The nucleon unpolarized generalized form factors and Mellin moments up to fourth order

Christian Kummer

with: C. Alexandrou, S. Bacchio, M. Constantinou, Y. Li, G. Spanoudes

Second Moment:
See C. Alexandrou Thu 2 pm

C. Alexandrou *et al.*, (2026), 2607.03458

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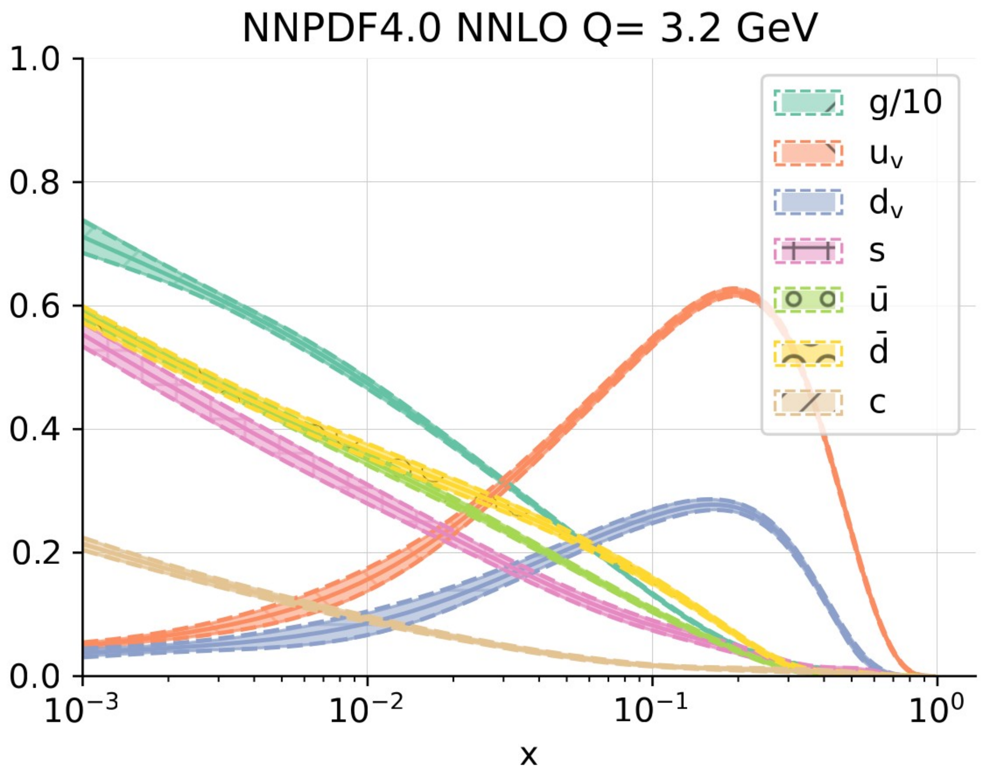
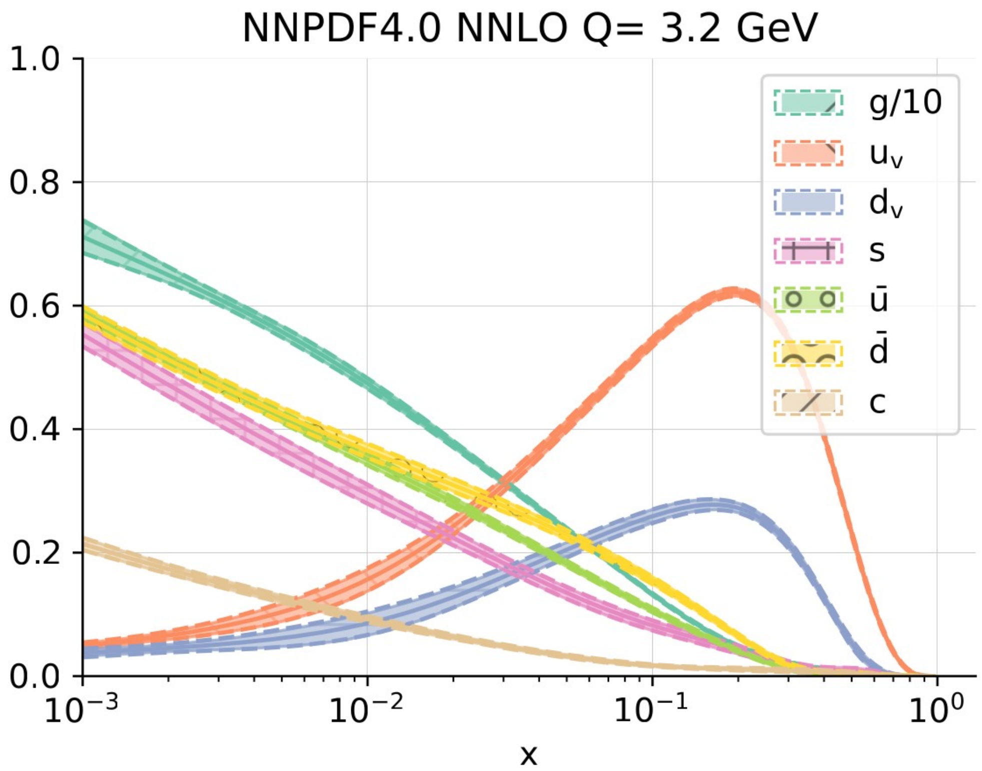


NNPDF, EPJC 82, 428 (2022)

Motivation

- Proton is an excellent probe for strong processes
- Internal structure highly non-perturbative ➡ Lattice QCD
- Interest in generalized parton distributions (GPDs)
- Direct calculation of Mellin moments



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Definitions

- Leading twist operators

$$\mathcal{O}^{\{\mu_1\mu_2\ldots\mu_n\}} = \overline{\psi}\,\gamma^{\{\mu_1}i\overleftrightarrow{D}^{\mu_2}\ldots i\overleftrightarrow{D}^{\mu_n\}}\,\psi$$

- Matrix elements

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$$\langle N(p')|\mathcal{O}^{\{\mu_1\mu_2\ldots\mu_n\}}|N(p)\rangle =$$
$$\overline{u}(p')\Bigg[\sum_{\substack{i=0\\ \text{even}}}^{n-1}\Big\{q^{\{\mu_1}\ldots q^{\mu_i}P^{\mu_{i+1}}\ldots P^{\mu_{n-1}}\gamma^{\mu_n\}}A_{n,i}(q^2)$$
$$+\, q^{\{\mu_1}\ldots q^{\mu_i}P^{\mu_{i+1}}\ldots P^{\mu_{n-1}}\frac{i\sigma^{\mu_n\}\alpha}q_\alpha}{2m_N}B_{n,i}(q^2)\Big\}$$
$$+\,\frac{q^{\{\mu_1}q^{\mu_2}\ldots q^{\mu_n\}}}{m_N}C_{n,0}(q^2)\Big|_{n\,\text{even}}\Bigg]u(p)$$

X.-D. Ji, JPG 24, 1181 (1998)

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- Connection to GPDs

$$\int_{-1}^{1} \mathrm{d}x\, x^{n-1} H(x,\xi,q^2) = \sum_{\substack{i=0\\ \text{even}}}^{n-1} A_{n,i}(q^2)(-2\xi)^i + C_{n0}(q^2)(-2\xi)^n\Big|_{n\ \text{even}}$$

$$\int_{-1}^{1} \mathrm{d}x\, x^{n-1} E(x,\xi,q^2) = \sum_{\substack{i=0\\ \text{even}}}^{n-1} B_{n,i}(q^2)(-2\xi)^i - C_{n0}(q^2)(-2\xi)^n\Big|_{n\ \text{even}}$$

