Progress on $Sp(4)$ Gauge Theory with $N_{\mathrm{f}} = 2$ Fundamental Möbius Domain Wall Fermions

Gianmarco Simonetti

In collaboration with
E. Bennett, P. Boyle, L. Del Debbio, N. Forzano, R. C. Hill, D. K. Hong, J.-W. Lee, C.-J. D. Lin, B. Lucini, M. Piai, D. Vadacchino, A. H. Verney-Provatas

Swansea University
The University of Edinburgh

Lattice 2026, July 26 - August 1, 2026 University of Maryland, College Park

Motivation

- $\boldsymbol{Sp(4)}$ **with two dynamical fermions**
  - template for strongly-coupled BSM dynamics based on the $SU(4) \to Sp(4)$ coset: minimal composite-Higgs and dark-matter models [Barnard et al., 1311.6562; Hochberg et al., 1411.3727]
  - previous lattice studies using Wilson fermions [Bennett et al., 1712.04220; Bennett et al., 1909.12662]

Motivation

- **$Sp(4)$ with two dynamical fermions**
  - template for strongly-coupled BSM dynamics based on the $SU(4) \to Sp(4)$ coset: minimal composite-Higgs and dark-matter models [Barnard et al., 1311.6562; Hochberg et al., 1411.3727]
  - previous lattice studies using Wilson fermions [Bennett et al., 1712.04220; Bennett et al., 1909.12662]
- **Mobius Domain Wall Fermions** [Brower, Neff, Orginos, 1206.5214]
  - control of the global symmetry breaking on the lattice
  - improved action with $\mathcal{O}(a^2)$ corrections
  - improvement over the Shamir formulation [Shamir, hep-lat/9303005]

The model: $Sp(4)$ with $N_{\mathrm{f}} = 2$ fundamental fermions

$$S = \int d^4x \, \left[-\frac{1}{2} \mathrm{Tr} \left(F_{\mu\nu} F^{\mu\nu} \right) + \sum_{\mathrm{f}=1}^{2} \bar{\psi}^{\mathrm{f}} \left(i\gamma^{\mu} D_{\mu} - m^{\mathrm{f}} \right) \psi^{\mathrm{f}} \right]$$

The model: $Sp(4)$ with $N_\mathrm{f} = 2$ fundamental fermions

$$S = \int d^4x \, \left[-\frac{1}{2} \mathrm{Tr} \left(F_{\mu\nu} F^{\mu\nu} \right) + \sum_{\mathrm{f}=1}^{2} \bar{\psi}^\mathrm{f} \left(i \gamma^\mu D_\mu - m^\mathrm{f} \right) \psi^\mathrm{f} \right]$$

Lattice action

$$S \equiv \boldsymbol{S_g[U]} + S_f[\Psi, \bar{\Psi}] + S_{\mathrm{PV}}[\phi, \phi^\dagger]$$

- $\boldsymbol{S_g[U]}$ is the Wilson gauge action, where U are the link variables, i.e. $Sp(2N)$ matrices with $N = 2$

$$S_g[U] \equiv \beta \sum_x \sum_{\mu<\nu} \left(1 - \frac{1}{2N} \mathrm{Re}\,\mathrm{Tr}\, P_{\mu\nu}(x) \right), \qquad \beta \equiv \frac{4N}{g_0^2}$$

The model: $Sp(4)$ with $N_{\rm f} = 2$ fundamental fermions

$$S = \int d^4x \left[-\frac{1}{2} \mathrm{Tr} \left(F_{\mu\nu} F^{\mu\nu} \right) + \sum_{\mathrm{f}=1}^{2} \bar{\psi}^{\mathrm{f}} \left(i \gamma^{\mu} D_{\mu} - m^{\mathrm{f}} \right) \psi^{\mathrm{f}} \right]$$

Lattice action

$$S \equiv S_g[U] + \boldsymbol{S_f[\Psi, \bar{\Psi}]} + S_{\rm PV}[\phi, \phi^{\dagger}]$$

- $\boldsymbol{S_f[\Psi, \bar{\Psi}]}$ is the fermionic action using domain wall fermions, where Ψ are five-dimensional fields. We consider degenerate flavours with mass m_0

$$S_f[\Psi, \bar{\Psi}] \equiv a^4 a_5 \sum_{\mathrm{f}=1}^{2} \sum_{x,y} \sum_{s,r=0}^{L_s-1} \bar{\Psi}^{\mathrm{f}}_{x,s} \, D^{M}_{x,s;\,y,r}(m_0; m_5, a_5, \alpha) \, \Psi^{\mathrm{f}}_{y,r}$$

