Zeros of the partition function for 12 flavor QCD and the spectral gap

Yannick Meurice

University of Iowa

Work done with Anas Saleh, Michael Hite, Paul Zelle and Diego Floor (arXiv: 2606.14642)

Supported by the Dept. of Energy under Award Number DE-SC0010113.

Lattice 2026, UMD, 7/28/2026

Talk Content

- Background and Motivations
- Partition function of QCD with 12 light flavors
- Scaling of the partition function zeros with volume and mass
- Zero gap versus mass gap
- Motivations and strategies to make progress with the Yang-Mills gap conjecture
- Tensor RG
- Conclusions
- For details and references see arXiv: 2606.14642

Motivations

- QCD: $\Gamma_\sigma \approx m_\sigma \approx \Lambda_{QCD}$
- Higgs: $\Gamma_H < 13 MeV \ll m_H(125 GeV) \ll \Lambda_{new}$ (many TeV ?) is the lightness of the Higgs (compared to the hypothetical Λ_{new}) the remnant of an approximate conformal symmetry?
- Examples of models: QCD with enough flavors
- RG flows, boundary of the conformal window: it's complicated!
- Incomplete list of references for N_f=12:
  - X.-Y. Jin and R. D. Mawhinney, 1203.5855 and 1304.0312
  - Etsuko Itou, 1212.1353 and 1307.6645
  - Hasenfratz and Schaich, 1610.10004
  - Lin, Ogawa, Ramos, 1510.05755
  - Fodor et al., 1607.06121
  - J. P. Klinger, R. Kaiser, O. Philipsen, and J. Schaible, 2603.20099
  - Z. Gelzer, Y. Liu, YM, and D. Sinclair, 1312.3906 and 1411.3360

Model: $SU(3)$ with 12 equal mass Dirac fermions

The fermionic part of the lattice action is

$$S_F = \frac{1}{2}\sum_x \bar{\psi}_x \left(\gamma_\mu \Delta^\mu + m_q\right)\psi_x + h.c., \tag{1}$$

The gauge part of the action is given by the Wilson action

$$S_G = \beta \sum_p \left[1 - \frac{1}{3}\mathrm{Re}\left(\mathrm{Tr}\, U_p\right)\right] \tag{2}$$

where $\beta = 6/g^2$. We use unimproved Kogut-Susskind's staggered formulation (based on D. Sinclair's code). The plaquette histograms are then used to build the partition function, written in terms of the density of states $\Omega(E)$ as

$$Z(\beta) = \int dE\, \Omega(E) e^{-\beta N_p E}. \tag{3}$$

This is done using the Ferrenberg-Swendsen algorithm.

Expected phase digram in the (m_q, β) plane

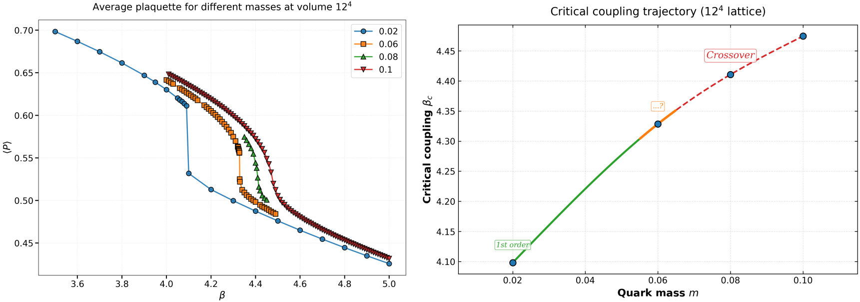


Figure: Left: Distribution of average plaquette values as a function of β for $L = 12$ and masses $m_q = 0.02,\ 0.06,\ 0.08$, and 0.1. Right: Expected phase diagram for 12 flavor $SU(3)$ in the (m_q, β) plane.