Operator Mixing

- Continuum: Lorentz group $SO(1,3)$
- Lattice: Hypercubic group $H(4)$

G. Beccarini *et al.*, NPB 456, 271 (1995)

M. Göckeler *et al.*, PRD 54, 5705 (1996)

Operator Mixing

- Continuum: Lorentz group *SO*(1,3)
- Lattice: Hypercubic group *H*(4)
- Mixing!
- In general: Avoid repeated indices
- Two derivatives $\mathcal{O}^{\{\mu\nu\rho\}}$
  with $\mu \neq \nu \neq \rho$
- Three derivatives
  $\mathcal{O}^{\{1234\}}$ $\quad \mathcal{O}^{\{44ij\}}$ $\quad \mathcal{O}^{\{44ij\}} - \mathcal{O}^{\{kkij\}}$

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  $\mathcal{O}^{\{1234\}}$ $\mathcal{O}^{\{44ij\}}$ $\mathcal{O}^{\{44ij\}} - \mathcal{O}^{\{kkij\}}$
- Special case: Forward limit
- Leads to boosted frame

G. Beccarini *et al.*, NPB 456, 271 (1995)
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Methods

- Ensemble at physical point

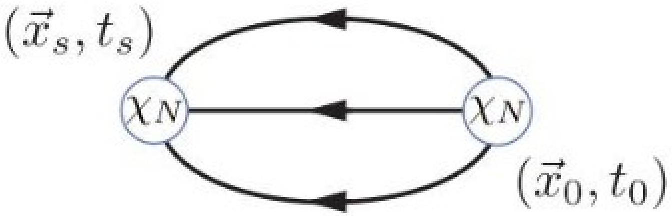
Ensemble	a [fm]	$(L/a)^3 \times T/a$	m_π [GeV]	L [fm]
cB211.072.64	0.0801(4)	$64^3 \times 128$	0.1393(7)	5.12(3)

C. Alexandrou *et al.*, PRD 98, 054518 (2018)

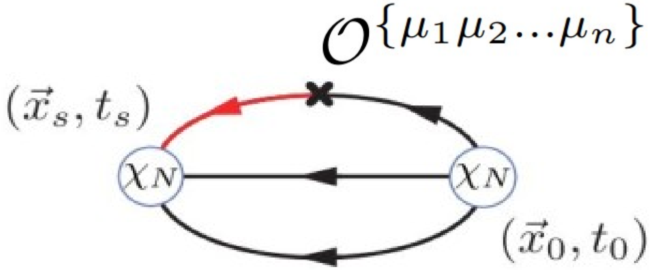
Methods

- Two- and three-point-functions
- Neglect disconnected

$$C(\Gamma_0, \vec{p}, t_s, t_0) = \sum_{\vec{x}_s} e^{-\mathrm{i}(\vec{x}_s - \vec{x}_0)\cdot\vec{p}} \times$$
$$\mathrm{tr}[\Gamma_0 \langle J_N(t_s, \vec{x}_s) \bar{J}_N(t_0, \vec{x}_0)\rangle]$$



$$C^{\{\mu_1\mu_2\ldots\mu_n\}}(\Gamma, \vec{q}, \vec{p}', t_s, t_{\mathrm{ins}}, t_0) =$$
$$\sum_{\vec{x}_{\mathrm{ins}}, \vec{x}_s} e^{-\mathrm{i}(\vec{x}_{\mathrm{ins}} - \vec{x}_0)\cdot\vec{q}} e^{-\mathrm{i}(\vec{x}_s - \vec{x}_0)\cdot\vec{p}'} \times$$
$$\mathrm{tr}[\Gamma \langle J_N(t_s, \vec{x}_s) \mathcal{O}^{\{\mu_1\mu_2\ldots\mu_n\}}(t_{\mathrm{ins}}, \vec{x}_{\mathrm{ins}}) \bar{J}_N(t_0, \vec{x}_0)\rangle]$$



Methods

- Statistics

$\vec{n}'$	N_{mom}	t_s/a	N_{conf}	N_{srcs}	N_{meas}
two derivatives					
$(1,1,0)$	12	8	735	1	8820
$(1,1,0)$	12	10	735	3	26460
$(1,1,0)$	12	12	735	9	79380
$(1,1,0)$	12	14	735	18	158760
three derivatives					
$(1,1,1)$	8	8	735	1	5880
$(1,1,1)$	8	10	735	3	17640
$(1,1,0)$	12	14	735	18	158760

Methods

- Construct Ratio to extract matrix element

$$R^{\mu\nu}(\Gamma;\vec{p}\,',\vec{p};t_s,t_{\text{ins}}) = \frac{C^{\mu\nu}(\Gamma;\vec{p}\,',\vec{p};t_s,t_{\text{ins}})}{C(\Gamma_0;\vec{p}\,';t_s)} \times$$

$$\sqrt{\frac{C(\Gamma_0,\vec{p};t_s-t_{\text{ins}})C(\Gamma_0,\vec{p}\,';t_{\text{ins}})C(\Gamma_0,\vec{p}\,';t_s)}{C(\Gamma_0,\vec{p}\,';t_s-t_{\text{ins}})C(\Gamma_0,\vec{p};t_{\text{ins}})C(\Gamma_0,\vec{p};t_s)}}$$

$$R^{\mu\nu}(\Gamma;\vec{p}\,',\vec{p};t_s,t_{\text{ins}}) \xrightarrow[t_{\text{ins}}\gg a]{t_s-t_{\text{ins}}\gg a} \Pi^{\mu\nu}(\Gamma;\vec{p}\,',\vec{p})$$

Methods

- Multiplicative renormalization

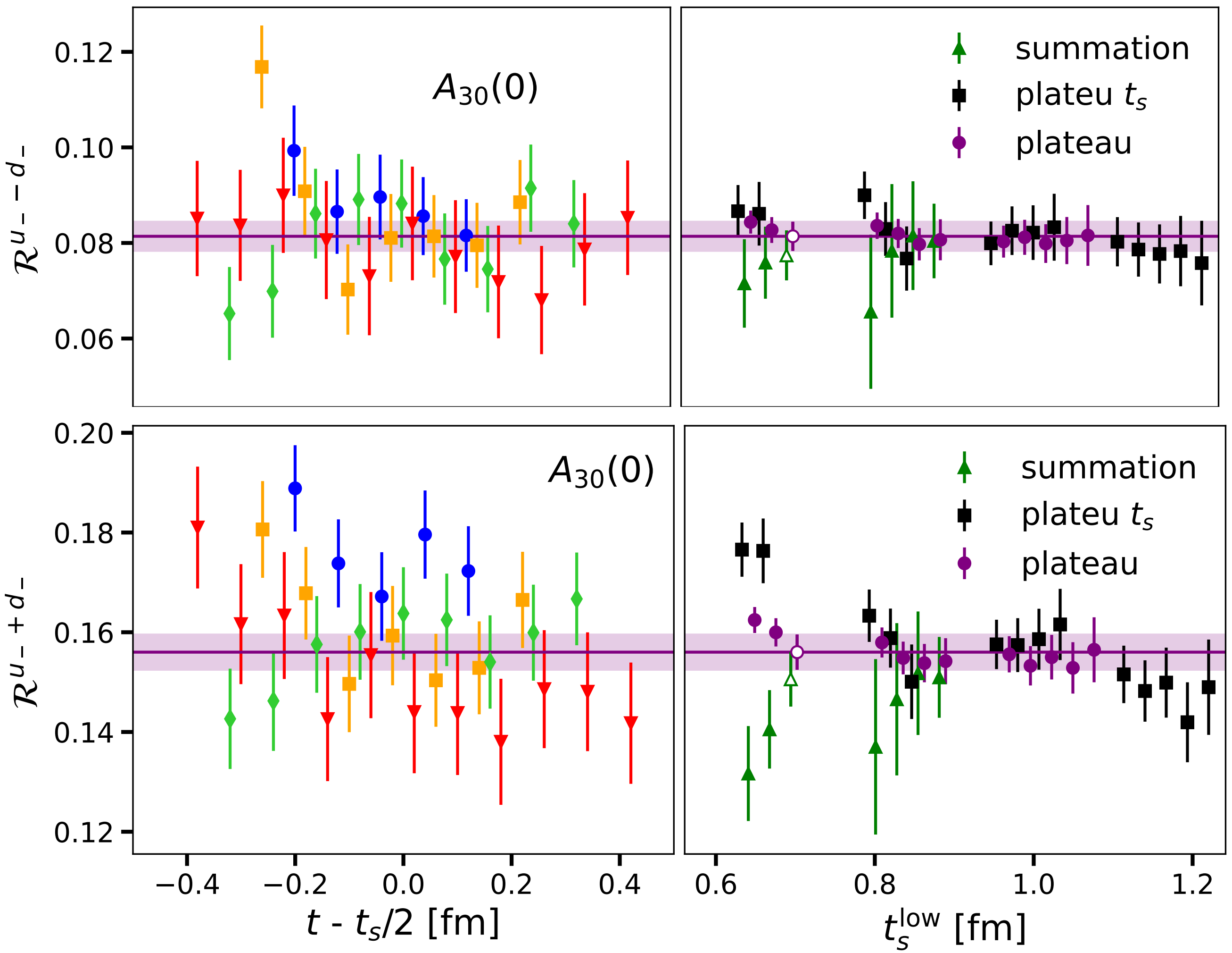
non-perturbative renormalization: $\mathrm{RI'\text{-}MOM}$