The model: $Sp(4)$ with $N_{\rm f} = 2$ fundamental fermions

$$S = \int d^4x \left[-\frac{1}{2} \mathrm{Tr}\left(F_{\mu\nu}F^{\mu\nu}\right) + \sum_{\rm f=1}^{2} \bar{\psi}^{\rm f} \left(i\gamma^{\mu} D_{\mu} - m^{\rm f}\right) \psi^{\rm f} \right]$$

Lattice action

$$S \equiv S_g[U] + S_f[\Psi, \bar{\Psi}] + \boldsymbol{S_{\rm PV}[\phi, \phi^{\dagger}]}$$

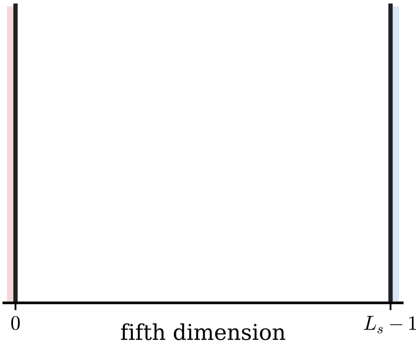
- $\boldsymbol{S_{\rm PV}[\Phi, \Phi^{\dagger}]}$ is the Pauli-Villars term needed to cancel the heavy bulk modes. It has the same form as S_f, but is defined on scalar fields with a mass $am_{\rm PV} = 1$

$$S_{\rm PV}[\Phi, \Phi^{\dagger}] \equiv a^4 a_5 \sum_{\rm f=1}^{2} \sum_{x,y} \sum_{s,r=0}^{L_s-1} \Phi^{\dagger \rm f}_{x,s} \, D^{M}_{x,s;\,y,r}(m_{\rm PV}; m_5, a_5, \alpha) \, \Phi^{\rm f}_{y,r}$$

Domain Wall Fermions

$$S_f \equiv a^4 a_5 \sum_{\mathrm{f}=1}^{2} \sum_{x,y} \sum_{s,r=0}^{\boldsymbol{L_s}-1} \bar{\Psi}^{\mathrm{f}}_{x,s}\, D^{M}_{x,s;\,y,r}(m_0; m_5, a_5, \alpha)\, \Psi^{\mathrm{f}}_{y,r}$$

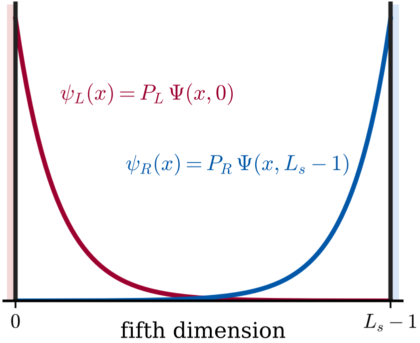
- The fermion action is formulated in five dimensions, with $\boldsymbol{L_s}$ the extent of the fifth dimension



Domain Wall Fermions

$$S_f \equiv a^4 a_5 \sum_{\mathrm{f}=1}^{2} \sum_{x,y} \sum_{s,r=0}^{L_s-1} \bar{\boldsymbol{\Psi}}^{\mathbf{f}}_{\boldsymbol{x},\boldsymbol{s}} \, D^M_{x,s;\,y,r}(m_0; m_5, a_5, \alpha) \, \boldsymbol{\Psi}^{\mathbf{f}}_{\boldsymbol{y},\boldsymbol{r}}$$

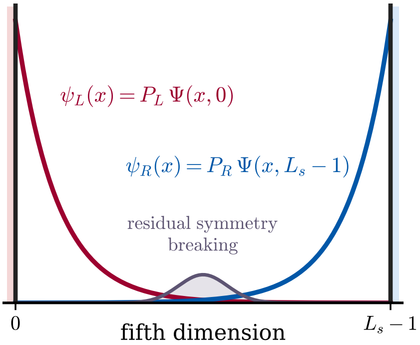
- The fermion action is formulated in five dimensions, with L_s the extent of the fifth dimension
- $\boldsymbol{\Psi}_{\boldsymbol{x},\boldsymbol{s}}$ are five-dimensional fermion fields; the physical four-dimensional field is $\psi = \psi_L + \psi_R$, with ψ_L and ψ_R localised on opposite walls of the fifth dimension



Domain Wall Fermions: residual mass

- The overlap between the two modes defines the extent of global symmetry breaking, described by the **residual mass**

