Searching for symmetric mass generation with staggered fermions in four dimensions

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July 30, 2026

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Lattice Model

$$S = \sum_x \bar{\psi}\left(\eta \cdot \Delta + iy\sigma\right)\psi + \frac{1}{2}\sum_x |\sigma|^2 - \frac{\kappa}{2}\sum_x \sigma\Box\sigma + \frac{\lambda}{4}\sum_x |\sigma|^4$$

ψ: two component staggered fermion field

- $\Delta_\mu \psi = \psi(x+\mu) - \psi(x-\mu)$
- $\eta^\mu(x) = (-1)^{\sum_{i=1}^{\mu-1} x_i}$

σ: scalar in adjoint representation of $SU(2)$

- $|\sigma|^2 = \mathrm{tr}(\sigma^\dagger \sigma)$
- $\Box\sigma = \Delta_\mu^2 \sigma = \sum_x \sigma(x+2\mu) + \sigma(x-2\mu) - 2\sigma(x)$

Weak and Strong Coupling Expansions

Weak coupling:

$$-\mathrm{Tr}\log\left(\eta\cdot\Delta + iy\sigma\right) = \sum_{n=1}(-1)^n\,\mathrm{Tr}\left(iy\,G(x,y)\sigma(y)\right)^n$$

$$S_{\mathrm{eff}} = \frac{y^2}{2}\sum_{x,y}|G(x,y)|^2\,\sigma^a(x)\sigma^b(y), \qquad G = (\eta\cdot\Delta)^{-1}$$

antiferromagnetic ground state

Strong coupling:

$$S_{\mathrm{eff}} = -\sum_x \log\sigma^2(x) + \frac{1}{2y^2|\sigma|^4}\sum_{x,y}|G^{-1}(x,y)|^2\sigma^a(x)\sigma^a(y)$$

non-zero $|\sigma|^2$ but per-site direction (lattice SMG)

Derivative Expansion

Expand fluctuations in $\sigma(x)$ around antiferromagnetic background as $\sigma(x) = \mu\varepsilon(x) n^a(x)\tau^a$ with $n^a(x)n^a(x) = 1$.

$$S_{\text{eff}} = -\frac{1}{2}\mathrm{Tr}\ln\left(-\Box + y^2\mu^2 + iy\mu\varepsilon(x)[\eta_\mu(x)\Delta_\mu, n(x)]\right)$$

After some algebraic manipulation and expanding in powers of $\frac{\Delta_\mu}{y\mu}$ the leading term is

$$-\frac{1}{4y^2\mu^2}\mathrm{Tr}(\varepsilon(x)\eta_\mu(x)\Delta_\mu n^a(x)\tau^a)^2 = -\frac{1}{4y^2\mu^2}\sum_x n^a(x)\Box n^a(x)$$

Hopf Defects

Next leading order contains a Skyrme term:

$$\frac{1}{16y^4\mu^4}\mathrm{Tr}\left(\epsilon^{abc}\eta_\mu(x)\Delta_\mu(x,y)n^a(y)\eta_\nu(x)\Delta_\nu(x,z)n^b(z)\right)^2 \xrightarrow{\text{continuum}} (\partial_\mu n \times \partial_\nu n)^2$$

- Hopf map – non-trivial map between S^2 vacuum manifold arising from constraint $n^a n^a = 1$ and the S^3 spacetime boundary at infinity
- If leading order term is suppressed by tuning κ, marginal Skyrme term dominates – Hopf defects diverge logarithmically with system size.
- Similar to XY model, logarithmic contribution to entropy may cancel large volume supression of action – condensate of defects may be a candidate for continuum SMG

Phase Diagram

- an unbroken symmetric phase at small y
- a SMG phase at large y
- a broken antiferromagnetic phase above κ_c
- the three phases meet at a point $P_c = (y_c, \kappa_c)$
- an unexplored frustrated phase for large negative κ