perturbative renormalization: $\overline{\mathrm{MS}}$

matching at $\bar{\mu} = 2 \ \mathrm{GeV}$

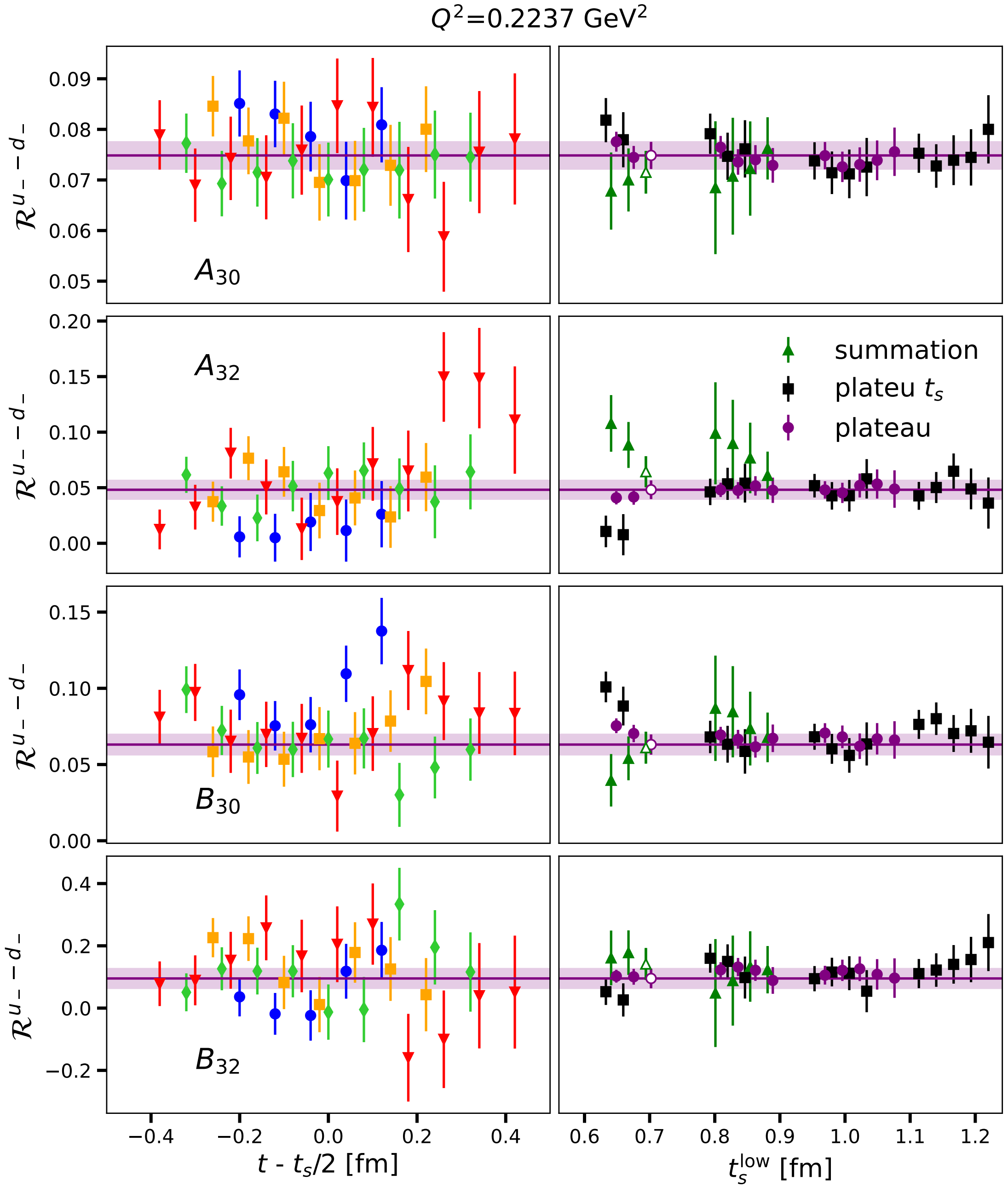
Results – Two-derivative operators

- Forward limit



Results – Two-derivative operators