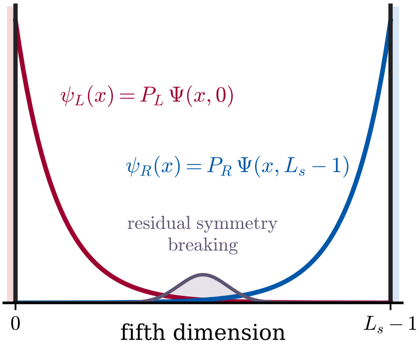
$$m_{\text{res}}(t) \equiv \frac{\sum_{\vec{x},\vec{y}} \langle O^1_{\text{PS},q}(\vec{y},t)\, O^1_{\text{PS}}(\vec{x},0)\rangle}{\sum_{\vec{x},\vec{y}} \langle O^1_{\text{PS}}(\vec{y},t)\, O^1_{\text{PS}}(\vec{x},0)\rangle} \xrightarrow[t \text{ large}]{} m_{\text{res}}$$



Domain Wall Fermions: residual mass

- The overlap between the two modes defines the extent of global symmetry breaking, described by the **residual mass**

$$m_{\mathrm{res}}(t) \equiv \frac{\sum_{\vec{x},\vec{y}} \langle O^1_{\mathrm{PS},q}(\vec{y},t)\, O^1_{\mathrm{PS}}(\vec{x},0)\rangle}{\sum_{\vec{x},\vec{y}} \langle O^1_{\mathrm{PS}}(\vec{y},t)\, O^1_{\mathrm{PS}}(\vec{x},0)\rangle} \xrightarrow[t\ \text{large}]{} m_{\mathrm{res}}$$



- The axial Ward identity for DWF is

$$\Delta_\mu \mathcal{A}^1_\mu(x) \simeq 2\big(am_0 + am_{\mathrm{res}}\big)\, O^1_{\mathrm{PS}}(x)$$

A working DWF mechanism requires

$$m_{\mathrm{res}} \ll m_0$$

Möbius Domain Wall Fermions parameters

Möbius kernel : $D^M(m_0; m_5, a_5, \alpha) =$

$$\begin{pmatrix} D_L & D_R P_R & 0 & \cdots & -m_0 D_R P_L \\ D_R P_L & D_L & D_R P_R & \cdots & 0 \\ 0 & D_R P_L & D_L & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & D_R P_R \\ -m_0 D_R P_R & 0 & 0 & \cdots & D_L \end{pmatrix}$$

$$D_L = \frac{a_5(\alpha+1)}{2} D^W(m_5) + 1, \qquad D_R = \frac{a_5(\alpha-1)}{2} D^W(m_5) - 1$$

$$P_{R,L} = \frac{1 \pm \gamma_5}{2}$$

Möbius Domain Wall Fermions parameters

- m_5: domain wall height entering the Wilson kernel $D_W(-m_5)$ and controlling the localisation of the light boundary modes

Möbius kernel : $D^M(m_0; m_5, a_5, \alpha) =$

$$\begin{pmatrix} D_L & D_R P_R & 0 & \cdots & -m_0 D_R P_L \\ D_R P_L & D_L & D_R P_R & \cdots & 0 \\ 0 & D_R P_L & D_L & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & D_R P_R \\ -m_0 D_R P_R & 0 & 0 & \cdots & D_L \end{pmatrix}$$

$$D_L = \frac{a_5(\alpha+1)}{2} D^W(m_5) + 1, \qquad P_{R,L} = \frac{1 \pm \gamma_5}{2}$$

$$D_R = \frac{a_5(\alpha-1)}{2} D^W(m_5) - 1$$

Möbius Domain Wall Fermions parameters

- m_5: domain wall height entering the Wilson kernel $D_W(-m_5)$ and controlling the localisation of the light boundary modes
- $\boldsymbol{a_5}$: effective fifth-direction lattice spacing

Möbius kernel : $D^M(m_0; m_5, \boldsymbol{a_5}, \alpha) =$

$$\begin{pmatrix} D_L & D_R P_R & 0 & \cdots & -m_0 D_R P_L \\ D_R P_L & D_L & D_R P_R & \cdots & 0 \\ 0 & D_R P_L & D_L & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & D_R P_R \\ -m_0 D_R P_R & 0 & 0 & \cdots & D_L \end{pmatrix}$$

$$D_L = \frac{\boldsymbol{a_5}(\alpha+1)}{2} D^W(m_5) + 1, \qquad P_{R,L} = \frac{1 \pm \gamma_5}{2}$$

$$D_R = \frac{\boldsymbol{a_5}(\alpha-1)}{2} D^W(m_5) - 1$$

Möbius Domain Wall Fermions parameters

- m_5: domain wall height entering the Wilson kernel $D_W(-m_5)$ and controlling the localisation of the light boundary modes
- a_5: effective fifth-direction lattice spacing
- α: Möbius rescaling parameter controlling the approximation to the sign function; $\alpha = 1$ reproduces the Shamir formulation