Determination of the partition function zeroes

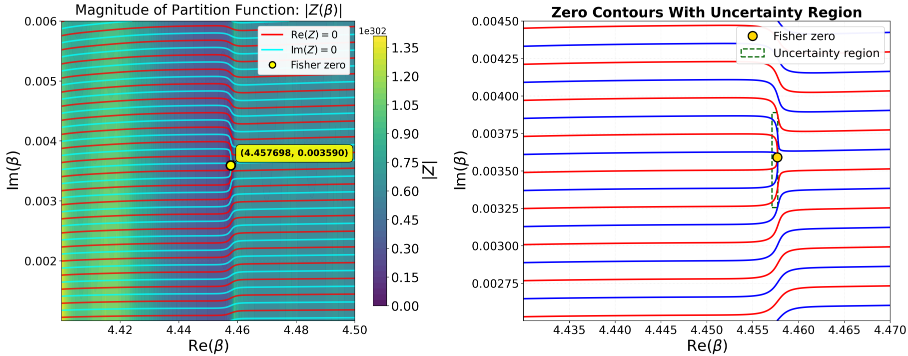
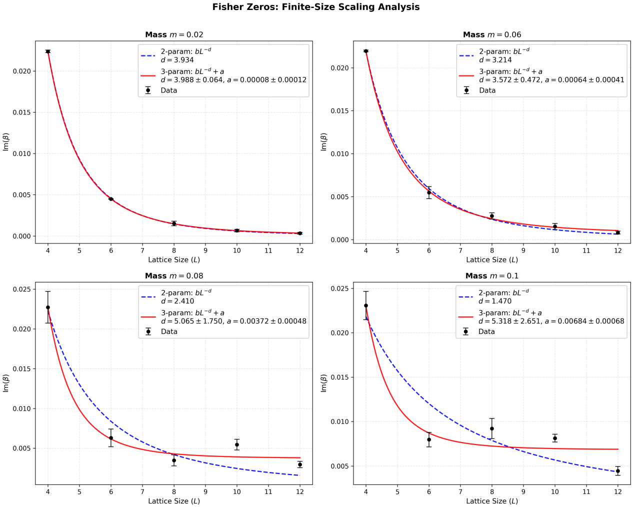


Figure: (Left) Heatmap of the of the magnitude of the partition function along with the real and imaginary zero contours for $L = 8,\ m_q = 0.08$. (Right) Zoomed in contour plot with an associated uncertainty region for the Fisher zero.

Lowest zero as a function of L with 4 masses

$Im\beta$: 2 parameter fit (blue, bL^{-d}) vs. 3 parameters (red, $bL^{-d} + a$)



Models for the zero gap ($\propto (m_q - m_q^c)^B$)

The better quality of the 3-parameter fits indicates that when $L \to \infty$ the lowest zero pinches the real axis for m_q= 0.02 but not for 0.06, 0.08 and 0.1 where their imaginary reach an asymptotic “gap" a. Scaling arguments (Itzykson et al. NPB 220) suggest

$$a \propto (m_q - m_q^c)^{\beta\delta}$$

For a Ising 4D mean field transition: β=1/2 and δ=3 so

$$a \propto (m_q - m_q^c)^{3/2}$$

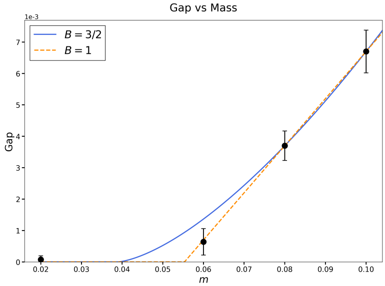


Figure: We considered the “fits" for the zero gap: $a = A(m_q - m_q^c)^B$ with $B = 1$ (from visualization) and $B = 3/2$ (from mean field). There are only 3 data points and A and m_q^c were determined to fit exactly for m_q= 0.08 and 0.1 while predicting a for $m_q = 0.06$. Clearly more data points are needed!

Zero gap and mass gap

- Our estimate of $m_q^c \simeq 0.05$ is consistent with extrapolations of results by J. P. Klinger, R. Kaiser, O. Philipsen, and J. Schaible, 2603.20099 for N_f up to 10 (also unimproved staggered fermions).
- Jin and Mawhinney found that the mass of the 0^{++} is consistent with the mean field prediction:

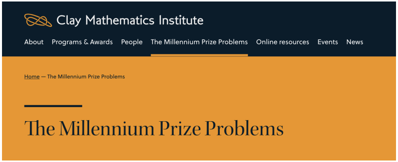
$$m_\sigma \propto (m_q - m_q^c)^{1/2}$$

- Mean field also predicts:

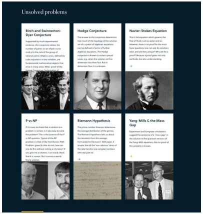
$$\text{Zero gap } \propto (m_q - m_q^c)^{3/2}$$

- Combining the two we get Zero gap $\propto$ (mass gap)^3
- Conjecture: In general, mass gap $\propto$ (Zero gap)^C

Millenium problem: prove the Yang-Mills gap



In order to celebrate mathematics in the new millennium, The Clay Mathematics Institute of Cambridge, Massachusetts (CMI) established seven *Prize Problems*.