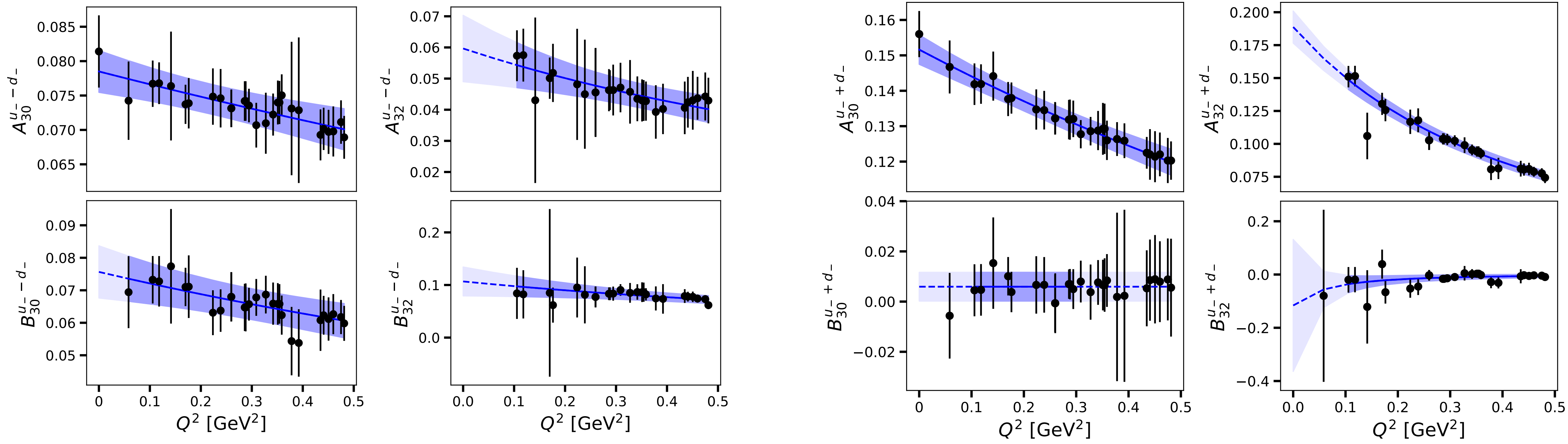
- Momentum transfer



Results – Two-derivative operators

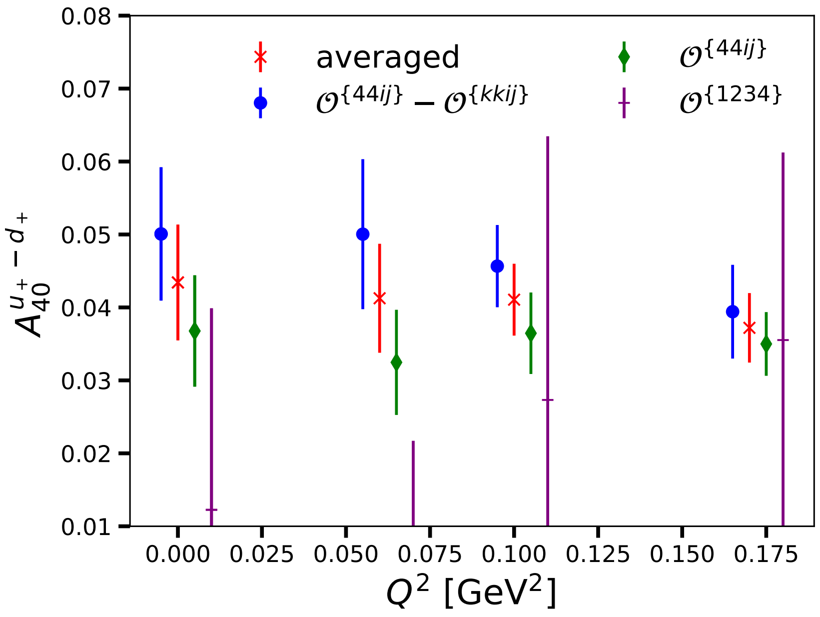
- Dipole fit

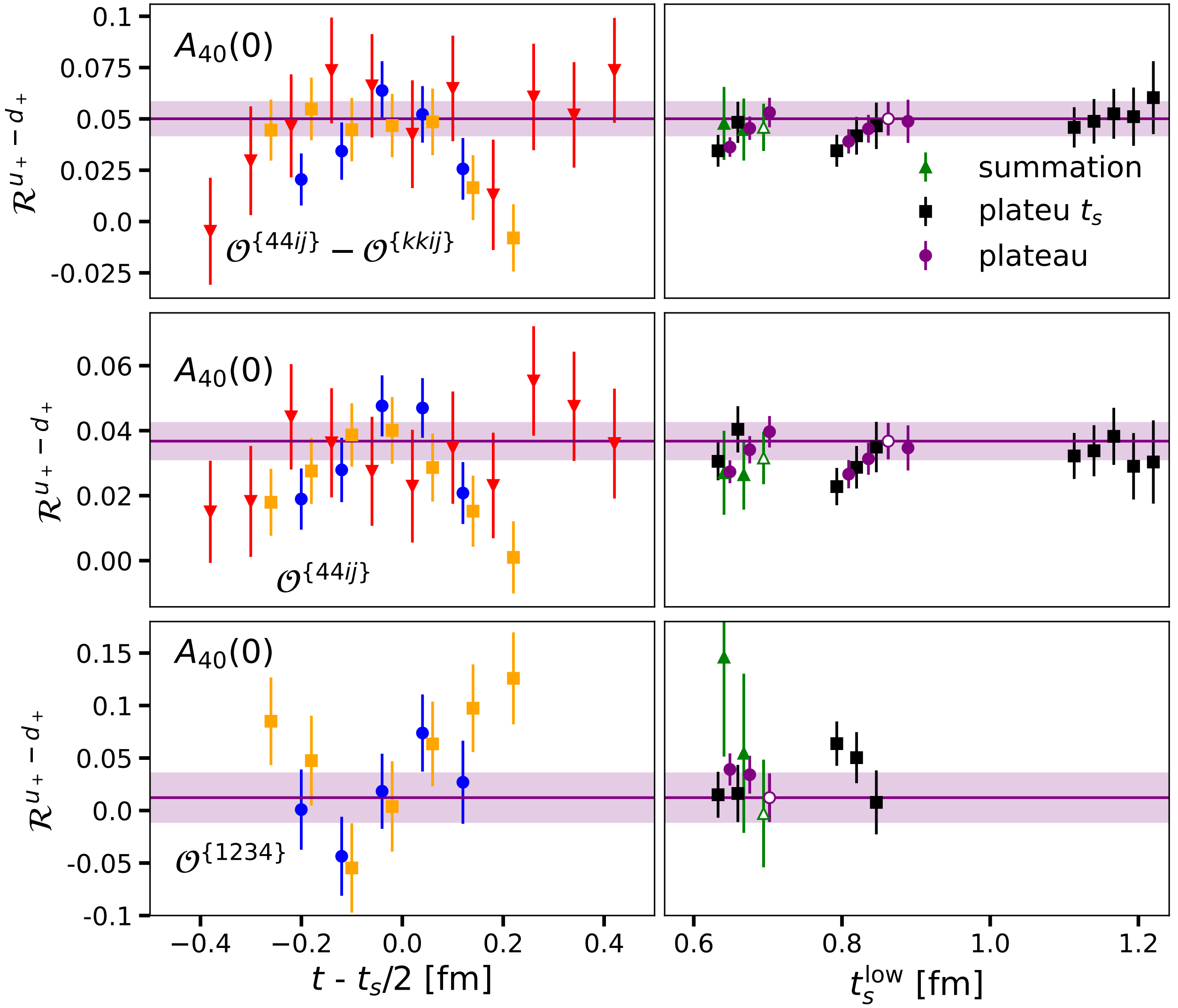
$$G(Q^2) = \frac{G(0)}{(1 + \frac{Q^2}{M^2})^2}$$



