# Reweighting using worldline formalism

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# Outline

- Motivation
- Proposed formalism
- Exploratory numerical studies
- Conclusion

# Motivation

- Complex Euclidean actions $\Rightarrow$ traditional Monte Carlo methods break down, eg.
  - Finite chemical potential $\mu > 0$ in QCD
  - QED with external electric background $\mathbf{E}_\mathrm{E} = -i\mathbf{E}_\mathrm{M}$
- **Reweighting**: correct the statistical weight under the unperturbed, positive action $S_0$

$$\langle \mathcal{O} \rangle = \frac{\langle \mathcal{O}\mathcal{V} \rangle_0}{\langle \mathcal{V} \rangle_0}$$

- Target cases:

$$\mathcal{V}[A, \mathcal{A}^{\mathrm{ext}}] \equiv e^{\Gamma[A,0] - \Gamma[A,\mathcal{A}^{\mathrm{ext}}]}, \quad \Gamma[A, \mathcal{A}^{\mathrm{ext}}] \equiv -\ln\det\left(\not{D}[A] + \not{\mathcal{A}}^{\mathrm{ext}}\right)$$

  $\Rightarrow$ Can we estimate this is in an efficient, systematic way for generic external potential?

- <u>Caveat</u>: Monte Carlo sampling efficiency limited by the overlap.

# Proposed formalism

## Worldline formalism

- In continuous spacetime, many tools to calculate

$$\Delta\Gamma[A,\mathcal{A}^{\mathrm{ext}}] \equiv \Gamma[A,\mathcal{A}^{\mathrm{ext}}] - \Gamma[A,0]$$

- Worldline formalism with heat-kernel regularization for QED

$$\Gamma_{t_h}[A] \equiv -\frac{1}{2}\int_{t_h}^{\infty}\frac{dT}{T}e^{-m^2T}\int_{x(0)=x(T)}\mathcal{D}x(\tau)\;\Phi[x,F]\mathcal{W}[x,A]e^{-\frac{1}{4}\int_0^T d\tau\,\dot{x}^2(\tau)}\,.$$

  $\mathcal{W}$ : Wilson loop; $\Phi$: spinor factor; $\mathcal{D}x$: spacetime trajectories.

- Can this be used in a lattice calculation?

  Yes, if the data are smooth enough for an interpolation.

# Proposed formalism

## Proposal

**Steps**:

1. Wilson-flow gauge configurations $A^{\mathrm{lat}}$ to flow-time $t = t_h$.

$\Rightarrow$ smooth, can be embedded in the continuous space $\tilde{A}(t|a)$.

2. $\boxed{2.}$ Evaluate $\Delta\Gamma_t$ using the worldline formalism with $\tilde{A}(t|a)$.

3. $\boxed{3.}$ Remove $t$-dependent divergence by adding counter terms at fixed lattice spacing $a$.

4. Take the $a \to 0$ and $t \to 0$ limits in turn to get the physical result $\langle \mathcal{O} \rangle$.

# Exploratory numerical studies

## Step 2: Embedding to continuous space

- Wilson-flowed $A^{\mathrm{lat}}(a,t)$: UV modes suppressed by $e^{-p^2 t}$.

- Whittaker-Shannon interpolation:

$$f(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^N} f(a\mathbf{n}) \prod_{i=1}^{N} \mathrm{sin}_{\mathrm{c}}\left(\frac{x_i - an_i}{a}\right), \quad \forall \mathbf{x} \in \mathbb{Z}^N .$$

perfect frequency domain reconstruction from limited bandwidth.

- Need to be careful with gauge degrees of freedom
  
  $\Rightarrow$ gauge-fixing before applying W-S.

- No wave function renormalization needed:

$$\tilde{A}(t|a) = A(t) + \mathrm{O}(a), \quad A(t) \equiv \lim_{a\to 0} A^{\mathrm{lat}}(a,t).$$

# Proposed formalism

## Step 3: counter terms and renormalization for $\Delta\Gamma$

- $1/t$ and $\ln t$ divergences inferred from the Seeley-DeWitt coefficients.
- Example: QED in an E&M background

$$\Delta\Gamma_t[\tilde{A}(t|a),\mathcal{A}^{\mathrm{ext}}] \underset{t\to 0}{\sim} -\frac{1}{32\pi^2}\ln\left(\frac{1}{t}\right)\int d^4x\left[\frac{2}{3}(\mathcal{F}^{\mathrm{ext}}_{\mu\nu})^2+\frac{4}{3}F^{\mu\nu}(t)\mathcal{F}^{\mathrm{ext}}_{\mu\nu}+\mathrm{O}(a)\right].$$