Möbius kernel : $D^M(m_0; m_5, a_5, \alpha) =$

$$\begin{pmatrix} D_L & D_R P_R & 0 & \cdots & -m_0 D_R P_L \\ D_R P_L & D_L & D_R P_R & \cdots & 0 \\ 0 & D_R P_L & D_L & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & D_R P_R \\ -m_0 D_R P_R & 0 & 0 & \cdots & D_L \end{pmatrix}$$

$$D_L = \frac{a_5(\alpha+1)}{2} D^W(m_5) + 1, \qquad P_{R,L} = \frac{1 \pm \gamma_5}{2}$$

$$D_R = \frac{a_5(\alpha-1)}{2} D^W(m_5) - 1$$

Tuning strategy and numerical results

Choose a region in (β, am_0) where lattice artefacts and the residual mass are under control. We use Shamir with $a_5/a = 1$, $am_5 = 1.8$ and $L_s = 8$

[Shamir, hep-lat/9303005]

Tuning strategy and numerical results

Choose a region in (β, am_0) where lattice artefacts and the residual mass are under control. We use Shamir with $a_5/a = 1$, $am_5 = 1.8$ and $L_s = 8$

[Shamir, hep-lat/9303005]

↓

Tune α, a_5/a and am_5 for selected values of β

Tuning strategy and numerical results

Choose a region in (β, am_0) where lattice artefacts and the residual mass are under control. We use Shamir with $a_5/a = 1$, $am_5 = 1.8$ and $L_s = 8$

[Shamir, hep-lat/9303005]

↓

Tune α, a_5/a and am_5 for selected values of β

↓

Tune α for different values of L_s keeping a_5/a and am_5 fixed

Tuning strategy and numerical results

Choose a region in (β, am_0) where lattice artefacts and the residual mass are under control. We use Shamir with $a_5/a = 1$, $am_5 = 1.8$ and $L_s = 8$

[Shamir, hep-lat/9303005]

↓

Tune α, a_5/a and am_5 for selected values of β

↓

Tune α for different values of L_s keeping a_5/a and am_5 fixed

Software used

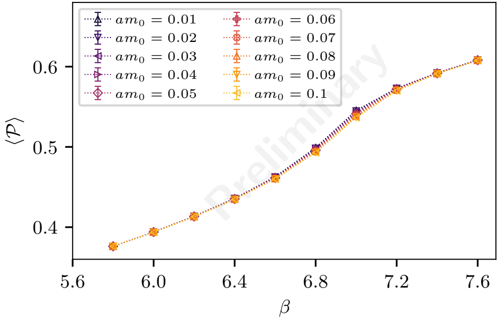
- Grid github.com/paboyle/Grid
- hmcdj github.com/telos-collaboration/hmcdj
  Software development and machines session
  Easier parameter scans with hmcdj, **Tuesday 16:10, A.H. Verney-Provatas**
- Hadrons github.com/aportelli/hadrons

Physical parameter scan: β and m_0

- Average plaquette as a function of the lattice coupling β

$$\langle P \rangle = \frac{1}{6V} \sum_{x,\mu<\nu} \frac{1}{2N} \mathrm{Re}\,\mathrm{Tr}\, U_{\mu\nu}(x)$$

No bulk phase transition effects



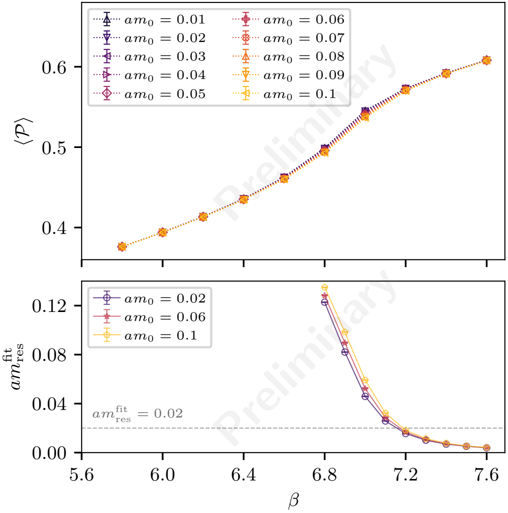
Physical parameter scan: β and m_0

- Average plaquette as a function of the lattice coupling β

$$\langle P \rangle = \frac{1}{6V} \sum_{x,\mu<\nu} \frac{1}{2N} \mathrm{Re}\,\mathrm{Tr}\, U_{\mu\nu}(x)$$