Results – Three-derivative operators

- Forward limit
- Comparison of different operators
- Note: *y*-axis scale





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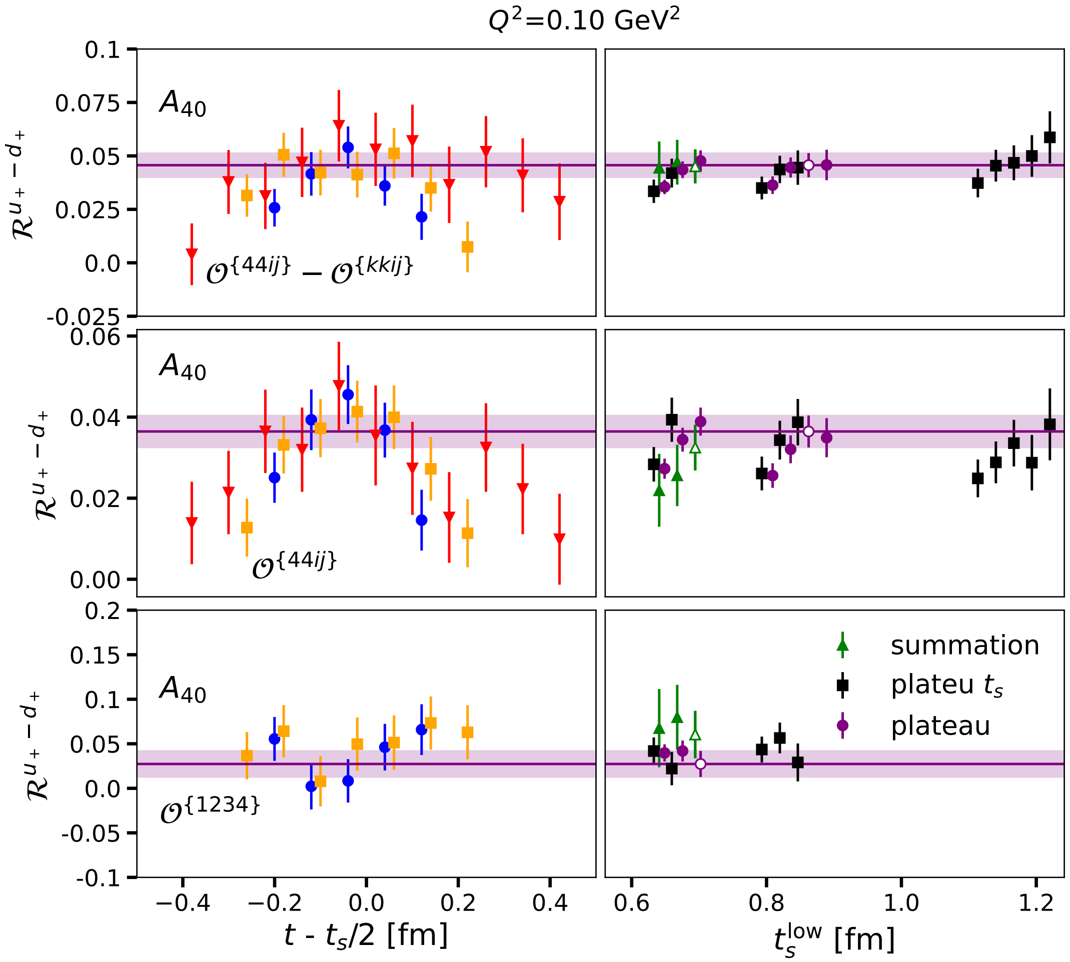
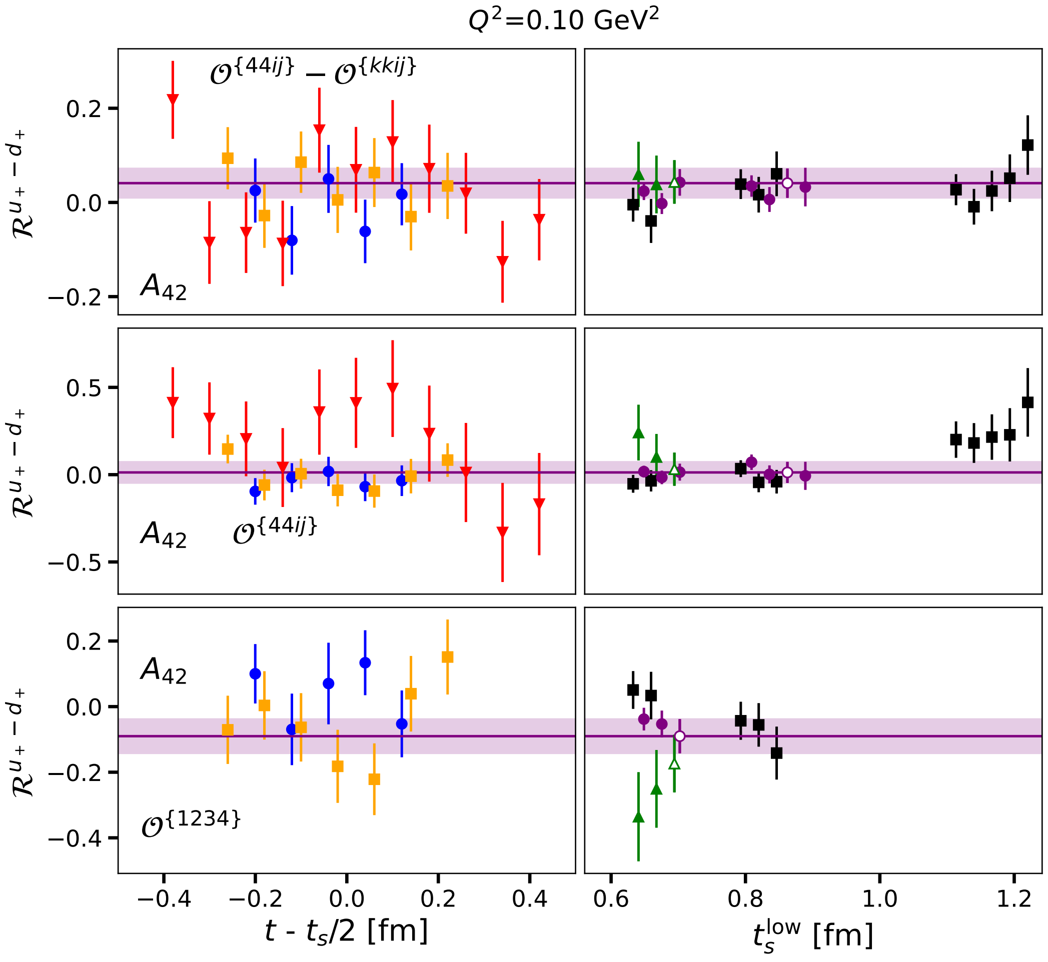
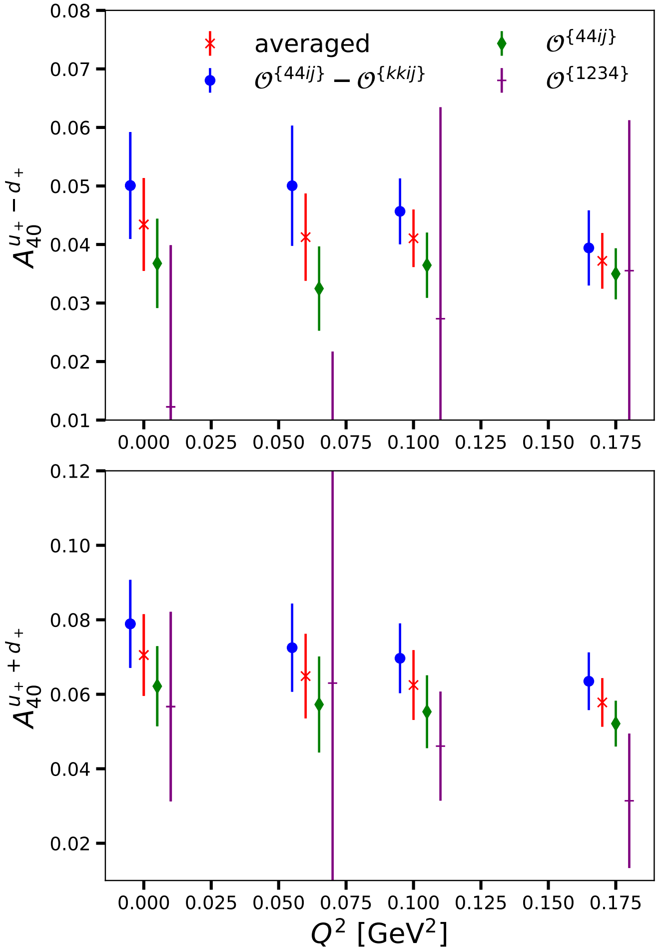
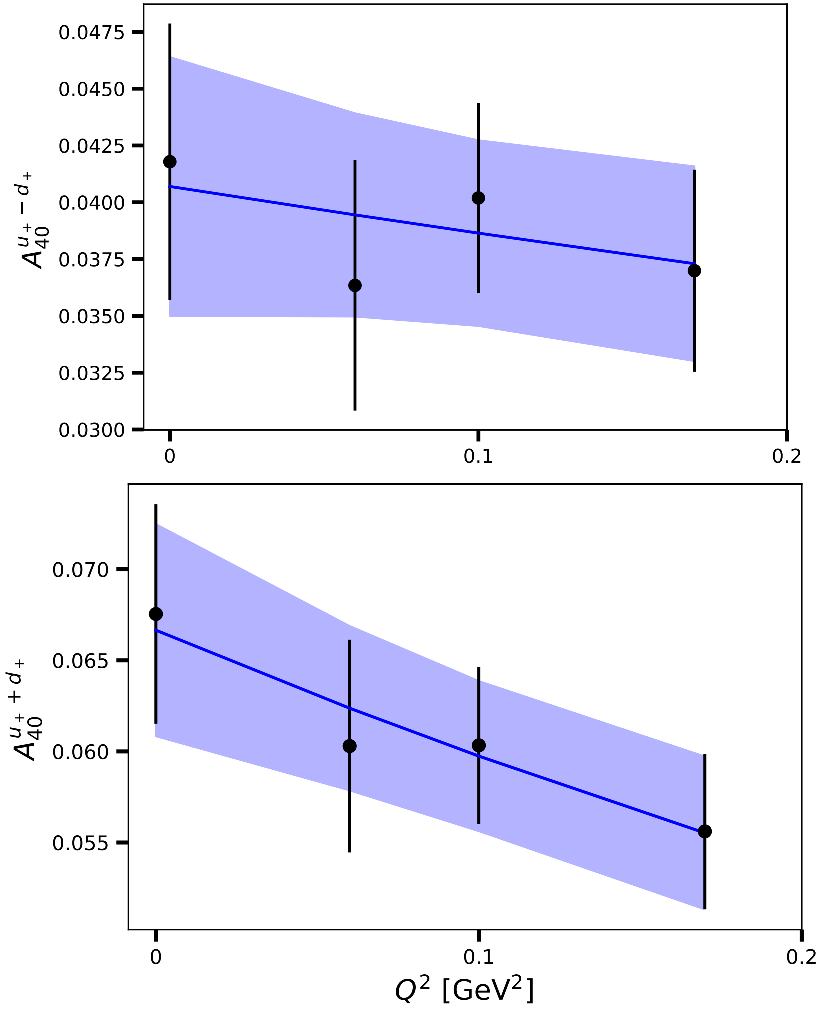




