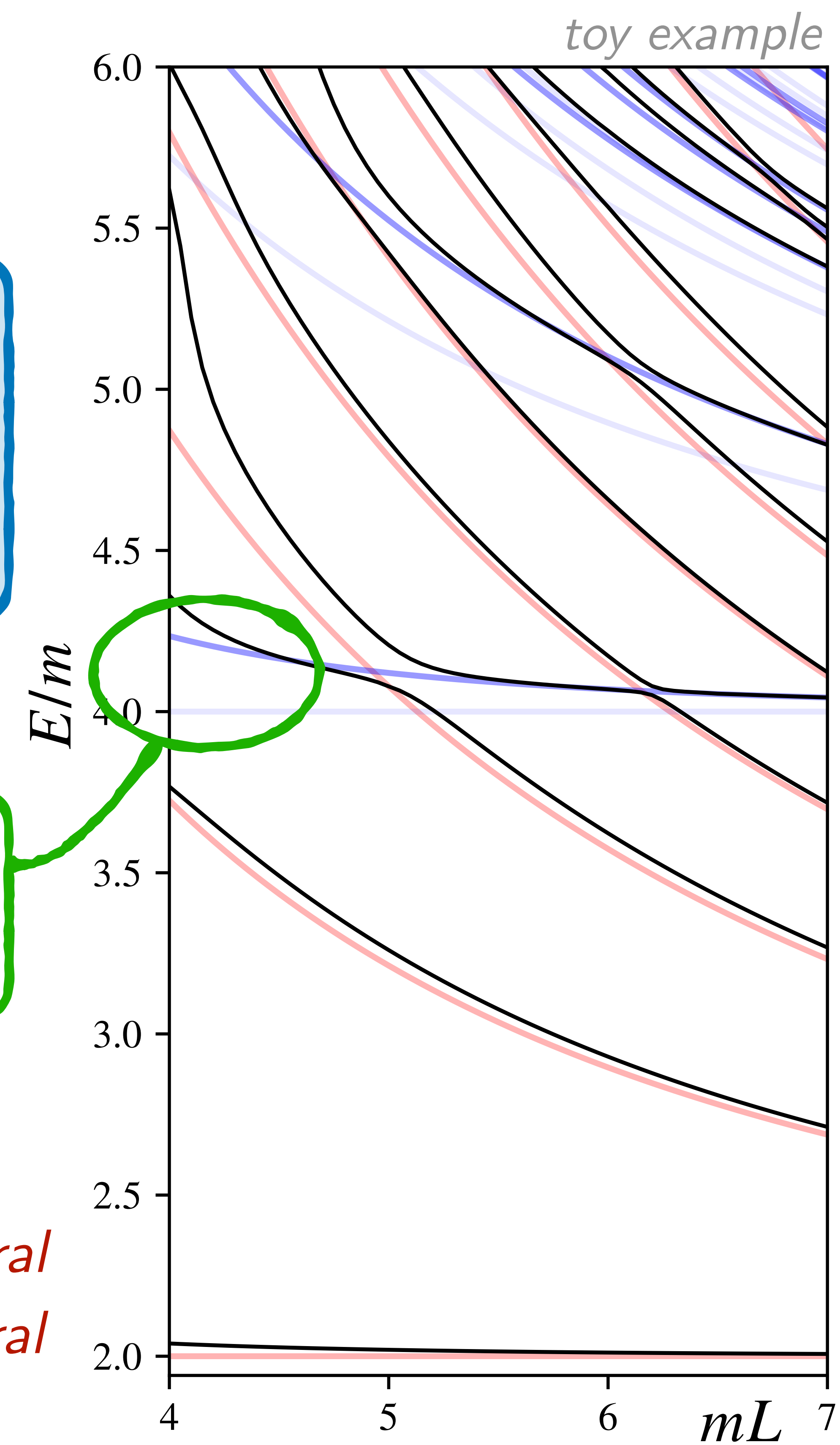
- ✳ N-particle framework reproduces same result
  - ✳ simpler than previous approach, *no τ integral*
  - ✳ simpler than Lüscher approach, *no $\mathbf{k}$ integral*