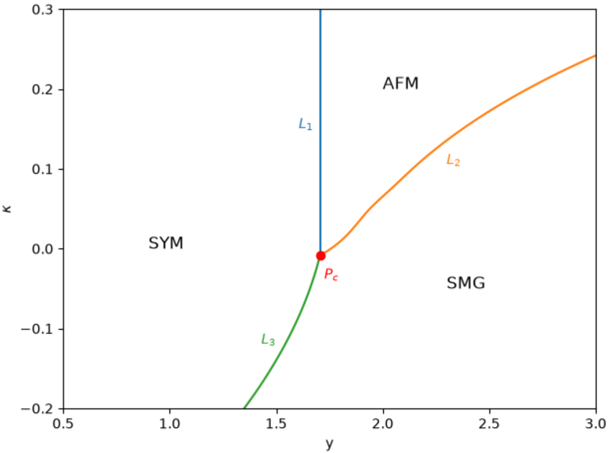


Figure: Phase diagram of the model.

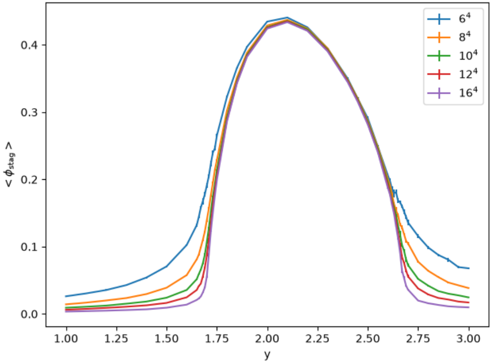
Main Observables

$$\phi = \frac{1}{V}\left|\sum_x \bar{\psi}(x)\psi(x)\right| \qquad \phi_{\mathrm{stag}} = \frac{1}{V}\left|\sum_x \varepsilon(x)\bar{\psi}(x)\psi(x)\right|$$

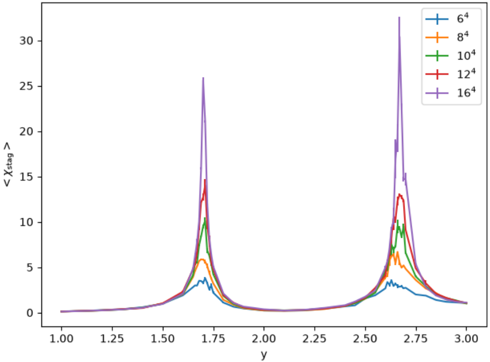
$$\Sigma = \frac{1}{V}\left|\sum_x \sigma(x)\right| \qquad \Sigma_{\mathrm{stag}} = \frac{1}{V}\left|\sum_x \varepsilon(x)\sigma(x)\right|$$

$$\chi = V\left(\langle\phi^2\rangle - \langle\phi\rangle^2\right) \qquad \chi_{\mathrm{stag}} = V\left(\langle\phi_{\mathrm{stag}}^2\rangle - \langle\phi_{\mathrm{stag}}\rangle^2\right)$$

Transitions above κ_c



(a) $\langle \phi_{\mathrm{stag}} \rangle$ at $\kappa = 0.2$



(b) $\langle \chi_{\mathrm{stag}} \rangle$ at $\kappa = 0.2$

Figure: Anti-ferromagnetic bilinear and its susceptibility vs y at $\kappa = 0.2$

Critical exponents

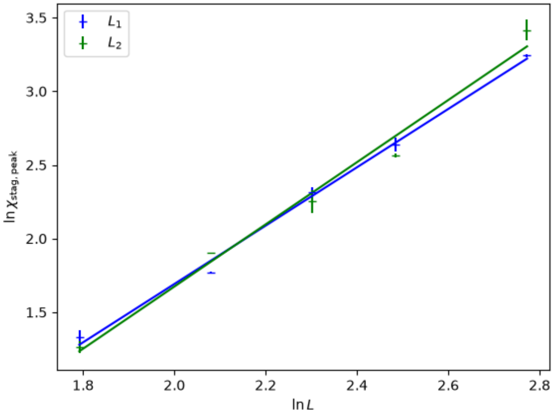


Figure: Peak $\chi_{\rm stag}$ versus L for $\kappa = 0.2$ and $\lambda = 1.0$ along the phase transition boundaries L_1 and L_2.