The Birch and Swinnerton-Dyer Conjecture
ANDREW WILES

The Hodge Conjecture
PIERRE DELIGNE

Existence and Smoothness of the Navier–Stokes Equation
CHARLES L. FEFFERMAN

The Poincaré Conjecture
JOHN MILNOR

The P versus NP Problem
STEPHEN COOK

The Riemann Hypothesis
ENRICO BOMBIERI

Quantum Yang–Mills Theory
ARTHUR JAFFE AND EDWARD WITTEN

M. Douglas review: Tensor RG is a promising strategy

nature reviews physics https://doi.org/10.1038/s42254-025-00909-2

Review article Check for updates

The Yang–Mills Millennium problem

Michael R. Douglas

Abstract

The Yang–Mills Millennium Prize problem is one of the great challenges of mathematical physics. In the quarter century since it was set, what progress has been made? This Review outlines the problem from a physics point of view, gives its physical background, explains its nature and significance as a problem in mathematics and surveys promising approaches from recent years.

Tensor RG

The block spin RG, although conceptually straightforward to explain, has many technical difficulties in implementation.

Of the many subsequent reformulations of the RG, one which many find particularly promising is the tensor RG. This was originally developed in condensed matter physics but is general; ref. 67 provides a review oriented towards QFT and gauge theory. In the tensor RG, the space of effective actions, and many of the steps involving approximations, are linear.

The starting point is to reformulate the partition function as a tensor network. This has been done for LGT 67,

67. Meurice, Y., Sakai, R. & Unmuth-Yockey, J. Tensor lattice field theory with applications to the renormalization group and quantum computing. *Rev. Mod. Phys*. **94**, 025005 (2022).

Key points of Michael Douglas Review

- Yang–Mills theory is the basis of the standard model of particle physics and describes the strong and weak forces.
- The Yang–Mills Millennium problem is to show that Yang–Mills theory is mathematically well defined and that it has the mass gap property.
- The issue of definition is to prove that the theory has a continuum limit, which is well defined at arbitrarily high energies. This requires renormalization, which has never been made rigorous in the needed generality.
- The mass gap property (no massless particles) is expected because it is true of real-world quantum chromodynamics and it is seen in numerical simulations. It is widely felt that no clear path is known towards proving it.
- Recent mathematical approaches include rigorous stochastic quantization and the rigorous strong coupling expansion. They are part of probability theory, and mathematicians are making significant advances.
- Numerical and computational methods are important in the physical study of Yang–Mills and likely to be used in any rigorous proof. Physicists could contribute significantly by developing more powerful computational renormalization group methods.

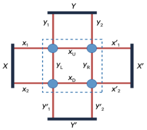


Fig. 3 | Graphical representation of a tensor renormalization group block spin step for the 2D nonlinear sigma model. The lower case *x*, *y* represent vectors in the vector space *V*, and upper case *X*, *Y* represent vectors obtained as a tensor product $V \otimes V$ followed by projection down to *V*. Figure reprinted with permission from ref. 67, APS.

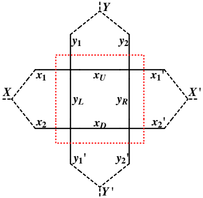
Motivations and strategies

- Great challenge for mathematical physicists but can something new be learned by lattice gauge theorists ?
- Possible by-product: better understanding of the way the gap can be reduced or closed by adding enough fermionic matter
- In a recent Nature Reviews Physics on the Yang-Mills Millennium problem, M. Douglas suggested that “Conceivably, the Tensor Renormalization Group (TRG) will be adequate, especially if it can accommodate duality arguments."
- Three possible strategies to make progress along these lines:
  1. Design a model with a smaller gap and where the TRG can be applied more easily (as for Dyson hierarchical model)
  2. Reproduce the basic steps in numerical calculations of the gap, using dual variables with the TRG and try to find upper and lower bounds on the gap.
  3. Connect the gap in the spectrum with the gap in the imaginary part of the zeros of the partition function (not pinching the real coupling constant axis.)