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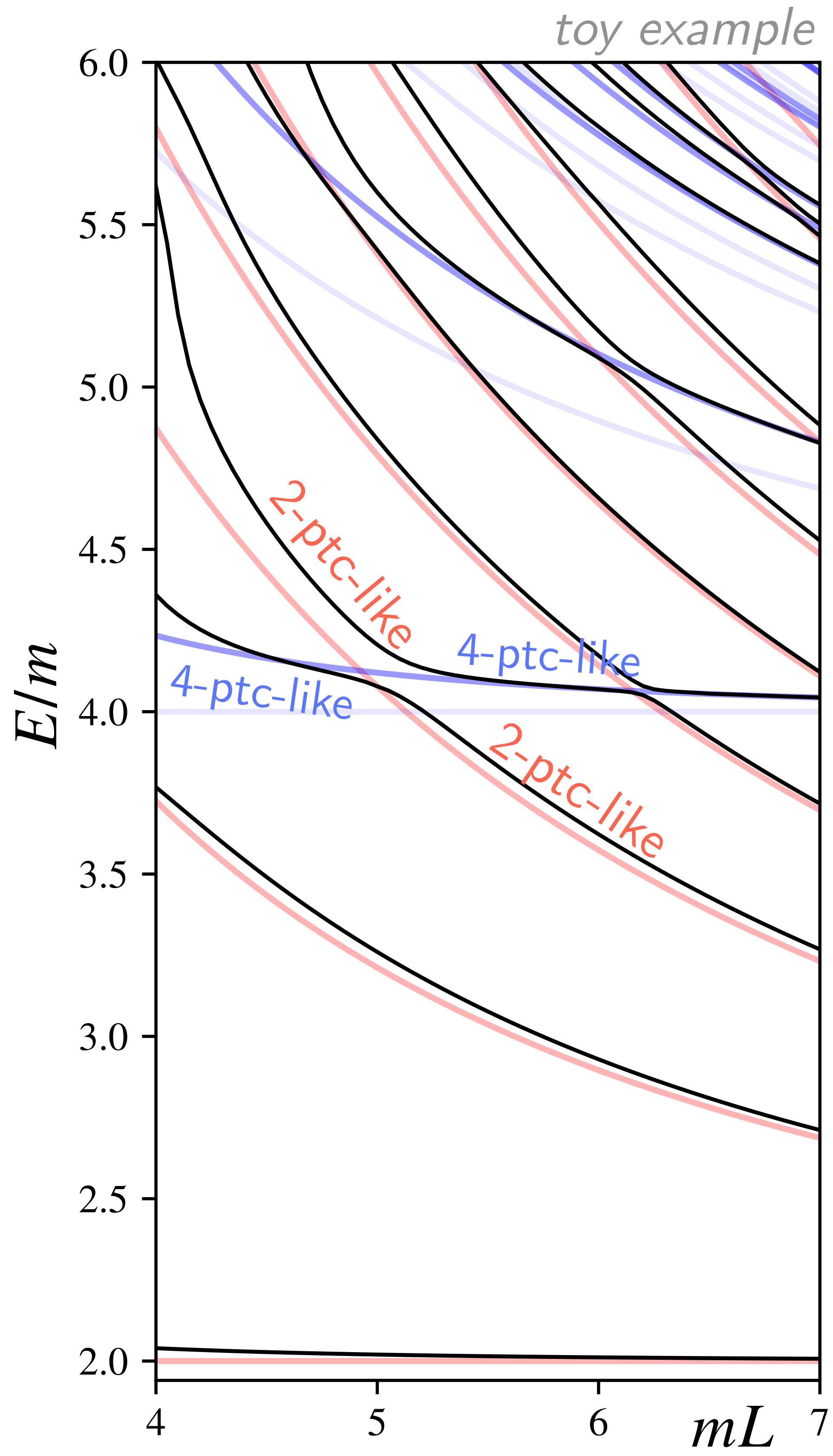