Results – Three-derivative operators

- Only A_{40} nonzero





Comparison with literature

- JAM and NNPDF are phenomenological studies

JAM, PRD 106, L051502 (2022)

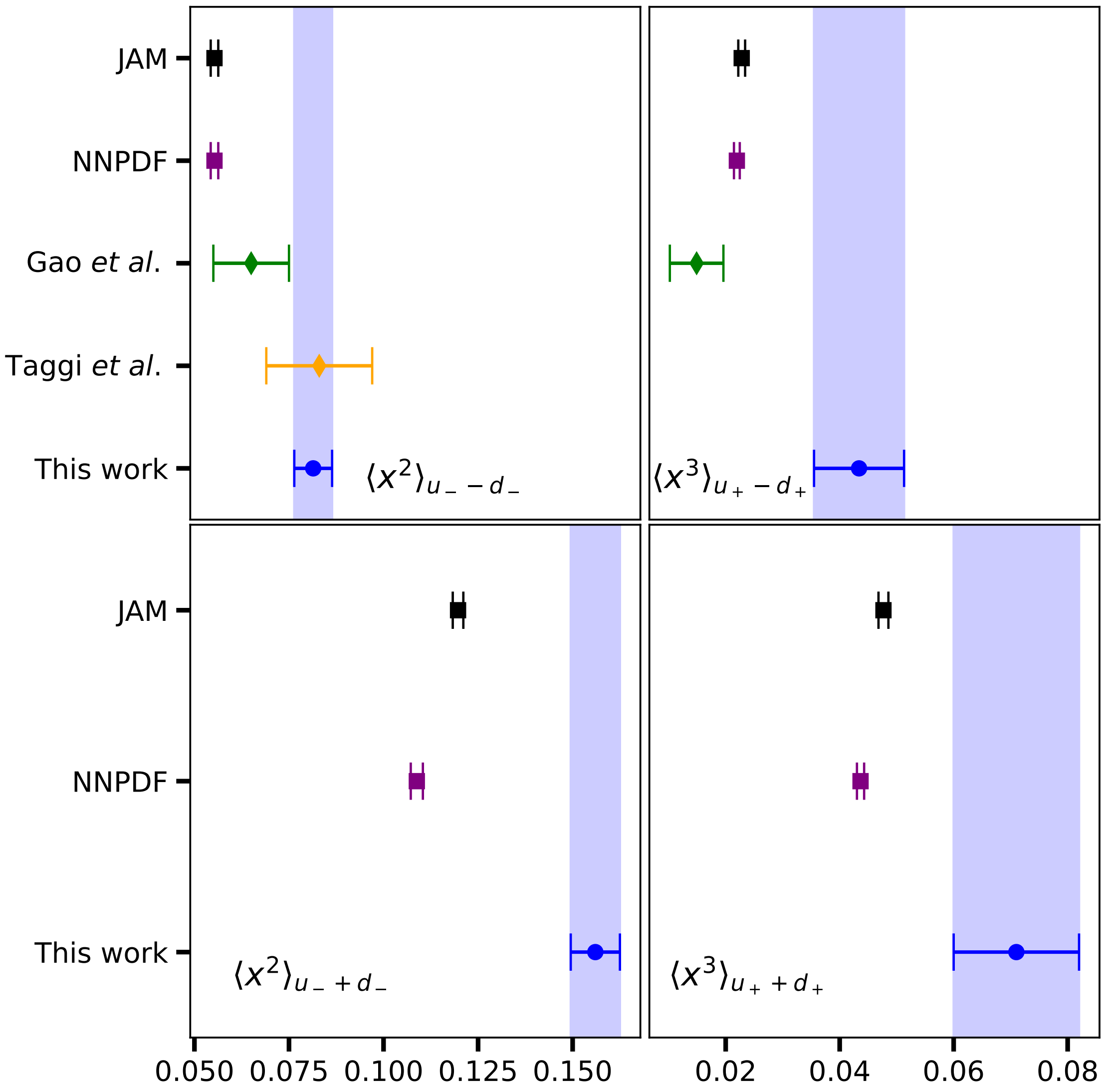
NNPDF, EPJC 82, 428 (2022)

- Gao uses non-local operators

X. Gao *et al.*, PRD 107, 074509 (2023)

- Taggi is the first direct computation third moment

E. Taggi *et al.*, (2026), 2605.02808



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PDF reconstruction

- Ansatz

$$q(x) = N x^{\alpha} (1 - x)^{\beta}$$

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$$\langle x^{n-1} \rangle = \int_0^1 \mathrm{d}x \, x^{n-1} [q(x) - (-1)^{n-1} \bar{q}(x)]$$

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- Even powers give valence distribution

$$\langle 1 \rangle_{q_-} = \langle 1 \rangle_{q_-}$$
$$\langle x^2 \rangle_{q_-} = \langle x^2 \rangle_{q_-}$$

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- Even powers give valence distribution
- Odd powers do not

$$\langle 1 \rangle_{q_-} = \langle 1 \rangle_{q_-}$$

$$\langle x^2 \rangle_{q_-} = \langle x^2 \rangle_{q_-}$$

$$\langle x \rangle_{q_-} = \langle x \rangle_{q_+} - 2 \langle x \rangle_{\bar{q}}$$

$$\langle x^3 \rangle_{q_-} = \langle x^3 \rangle_{q_+} - 2 \langle x^3 \rangle_{\bar{q}}$$