The Tensor Renormalization Group (TRG) method

- Exact blocking (spin and gauge, PRB 87 064422, PRD 88 056005)

Unique feature: the blocking separates the degrees of freedom inside the block (integrated over), from those kept to communicate with the neighboring blocks. The only approximation is the truncation in the number of "states" kept.



- Applies to many lattice models: Ising model, $O(2)$ model, $O(3)$ model, $SU(2)$ principal chiral model, Abelian and $SU(2)$ gauge theories, ϕ^4, fermions ..
- Solution of sign problems: complex temperature (PRD 89, 016008), chemical potential (PRA 90, 063603).
- Reviews: YM, Ryo Sakai, Judah Unmuth-Yockey, Tensor lattice field theory for renormalization and quantum computing, arXiv:2010.06539, Rev. Mod. Phys. 94, 025005 (2022). Shinichiro Akiyama, YM, Ryo Sakai, Tensor Renormalization Group for fermions, arXiv:2401.08542, J.Phys. CM 36 (2024).

TRG inspired talks at Lattice 2026

- Lattice Field theories with Tensor Networks, Adwait Naravane
- Tensor renormalization group approach to EN, Gota Tanaka
- Tensor Renormalization Group Study of Step-Scaling Function in the CP(1) Model with a theta Term, Hayato Aizawa
- Tensor-Network Study of Confining Flux Tubes and Effective Strings in (2+1)D Compact U(1) Gauge Theory, Alessio Negro
- Tensor Network States and Lattice Gauge Theories, Erez Zohar
- Gauge-invariant PEPS for pure Z_2 ..., Etsuko Itou and T. Okubo
- Tensor renormalization group for four-dimensional SU(N) gauge theories, Katsumasa Nakayama
- Tensor renormalization group analysis of correlators in Yang-Mills theory, Takaaki Kuwahar
- Phase diagram of the single-flavor GNW model ... Jian-Gang Kong, Shinichiro Akiyama, Tao Shin, Zhiyuan Xie
- Tensor renormalization group study of cold and dense 2 and 3 color QCD ..., Yuto Sugimoto, Shinichiro Akiyama, Yoshinobu Kuramashi

Conclusions

- We calculated the density of states and the complex zeros of the partition function for $SU(3)$ with $N_f = 12$, m_q= 0.02. 0.06, 0.08 and 0.1, for L^4 sites with L=4, 6, 8, 10 and 12.
- When $L \to \infty$ the lowest zero pinches the real axis for m_q= 0.02 but not for 0.06, 0.08 and 0.1 where their imaginary part apparently reaches an asymptotic “gap" $\propto (m_q - m_q^c)^B$
- $m_q^c \simeq 0.05$ is consistent with extrapolations of J. P. Klinger et al. 2603.20099
- More data near that value is needed to confirm or reject the mean field value $B = 3/2$
- Conjecture: In general, mass gap $\propto$ (Zero gap)^C
- Tensor RG for zero gap?
- Thanks!

Additional slides

Dynamical mass generation

- Very appealing: a lot of structure from simple input
- Very common: in the standard model, more than 98 percent of the mass of the proton comes from quark-gluon interactions (QCD).
- Can be described by an effective theory: a Yukawa coupling $g_{\sigma NN}$ between the σ and the Nucleons:

$$m_N \sim g_{\sigma NN} < \sigma >$$

- In the standard model

$$m_f \sim g_{Hff} < H >$$

  f: quark or lepton, H: Higgs. Is the Higgs also composite?

-
$$\Gamma_\sigma \approx m_\sigma \approx \Lambda_{QCD}$$

  but

$$\Gamma_H < 13 MeV \ll m_H(125 GeV) \ll \Lambda_{new}??$$

- If $\Lambda_{new} >> \text{a few}$TeV, is the lightness of the Higgs the remnant of an approximate conformal symmetry?