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@ $E_n(L)$

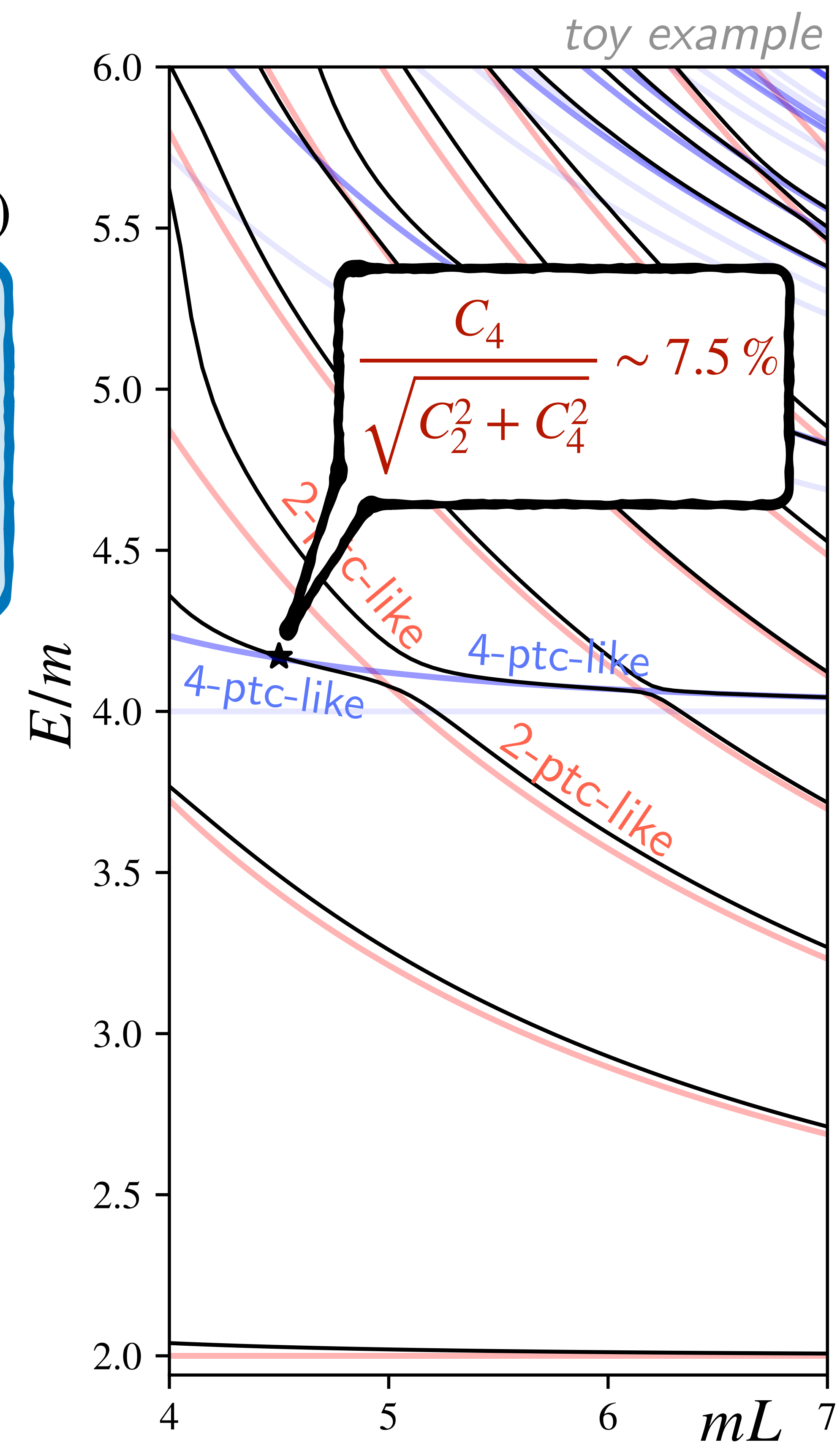
$$\left[\begin{pmatrix} S_{0,2}^{-1} & 0 \\ 0 & S_4^{-1} \end{pmatrix} - \begin{pmatrix} g_{2\leftarrow 2} & g_{2\leftarrow 4} \\ g_{4\leftarrow 2} & g_{4\leftarrow 4} \end{pmatrix}\right] \begin{bmatrix} C_2 \\ C_4 \end{bmatrix} = 0$$

✳ Main result:

the eigenvector associated with the zero eigenvalue tells us how much each particle sector contributes to a finite-volume state

Hansen & Sharpe (2012)