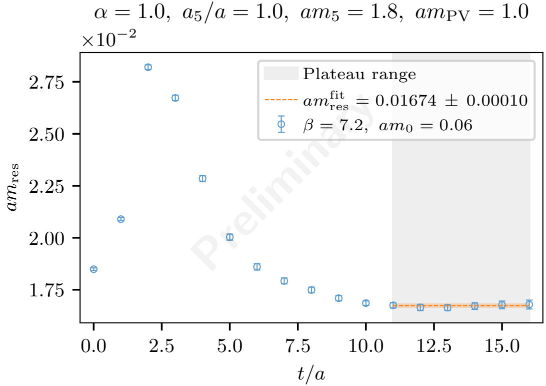
No bulk phase transition effects

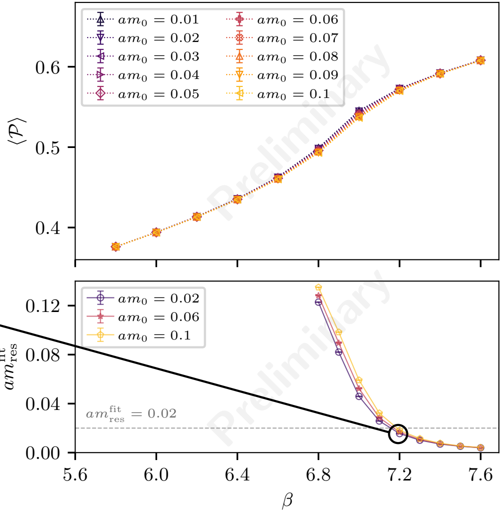
- Residual-mass as a function of β: we choose $\beta \geq 7.4$ such that am_{res} is about one order of magnitude smaller than am_0.



Physical parameter scan: β and m_0

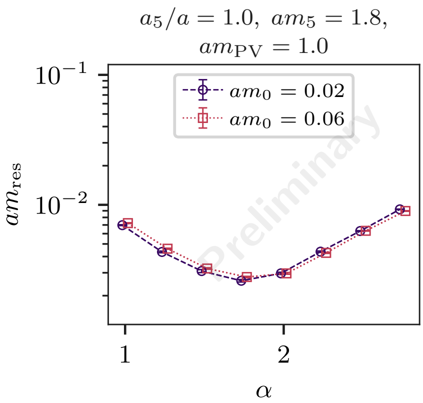
- Average plaquette as a function of





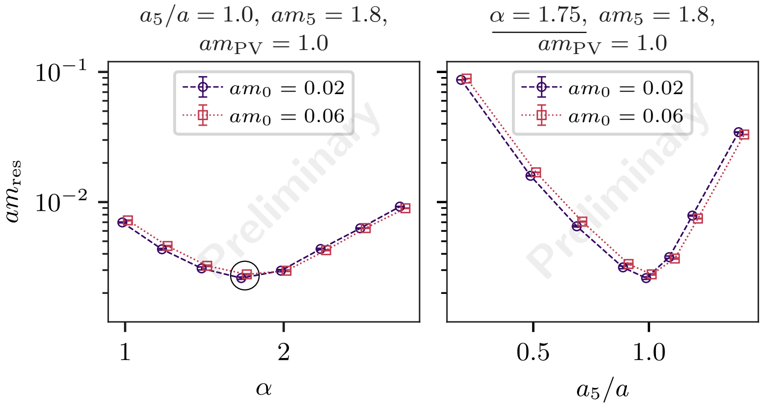
- smaller than am_0.