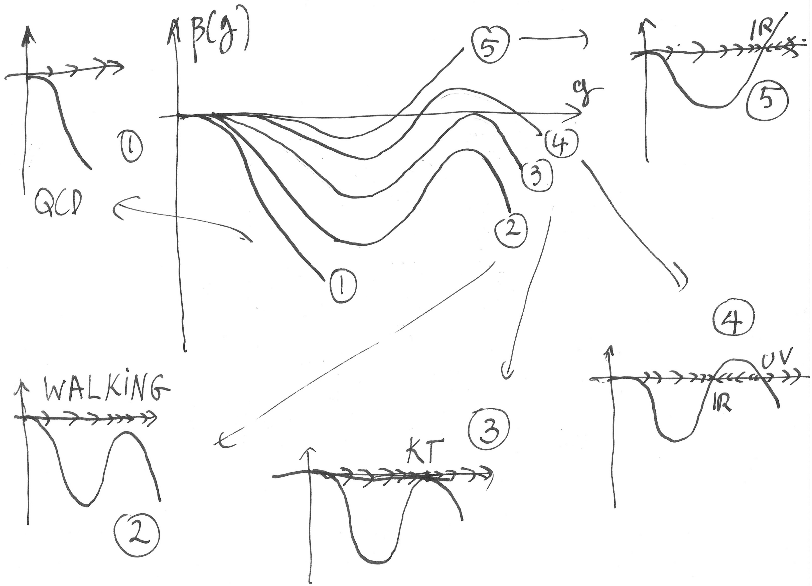
Simple models: $SU(3)$ gauge group with N_f light (or massless) fundamental Dirac quarks

- Familiar setup but the asymptotic scaling can be slower than QCD
- 2-loop perturbative beta function: nontrivial zero for $N_f > 8$

N_f	α_c
9	5.23599
10	2.20761
11	1.2342
12	0.753982
13	0.467897
14	0.278017
15	0.1428
16	0.0416105
QED	$0.0073 \simeq 1/137$

- It is doubtful that we can trust perturbation theory near the perturbative conformal fixed point α_c for $N_f \simeq 12$ massless flavors.
- Could $m_H \ll \Lambda_{new}$ be the remnant of some conformal symmetry?

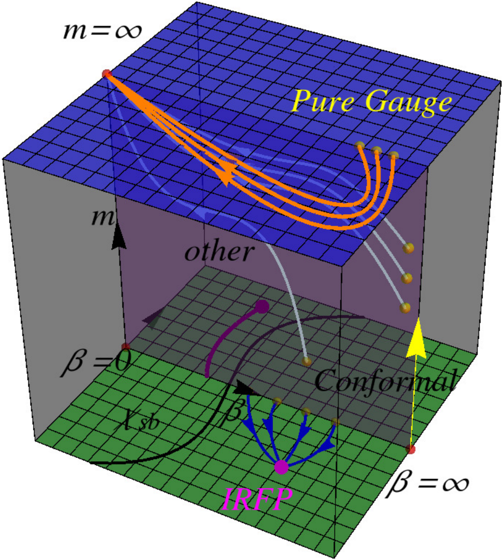
Beta functions beyond perturbation theory (schematic)



N_f=12 and $m = 0$

- The non trivial zero is at $\alpha_s \simeq 0.75 \sim 100\alpha_{QED}$, we need the lattice
- A majority of lattice practitioners think that for N_f=12 and $m = 0$:
  1) There is no confinement
  2) Chiral symmetry is unbroken
  3) The theory is conformal
  (see e. g. Hasenfratz and Schaich 1610.10004 , and Lin, Ogawa, Ramos 1510.05755)
  while a minority believes the exact opposite.
  (see e.g. Fodor et al. 1607.06121)
- A third logical possibility (confinement with unbroken chiral symmetry) seems excluded by 't Hooft anomaly matching + decoupling condition
- Different parts of the RG flows can be connected to hypothetical physical properties, but no realistic attempts are made here
- Recent reviews: DeGrand RMP88, Nogradi and Patella IJMPA 31

Schematic RG flows ($\beta = 6/g^2$)



RG flows from UV to IR (massless case)

1) Setting the scale (initial conditions, far UV)
For a sufficiently high scale we can use the universal perturbative running/dimensional transmutation. The QCD analog is $\alpha_s(M_Z^2) \simeq 0.1$. It is a physical input.