Briceño, Hansen & Walker-Loud (2015)
Hansen, Romero-Lopez & Sharpe (2021)



How to use this framework

LO QC

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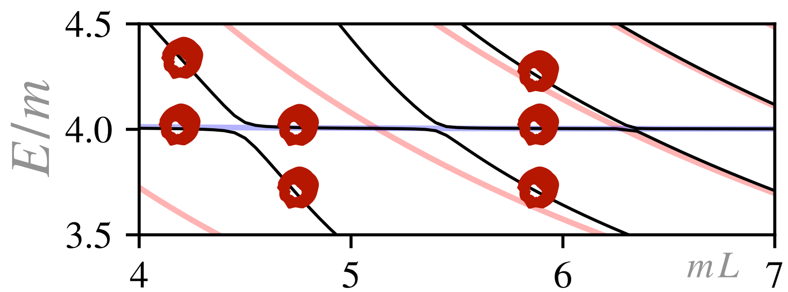
$|E_n, L\rangle$



fit

$$\begin{pmatrix} g_{2\leftarrow 2} & & \cdots & \\ & \ddots & & \\ \vdots & & g_{n\leftarrow i} & \\ & & & \ddots \end{pmatrix}$$

lattice data *bare couplings*

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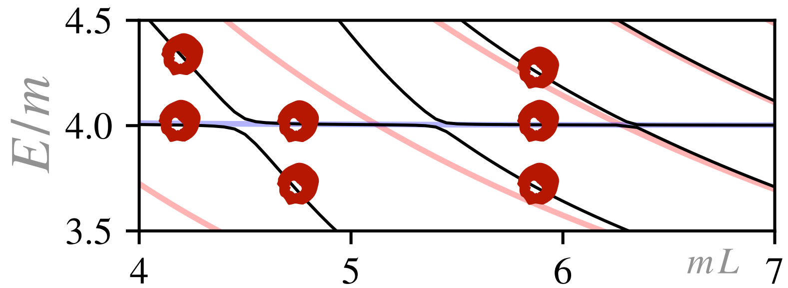

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predict

$$\begin{pmatrix} C_2 \\ \vdots \\ C_n \\ \vdots \end{pmatrix} \begin{array}{l} |\,2\text{-ptc, out}\rangle_{\mathrm{KLP}*} \\ \\ |\,n\text{-ptc, out}\rangle_{\mathrm{KLP}} \\ \\ \end{array}$$

lattice data

bare couplings

multi-particle admixtures

at the level of K-matrices,
this framework provides a *systematic estimate* for n-particle effects

*K-matrix like piece

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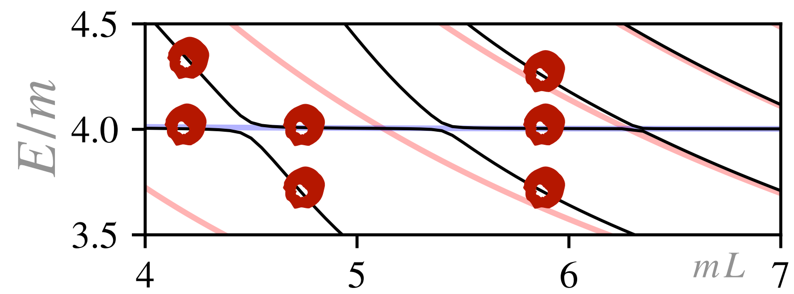

fit

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$$\begin{pmatrix} C_2 \\ \vdots \\ C_n \\ \vdots \end{pmatrix} \begin{matrix} |\,2\text{-ptc, out}\rangle_{\mathrm{KLP}*} \\ \\ |\,n\text{-ptc, out}\rangle_{\mathrm{KLP}} \\ \end{matrix}$$

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unitarity, integral equations

propagate as uncertainty

$\mathcal{A}_{2-\mathrm{ptc}}$ $(\mathrm{err})_{n-\mathrm{ptc}}$ $\cdots$

decay amplitudes

*K-matrix like piece

Scope

strengths	limitations

1. General: applicable to arbitrary particle multiplicity & higher energies
2. Practical: captures leading FV effects without solving full many-body problem
3. Systematic: controlled expansion with known domain of validity

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limitations

Perturbative treatment breaks down near

1. strongly coupled resonances
2. kinematic thresholds

where rescattering must be resummed to all orders

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limitations

Perturbative treatment breaks down near

1. strongly coupled resonances
2. kinematic thresholds

where rescattering must be resummed to all orders

Hadronic processes: dynamics must be treated non-perturbatively!