- Wave-function renormalization for the external potential

$$\Delta\Gamma_t^{1-\mathrm{loop,R}}[\tilde{A}(t|a),\mathcal{A}^{\mathrm{ext}}]\equiv \Delta\Gamma_t[\tilde{A}(t|a),\mathcal{A}^{\mathrm{ext,R}}(\mu)]\;+\;\delta Z_{\mathrm{ext}}(\mu,t)\int d^4x\tilde{F}_{\mu\nu}(t|a)\mathcal{F}^{\mathrm{ext,R}}_{\mu\nu}(\mu)\,,$$

where $\delta Z_{\mathrm{ext}}(\mu,t)=-\frac{1}{24\pi^2}\ln(\mu^2 t)$.

- Repeat the procedure on each configuration for a clean $a\to 0$ limit at fixed physical $t>0$.

# Exploratory numerical studies

$$\Gamma_t[A] \equiv -\frac{1}{2}\int_t^\infty \frac{dT}{T} e^{-m^2 T} \int_{x(0)=x(T)} \mathcal{D}x \; \Phi[x,F]\mathcal{W}[x,A] e^{-\frac{1}{4}\int_0^T d\tau\, \dot{x}^2} \,,$$

$\Rightarrow$ Master integral:

$$I = \lim_{M\to\infty} I_M \,,$$

where

$$I_M \equiv \int_0^\infty dT\; h(T) e^{-m^2 T} \prod_\mu \int_{-\infty}^\infty da_{\mu,0}$$
$$\prod_{k=1}^M \int_{-\infty}^\infty da_{\mu,k} \int_{-\infty}^\infty db_{\mu,k}\; g\left(a_{\mu,0}, \{a_{\mu,i}, b_{\mu,i}\}_{1\le i\le M}\right) e^{-\frac{\pi^2}{2T}\sum_{n=1}^M n^2\left(a_{\nu,n}^2 + b_{\nu,n}^2\right)} \,,$$

with close-loop Fourier expansion

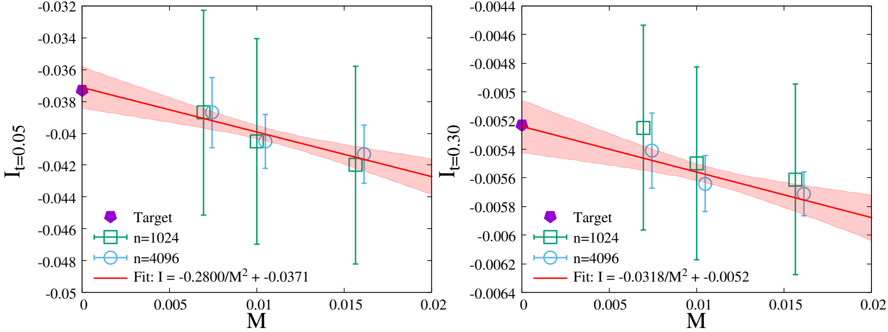
$$x_\mu^{(M)}(\tau) \equiv \frac{a_{\mu,0}}{2} + \sum_{n=0}^M a_{\mu,n}\cos\omega n\tau + b_{\mu,n}\sin\omega n\tau \,.$$

$\Rightarrow$ feasible path evaluation thanks to high-bandwidth hardware!

# Exploratory numerical studies

## Error scaling and the $M \to \infty$ limit

System: $8^4$ QED with an external field $A_y(x) = 0.5sin(kx) - 0.3sin(2kx) + 0.2sin(3kx)$



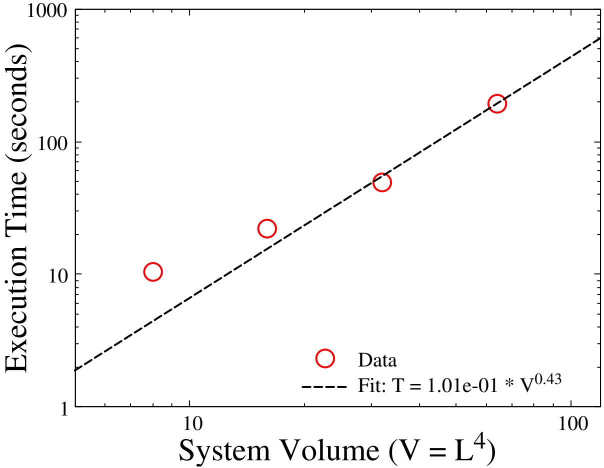

($M$: # Fourier modes; $n$: # sampled paths)

- Randomized Quasi-Monte Carlo sampling for each data point (8 batches each for error scaling test).
- Expected $1/M^2$-scaling after 1st-order Richardson extrapolation confirmed.