Möbius parameter scan: α, a_5, m_5



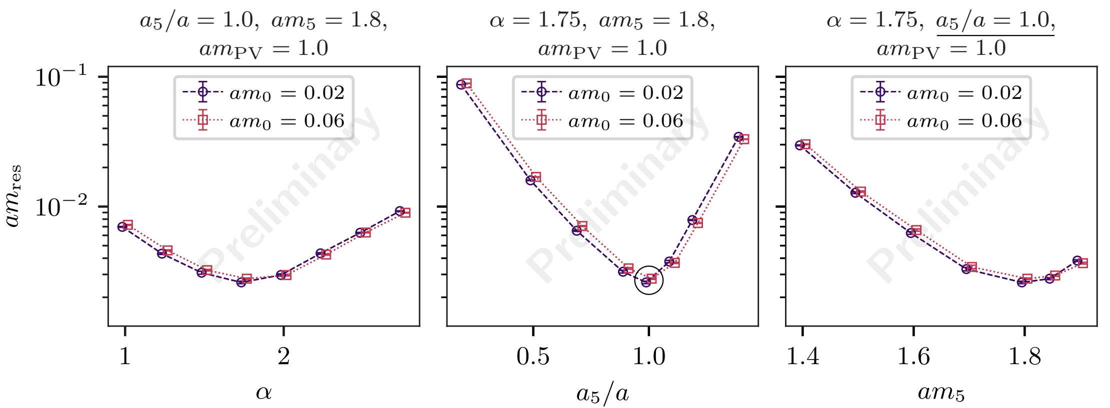
Residual-mass scan at $\beta = 7.4$, $L_s = 8$ and $\tilde{V} = 32 \times 16^3$

Möbius parameter scan: α, a_5, m_5



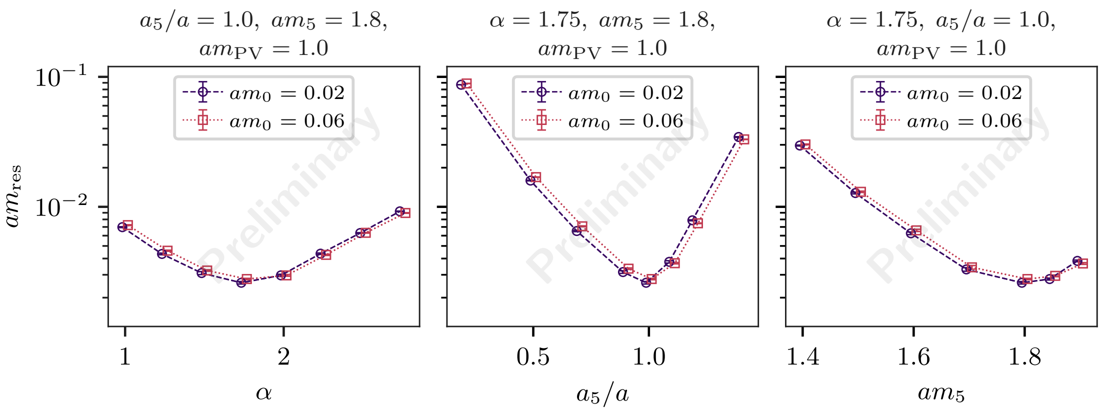
Residual-mass scan at $\beta = 7.4$, $L_s = 8$ and $\tilde{V} = 32 \times 16^3$

Möbius parameter scan: α, a_5, m_5



Residual-mass scan at $\beta = 7.4$, $L_s = 8$ and $\tilde{V} = 32 \times 16^3$

Möbius parameter scan: α, a_5, m_5



Residual-mass scan at $\beta = 7.4$, $L_s = 8$ and $\tilde{V} = 32 \times 16^3$

Tuned parameters: $\alpha = 1.75$, $a_5/a = 1.0$, $am_5 = 1.8$

Scan over L_s

[Antonio et al., 10.1103/PhysRevD.77.014509; Brower, Neff, Orginos, 1206.5214]

$$m_{\mathrm{res}}(L_s) \simeq c_{\mathrm{ext}} e^{-\boldsymbol{\lambda_c} L_s} + \frac{c_{\mathrm{loc}}}{L_s^{\nu}}$$

- $\boldsymbol{\lambda_c}$: mobility-edge scale controlling the exponential decay

Scan over L_s

[Antonio et al., 10.1103/PhysRevD.77.014509; Brower, Neff, Orginos, 1206.5214]

$$m_{\text{res}}(L_s) \simeq \boldsymbol{c}_{\textbf{ext}} e^{-\lambda_c L_s} + \frac{c_{\text{loc}}}{L_s^{\nu}}$$

- λ_c: mobility-edge scale controlling the exponential decay
- $\boldsymbol{c}_{\textbf{ext}}$: set by extended modes near the mobility edge, $\lambda_c \lesssim \lambda \lesssim \lambda_c + \frac{1}{L_s}$; exponentially suppressed at large L_s

Scan over L_s

[Antonio et al., 10.1103/PhysRevD.77.014509; Brower, Neff, Orginos, 1206.5214]

$$m_{\text{res}}(L_s) \simeq c_{\text{ext}} e^{-\lambda_c L_s} + \frac{\boldsymbol{c}_{\textbf{loc}}}{L_s^{\nu}}$$