Expect peak susceptibility to scale as a power of L at the phase transition

$$\chi_{\rm peak} \sim L^{2-\eta}$$

Extract η via linear least squares fit of plot $\ln \chi_{\rm peak}$ versus $\ln L$.

$$\eta_1 = 0.019 \pm 0.077$$

$$\eta_2 = -0.11 \pm 0.15$$

Location of point P_c

Can extrapolate position of merged point via linear fit of transition boundaries close to it:

- $\kappa = 0.025$
- $\kappa = 0.05$
- $\kappa = 0.075$
- $\kappa = 0.1$
- $\kappa = 0.15$

$$y_c = 1.706 \pm 0.003$$
$$\kappa_c = -0.01 \pm 0.006$$

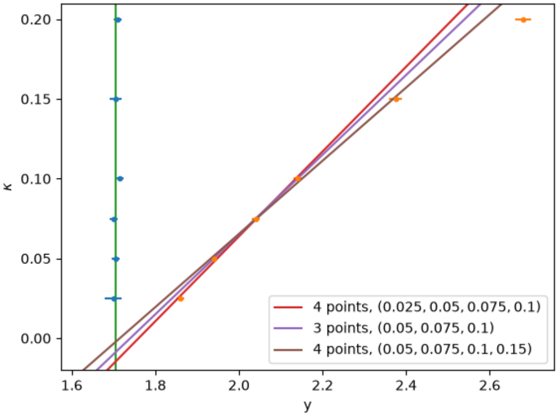
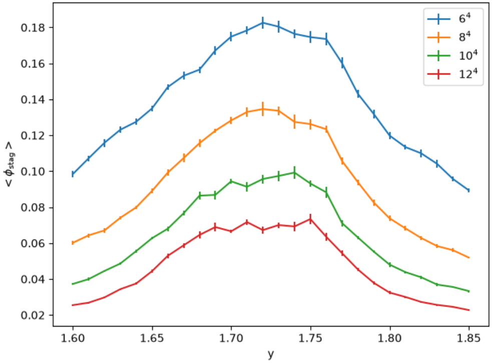
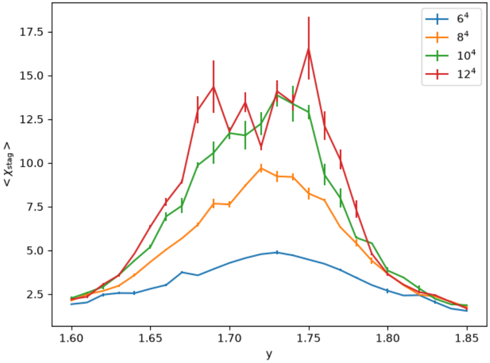


Figure: Location of the two phase transitions and their crossing point

Fermion Observables at κ_c



(a) $\langle \phi_{\text{stag}} \rangle$ at $\kappa = -0.01$



(b) $\langle \chi_{\text{stag}} \rangle$ at $\kappa = -0.01$

Figure: Anti-ferromagnetic bilinear and its susceptibility vs y at $\kappa = -0.01$

Critical exponents at P_c

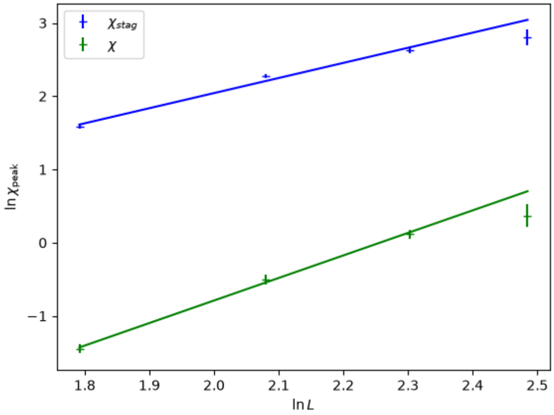


Figure: Peak χ and χ_{stag} versus L for $\kappa = -0.01$ and $\lambda = 1.0$.