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Proposal: a hybrid quantisation condition

LO QC captures $O(g_{j\leftarrow i})$ effects in all sectors:

$$\det\left[\begin{pmatrix} (S^{\mathrm{LO}}_{2\leftarrow 2})^{-1} & & \cdots & \\ & \ddots & & \\ \vdots & & (S^{\mathrm{LO}}_{j\leftarrow i})^{-1} & \\ & & & \ddots \end{pmatrix} - \begin{pmatrix} B^{\mathrm{c,LO}}_{2\leftarrow 2} & & \cdots & \\ & \ddots & & \\ \vdots & & B^{\mathrm{c,LO}}_{j\leftarrow i} & \\ & & & \ddots \end{pmatrix}\right] = 0$$

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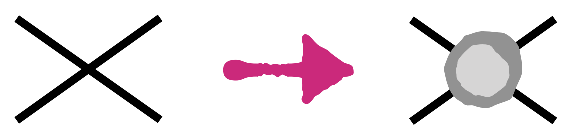
Instead **treat two-particle sector non-perturbatively**

Replace: LO interaction  all-orders resummed $\mathscr{K}_{\mathrm{BS}}$ & R_0

$g_{2\leftarrow 2}$ $\mathscr{K}_2$

$$\det\left[\begin{pmatrix} F_2^{-1} & & \cdots & \\ & \ddots & & \\ \vdots & & (S^{\mathrm{LO}}_{j\leftarrow i})^{-1} & \\ & & & \ddots \end{pmatrix} - \begin{pmatrix} \mathscr{K}_2 & & \cdots & \\ & \ddots & & \\ \vdots & & B^{\mathrm{c,LO}}_{j\leftarrow i} & \\ & & & \ddots \end{pmatrix}\right] = 0$$

$\mathscr{K}_2$: (non-standard) 2-particle K-matrix

$F_2 \sim$ Lüscher zeta function *Lüscher (1986)*

Scope (hybrid)

strengths

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potential test cases

system	non-perturbative channels	perturbative channels
$K\pi\pi$, $N\pi\pi$	$K^* \leftrightarrow K\pi$, $\Delta \leftrightarrow N\pi$	$K\pi\pi$, $N\pi\pi$ cross-check?

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hadronic D decays	$D \to \pi\pi, K\overline{K}$	4π, 6π, … new!

Summary & outlook

Q: can we neglect higher multi-particle effects?

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- *simple*, perturbative coupled-channel description of N particles
- captures LO finite-volume effects (reproduces known results)
- ★ quantifies size of LO n-particle contributions
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Needs to be tested on lattice data!

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Where did the hats go? backup

Finite-volume two-point function:

$$C_L(E,\mathbf{P}) = C_L^{\text{reg}} + i\mathscr{L}\frac{1}{S^{-1} - B^{\mathsf{c}}}\mathscr{R}^\dagger$$

$$= C_L^{\text{reg}} + i\sum_{\psi,\psi'}\Big\langle 0\,\Big|\,\hat{\mathscr{L}}\underbrace{\Big|\,\psi\Big\rangle\Big\langle\psi\,\Big|}\frac{1}{\hat{S}^{-1} - \hat{B}^{\mathsf{c}}}\underbrace{\Big|\,\psi'\Big\rangle\Big\langle\psi'\,\Big|}\hat{\mathscr{R}}^\dagger\Big|\,0\Big\rangle$$

complete sets of
finite volume states

To-do: in a given theory

1. Construct $\{\,|\,\psi\rangle\}$, a complete set of FV eigenstates of $H_0(\mathbf{P},L)$ or even $\{\,|\,\psi,\Lambda\rangle\}$ with $\Lambda \in \mathrm{Irrep}(\mathrm{LG}(\mathbf{P}))$
2. Fix kinematic range to $E_{\min} < E < E_{\max}$
3. Compute matrix elements of S_0, V^{c} and V^{d}