- λ_c: mobility-edge scale controlling the exponential decay
- c_{ext}: set by extended modes near the mobility edge, $\lambda_c \lesssim \lambda \lesssim \lambda_c + \frac{1}{L_s}$; exponentially suppressed at large L_s
- $\boldsymbol{c}_{\textbf{loc}}$: set by the density of localised near-zero modes with $0 \lesssim \lambda \lesssim \frac{1}{\alpha L_s}$

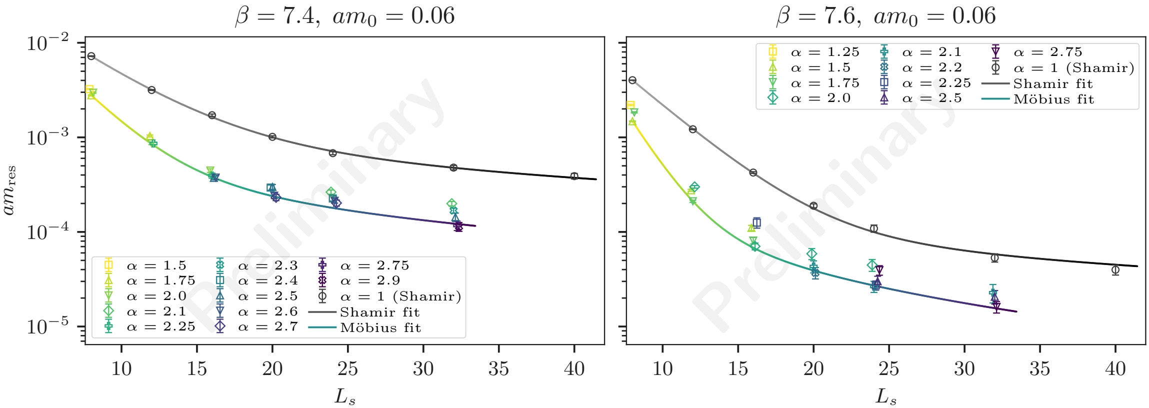
Scan over L_s

[Antonio et al., 10.1103/PhysRevD.77.014509; Brower, Neff, Orginos, 1206.5214]

$$m_{\text{res}}(L_s) \simeq c_{\text{ext}} e^{-\lambda_c L_s} + \frac{c_{\text{loc}}}{L_s^{\boldsymbol{\nu}}}$$

- λ_c: mobility-edge scale controlling the exponential decay
- c_{ext}: set by extended modes near the mobility edge, $\lambda_c \lesssim \lambda \lesssim \lambda_c + \frac{1}{L_s}$; exponentially suppressed at large L_s
- c_{loc}: set by the density of localised near-zero modes with $0 \lesssim \lambda \lesssim \frac{1}{\alpha L_s}$
- $\boldsymbol{\nu} = 1$ for Shamir, while $\boldsymbol{\nu} = 2$ for a tuned Möbius formulation

Scan over L_s



Shamir: fixed $\nu = 1$

Möbius: ν is a parameter of the fit

Möbius min. $m_{\rm res}$	β	am_0	ν
	7.4	0.06	1.25(30)
	7.6	0.06	1.90(32)

Spectrum

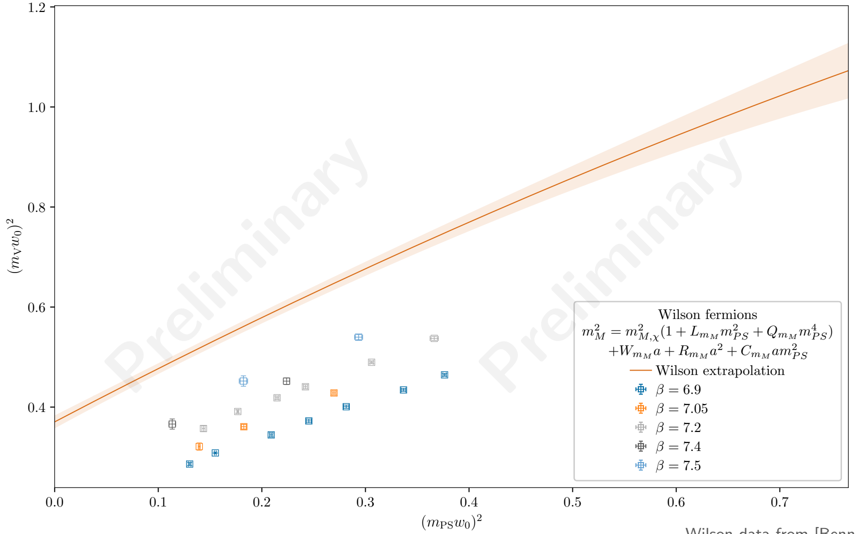
- We perform spectroscopy for pseudoscalar observables and the vector mass, using

$$C_{\mathrm{ch}_1,\mathrm{ch}_2}(t) = \sum_{\vec{x}} \langle O_{\mathrm{ch}_1}(\vec{x},t)\, O^\dagger_{\mathrm{ch}_2}(\vec{0},0)\rangle$$