Extract η via linear least squares fit of plot $\ln \chi_{\mathrm{peak}}$ versus $\ln L$.

$$\chi_{\mathrm{stag}} : \eta = -0.06 \pm 0.22$$
$$\chi : \eta' = -1.08 \pm 0.15$$

Peak widths and data collapse

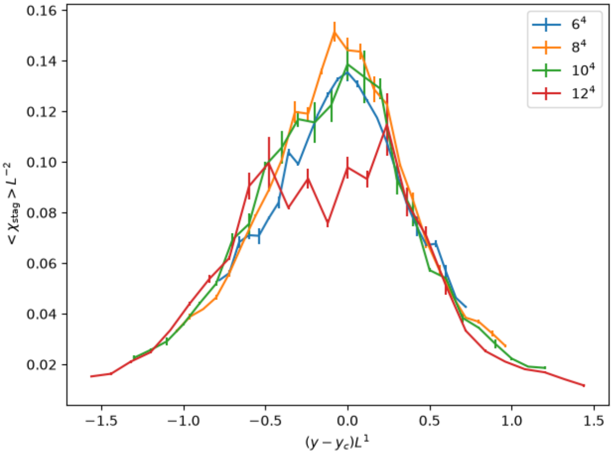


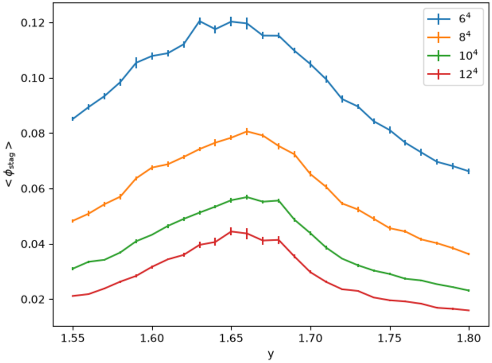
Figure: Curve collapse plot for $< \chi_{\mathrm{stag}} >$ for $\kappa = -0.01$ and $\lambda = 1.0$

Can collapse the susceptibility data onto a single curve for all volumes by assuming that in the critical region

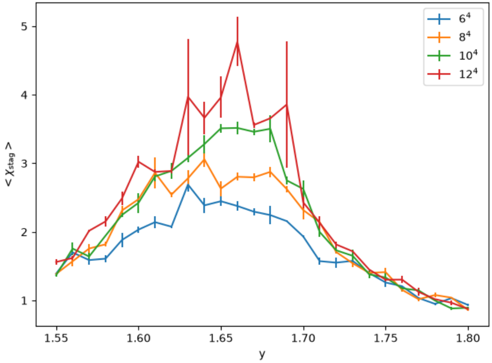
$$y - y_c \sim \frac{1}{L}.$$

Unusual fixed point not consistent with mean field theory

Behavior at $\kappa < \kappa_c$



(a) $<\phi_{\rm stag}>$ vs y at $\kappa = -0.05$



(b) $<\chi_{\rm stag}>$ vs y at $\kappa = -0.05$

Figure: $<\phi_{\rm stag}>$ and $<\chi_{\rm stag}>$ for $\kappa = -0.05$ and $\lambda = 1.0$

Conclusions

- Three phases: free (symmetric) phase, broken antiferromagnetic phase, SMG phase
- Transitions bordering AFM phase consistent with mean field theory
- Candidate multicritical point P_c where phase boundaries merge
- Evidence that a single critical line continues from P_c into negative κ

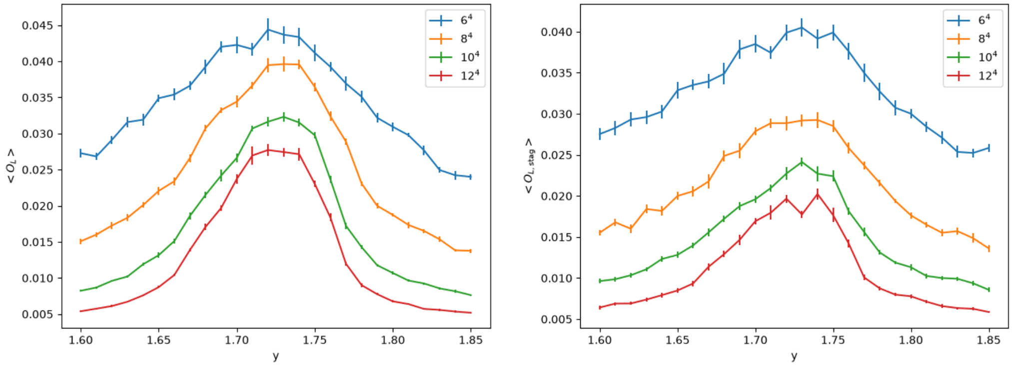
At P_c:

- ferromagnetic and antiferromagnetic order parameters all vanish
- non-zero four fermion condensate
- fermion susceptibilities diverge with non-trivial exponents

Questions

Extra: One-link operators

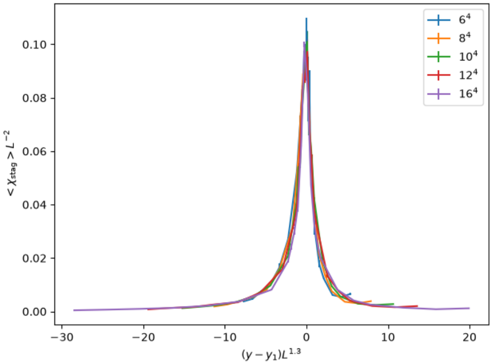
$$O_L = \sum_{\mu} \xi_\mu(x)\bar{\chi}(x)\left[\chi(x+\mu) + \chi(x-\mu)\right]$$



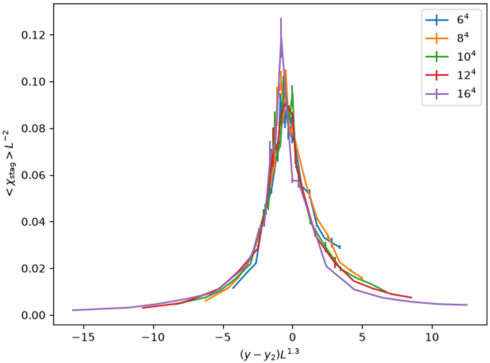
(a) $< O_L >$ vs y at $\kappa = -0.01$ and $\lambda = 1.0$

(b) $< O_{L,\mathrm{stag}} >$ vs y at $\kappa = -0.01$ and $\lambda = 1.0$

Extra: Peak widths at $\kappa > \kappa_c$



(a) Transition L_1 from symmetric to antiferromagnetic phases

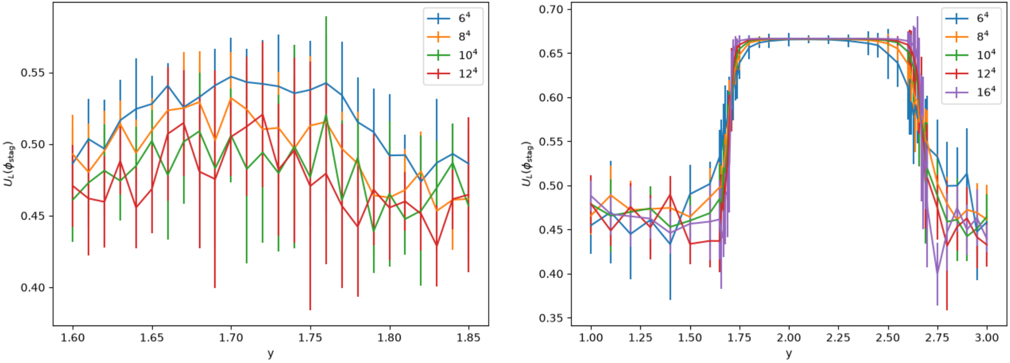


(b) Transition L_2 from antiferromagnetic to SMG phases

Figure: Curve collapse plots for $< \chi_{\mathrm{stag}} >$ for $\kappa = 0.2$ and $\lambda = 1.0$

Extra: Binder cumulant

$$U_L = 1 - \frac{< \phi^4_{\mathrm{stag}} >}{3 < \phi^2_{\mathrm{stag}} >^2}$$



(a) Binder cumulant at $\kappa = -0.01$ and $\lambda = 1.0$ (b) Binder cumulant at $\kappa = 0.2$ and $\lambda = 1.0$