# Exploratory numerical studies

- Wall-time scaling with the volume on one Nvidia A100 GPU.

  Example: 4d-QCD with external finite potential at $L = 8, 16, 32, 64$.



# Conclusion

## Summary

- A reweighting method based on application of worldline formalism to smoothly-interpolated lattice fields is proposed.
- Established a workflow for getting properly renormalized quantity.

## Outlook

- Potential applications:
  - Finite chemical potential QCD at high temperature, comparing with results from Taylor expansion [Borsányi et al, PRL '21] and extending to higher $\mu$.
  - Strong-field QED: investigate the perturbative expansions in $\alpha\chi^{2/3}$ with $\chi = (e/m^3)\sqrt{-(F_{\mu\nu}p^\nu)^2}$ [Mironov et al, PRD '20].

# Back-up slides

# Notation

- Consider a gauge theory with fermions with a background field

$$\mathcal{Z}[\mathcal{A}^{\mathrm{ext}}] = \int \mathcal{D}[A,\psi,\bar{\psi}] e^{-S_{\mathrm{g}}[A]-S_{\mathrm{f}}[\mathcal{A}^{\mathrm{ext}},A,\psi,\bar{\psi}]} \,.$$

  where

$$S_{\mathrm{f}}[\mathcal{A}^{\mathrm{ext}},A,\psi,\bar{\psi}] = \bar{\psi}\not{D}[A]\psi + \bar{\psi}\not{\mathcal{A}}^{\mathrm{ext}}\psi\,, \quad \not{D} \equiv \not{\partial} + i\not{A} + m\,.$$

- Integrating out the fermion degrees of freedom,

$$\mathcal{Z}[\mathcal{A}^{\mathrm{ext}}] = \int \mathcal{D}[A] e^{-S_{\mathrm{g}}[A]-\Gamma[A,0]}\, \mathcal{V}[A,\mathcal{A}^{\mathrm{ext}}] = \left\langle \mathcal{V}[A,\mathcal{A}^{\mathrm{ext}}] \right\rangle_0 \,,$$

  where we define the reweighting factor in terms of the effective action

$$\mathcal{V}[A,\mathcal{A}^{\mathrm{ext}}] \equiv e^{-\Delta\Gamma[A,\mathcal{A}^{\mathrm{ext}}]}\,,$$

$$\Delta\Gamma[A,\mathcal{A}^{\mathrm{ext}}] = \Gamma[A,\mathcal{A}^{\mathrm{ext}}] - \Gamma[A,0]\,, \quad \Gamma[A,\mathcal{A}^{\mathrm{ext}}] \equiv -\ln\det\left(\not{D}[A] + \not{\mathcal{A}}^{\mathrm{ext}}\right)\,.$$

# Proposed formalism

## Wilson flow

- The flow equation ($[t] = 2$ is the flow time):

$$\frac{\partial}{\partial t} B_\mu = D_\nu G_{\nu\mu}\,, \quad B_\mu \,|_{t=0} = A_\mu\,, \quad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]\,.$$

- Our convention: the coupling constant is absorbed in the field, s.t. $D_\mu = \partial_\mu + [B_\mu, \cdot]$
- The UV fluctuations are $e^{-p^2 t}$ suppressed $\Rightarrow$ no wave-function renormalization is required.
- Smooth $a \to 0$ limit on the lattice (fixed physical $t$).

# Proposed formalism

## Worldline formalism

- QED effective action under worldline formalism

$$\Gamma[A] \equiv -\frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_{x(0)=x(T)} \mathcal{D}x \; \Phi[x,F]\mathcal{W}[x,A] e^{-\frac{1}{4}\int_0^T d\tau \; \dot{x}^2} \, ,$$

where the Wilson loop and the spinor factor are defined as

$$\mathcal{W}[x,A] \equiv e^{-i\int_0^T d\tau \dot{x}^\mu A_\mu(x(\tau))} \, ,$$

$$\Phi[x,F] = \mathrm{Tr}\mathcal{P} \exp\left[\frac{i}{2}\sigma^{\mu\nu}\int_0^T d\tau F_{\mu\nu}(x(\tau))\right] \, .$$