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PDF reconstruction

- Comparison between approaches with JAM data
- JAM data: $u_- - d_-$
- Calculate moments $\langle x \rangle_{q-}$, $\langle x^2 \rangle_{q-}$, $\langle x^3 \rangle_{q-}$
  and reconstruct PDF: $(u-d)^R_-$
- Calculate moments $\langle x \rangle_{q+}$, $\langle x^2 \rangle_{q-}$, $\langle x^3 \rangle_{q+}$
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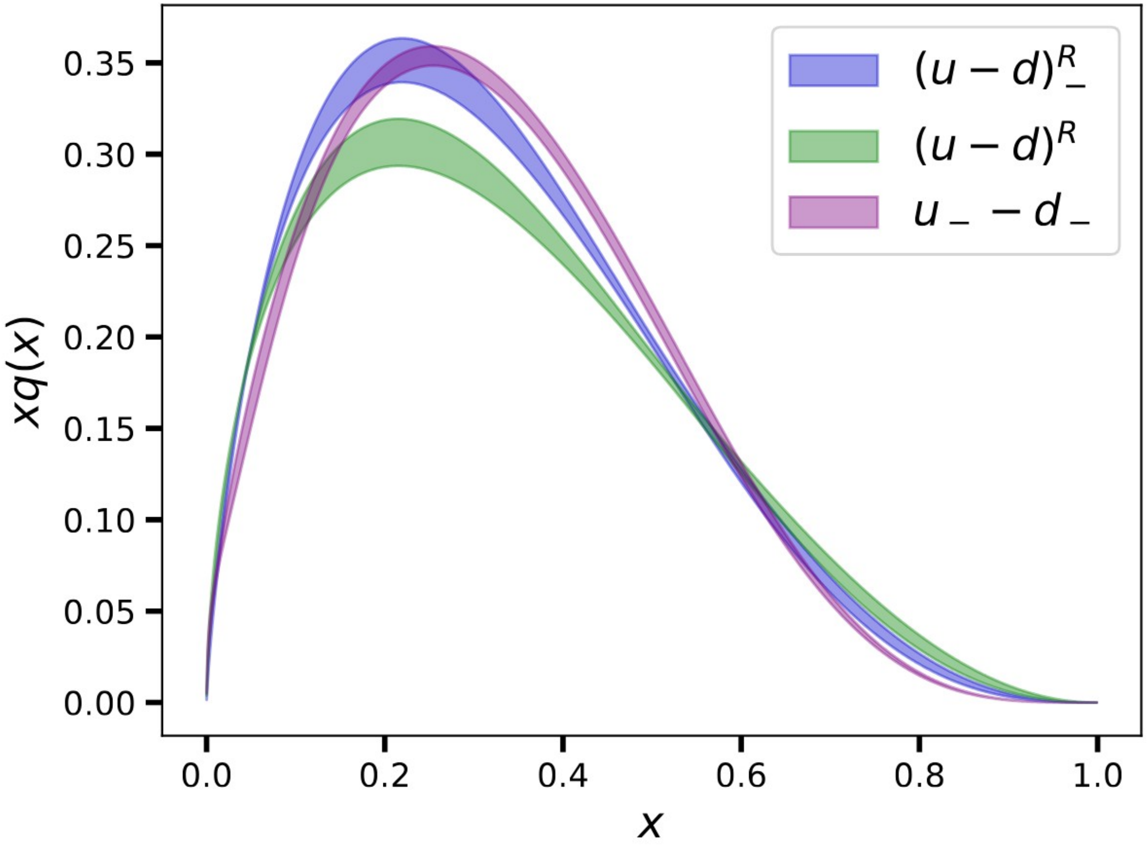


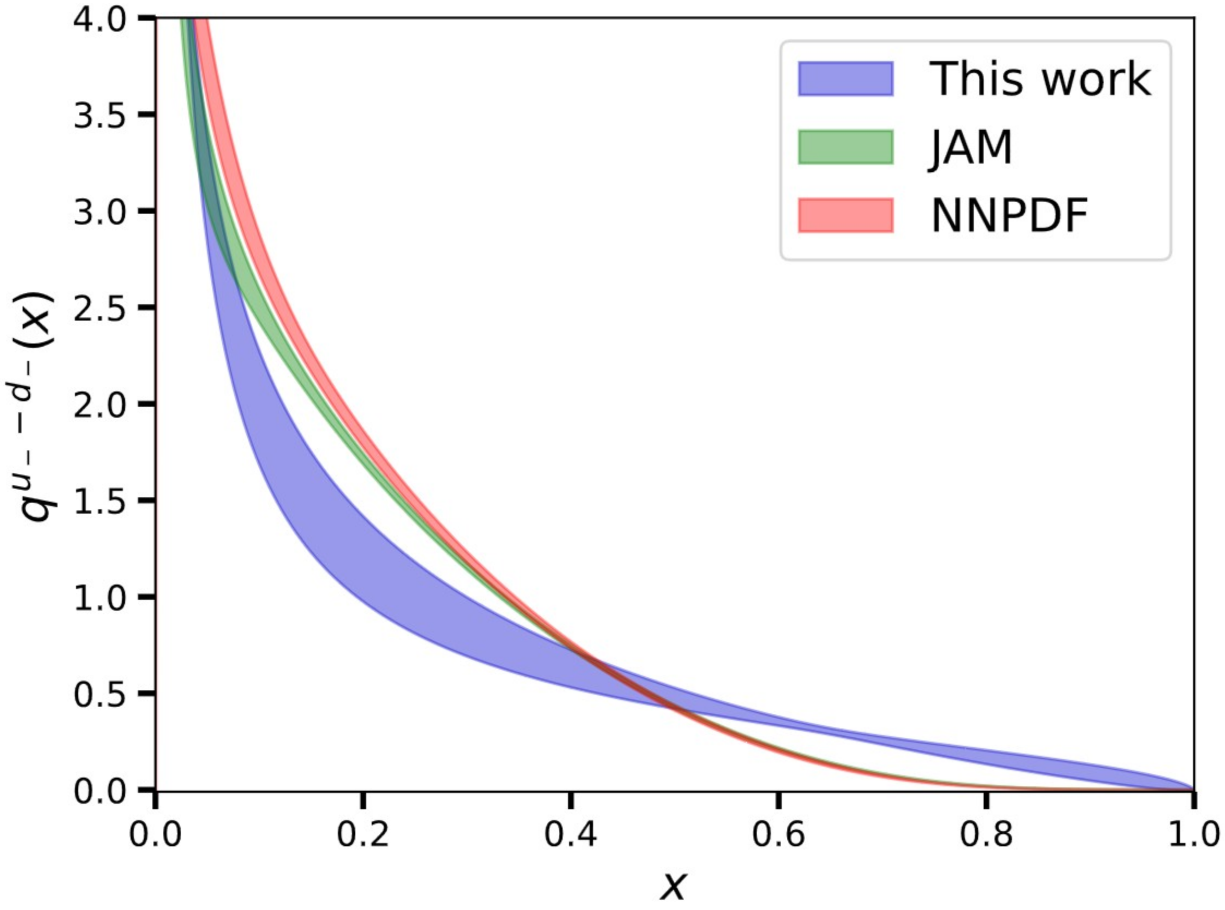
JAM, PRD 106, L051502 (2022)

PDF reconstruction

- Ansatz

$$q(x) = N x^{\alpha} (1 - x)^{\beta}$$

- Include JAM data for $\langle x^n \rangle_{\bar{q}}$ in reconstruction



JAM, PRD 106, L051502 (2022)

NNPDF, EPJC 82, 428 (2022)

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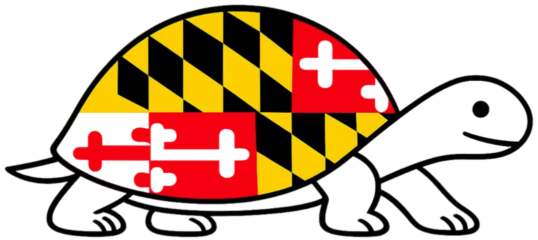
Conclusions