2) The intermediate scale
Using the reference scale in 1), we then reach a physical scale (in TeV) where we are far from both fixed points. From a computational point of view, things look maximally nonlinear/multidimensional in both directions. It is challenging to capture the essential features with small lattices and one-dimensional RG flow approximations.

3) The deep IR scale (assuming $m = 0$ and an attractive IRFP)
As we continue most of the irrelevant features get washed out and if we run all the way down, the intermediate scale does not appear anymore. For model building applications, the standard model interactions will break the conformal symmetry at the EW scale.

Introductions to Tensor Field Theory

REVIEWS OF MODERN PHYSICS, VOLUME 94, APRIL–JUNE 2022

Tensor lattice field theory for renormalization and quantum computing

Yannick Meurice
Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52242, USA

Ryo Sakai
Department of Physics and Astronomy, The University of Iowa, Iowa City, Iowa 52242, USA and Department of Physics, Syracuse University, Syracuse, New York 13244, USA

Judah Unmuth-Yockey
Department of Physics, Syracuse University, Syracuse, New York 13244, USA and Fermi National Accelerator Laboratory, Batavia, Illinois 60510, USA

(published 26 May 2022)

The successes and limitations of statistical sampling for a sequence of models studied in the context of lattice QCD are discussed and the need for new methods to deal with finite-density and real-time evolution is emphasized. It is shown that these lattice models can be reformulated using tensorial methods where the field integrations in the path-integral formalism are replaced by discrete sums. These formulations involve various types of duality and provide exact coarse-graining formulas that can be combined with truncations to obtain practical implementations of the Wilson renormalization group program. Tensor reformulations are naturally discrete and provide manageable transfer matrices. Truncations with the time continuum limit are combined, and Hamiltonians suitable for performing quantum simulation experiments, for instance, using cold atoms, or to be programmed on existing quantum computers, are derived. Recent progress concerning the tensor field theory treatment of noncompact scalar models, supersymmetric models, economical four-dimensional algorithms, noise-robust enforcement of Gauss's law, symmetry preserving truncations, and topological considerations are reviewed. Connections with other tensor network approaches are also discussed.

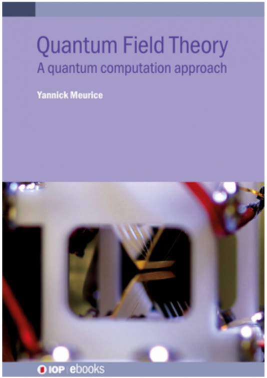
DOI: 10.1103/RevModPhys.94.025005

CONTENTS

0034-6861/2022/94(2)/025005(65) 025005-1

IOP Publishing | Bookstore

Home | Quantum Field Theory



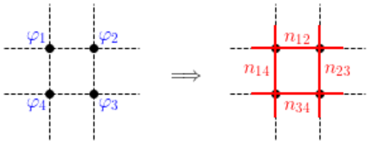
TLFT: From compact to discrete (O(2) example)

$$Z_{O(2)} = \prod_x \int_{-\pi}^{\pi} \frac{d\varphi_x}{2\pi} e^{\beta \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x)} = \mathrm{Tr} \prod_x T^{(x)}_{n_{x-\hat{1},1}, n_{x,1}, \dots, n_{x,D}}.$$

$$e^{\beta \cos(\varphi_{x+\hat{\mu}} - \varphi_x)} = \sum_{n_{x,\mu}=-\infty}^{\infty} e^{i n_{x,\mu} \varphi_{x+\hat{\mu}}} I_{n_{x,\mu}}(\beta) e^{-i n_{x,\mu} \varphi_x}.$$

$$\text{Tensor}: \ T^{(x)}_{n_{x-\hat{1},1}, n_{x,1}, \dots, n_{x-\hat{D},D}, n_{x,D}} = \sqrt{I_{n_{x-\hat{1},1}} I_{n_{x,1}}, \dots, I_{n_{x-\hat{D},D}} I_{n_{x,D}}} \times \delta_{n_{x,\mathrm{out}}, n_{x,\mathrm{in}}},$$

$$\prod_x \int_{-\pi}^{\pi} d\varphi_x \Longrightarrow \sum_{\{n\}}$$



The gauged version is the Abelian Higgs model.