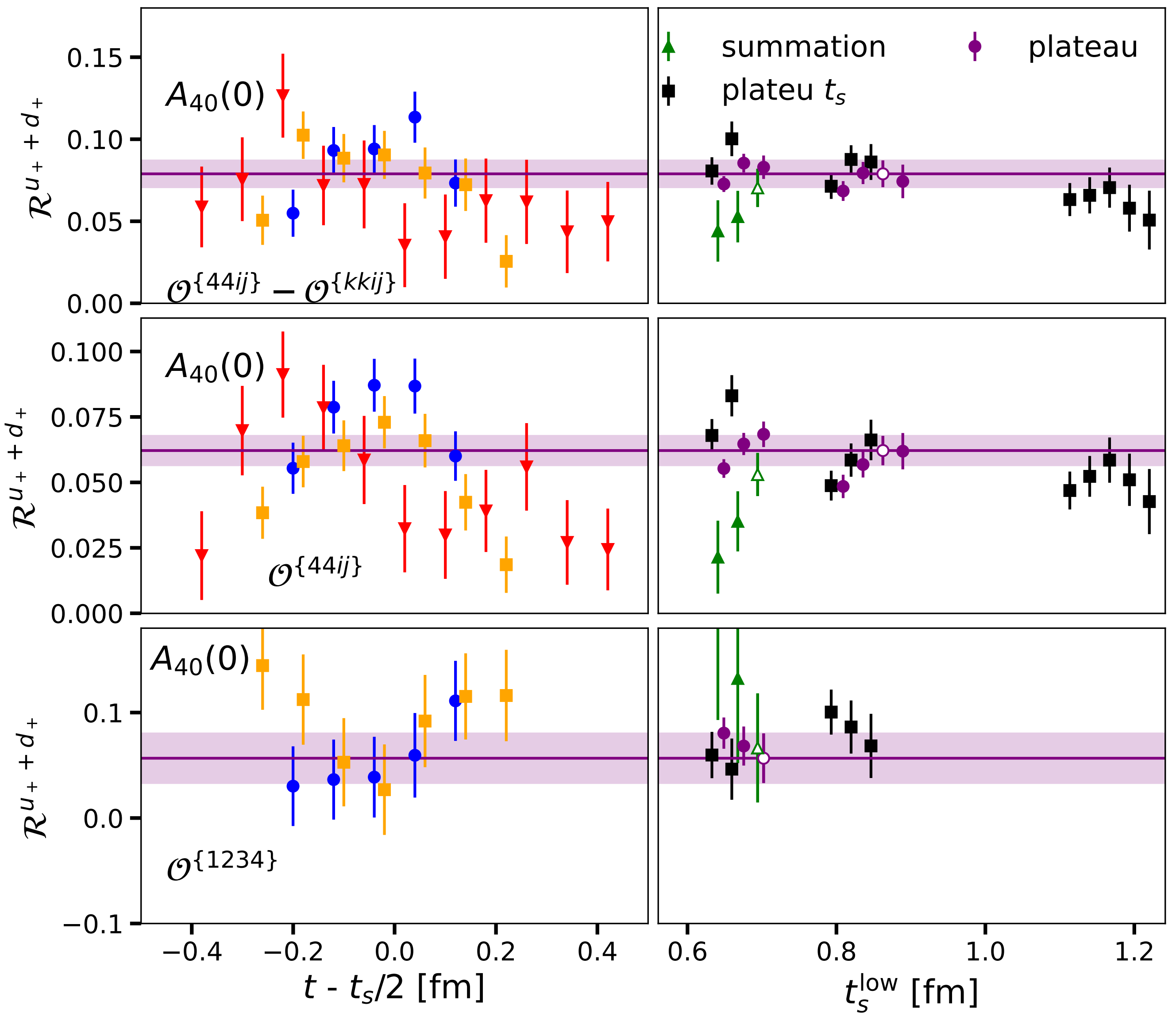
- Direct calculation of third nucleon moment
- Direct calculation of fourth moment
- Determination of GFFs
- Reconstruction of isovector PDF

Backup



Results

- Forward limit
- Three derivative operators
- isoscalar



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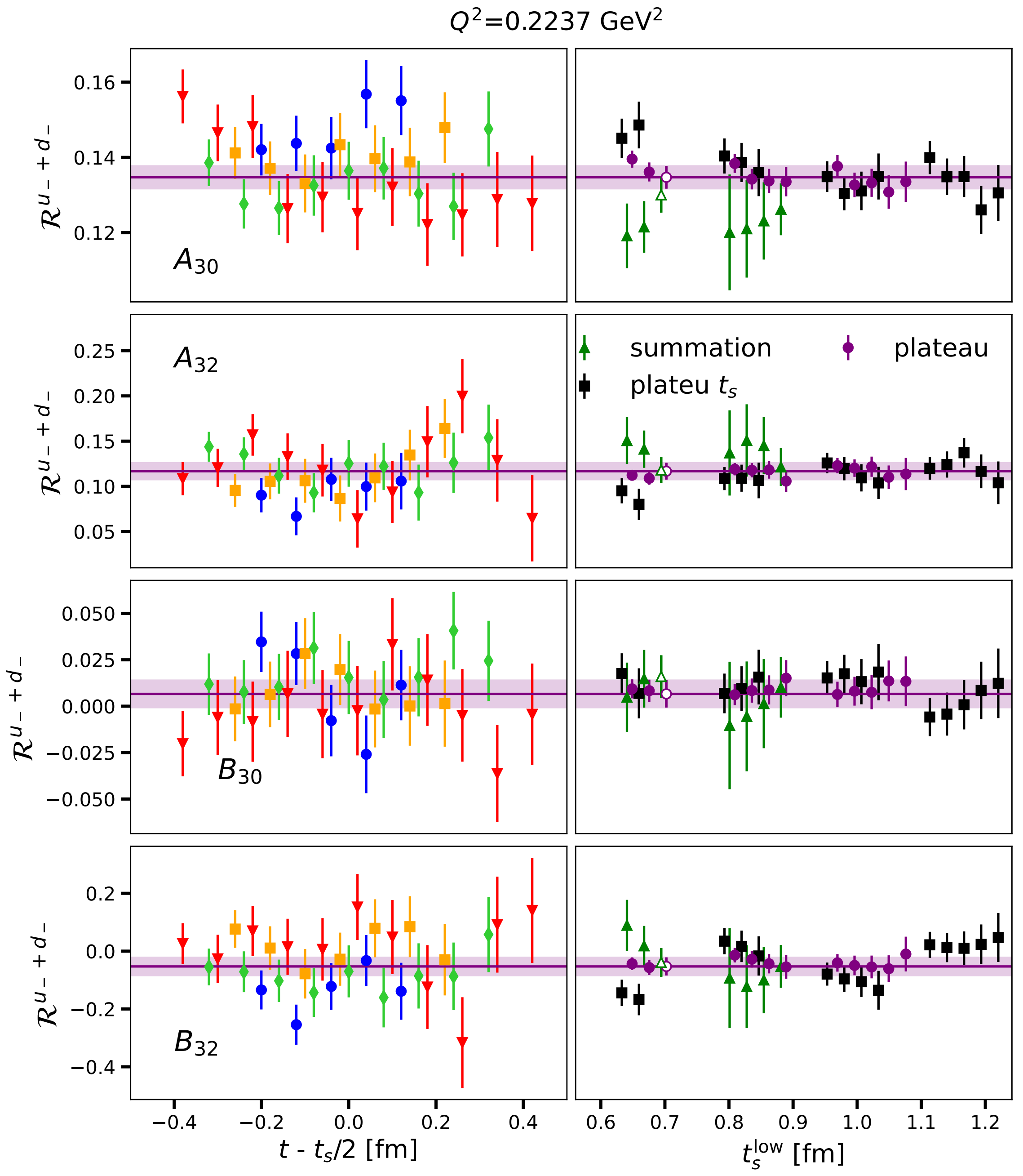
Results

- Momentum transfer
- Two-derivative operators
- isoscalar



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Results

- Three-derivative operators, isoscalar