Channel	**Operator** $\mathcal{O}$	J^P	$Sp(4)$
PS	$\psi_1\gamma_5\psi_2$	0^-	5
V	$\psi_1\gamma_\mu\psi_2$	1^-	10
AV	$\psi_1\gamma_5\gamma_\mu\psi_2$	1^+	5

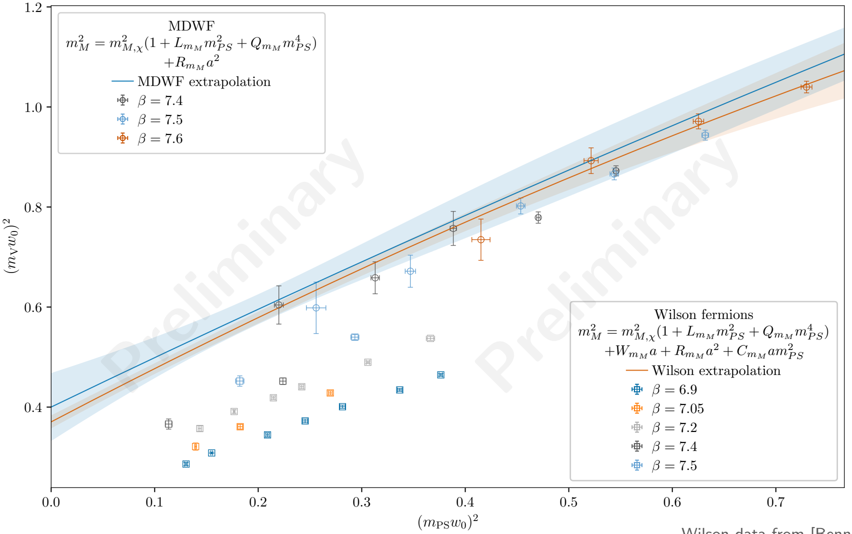
- For large Euclidean time separations, we extract the masses and amplitudes from the effective mass and perform correlated fits
- We set the scale by computing w_0 from the Wilson flow [Lüscher, 1006.4518]
- We perform the continuum and massless limit extrapolations of am_{V} and af_{PS} and compare with previous Wilson-fermion results
  [Rupak, Shoresh, hep-lat/0201019; Symanzik, Nucl. Phys. B226 (1983) 187]

m_V extrapolation



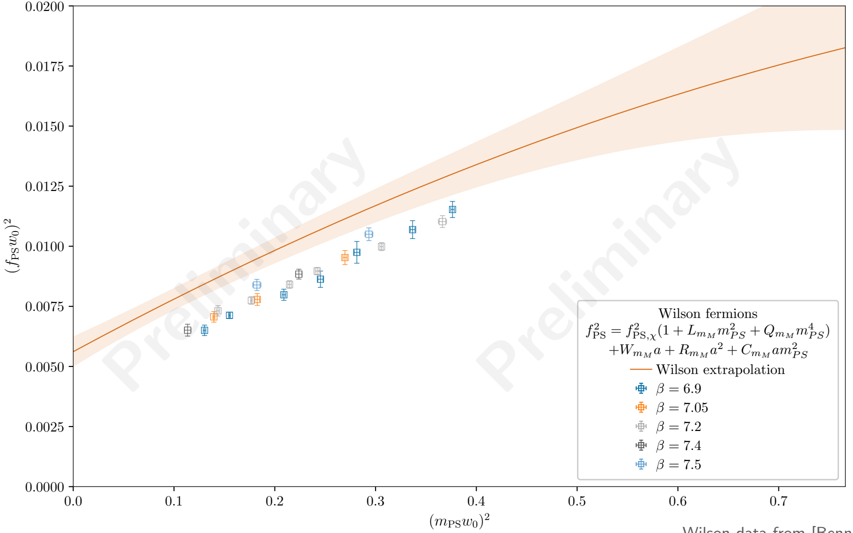
Wilson data from [Bennett et al., 2606.14546]

m_V extrapolation



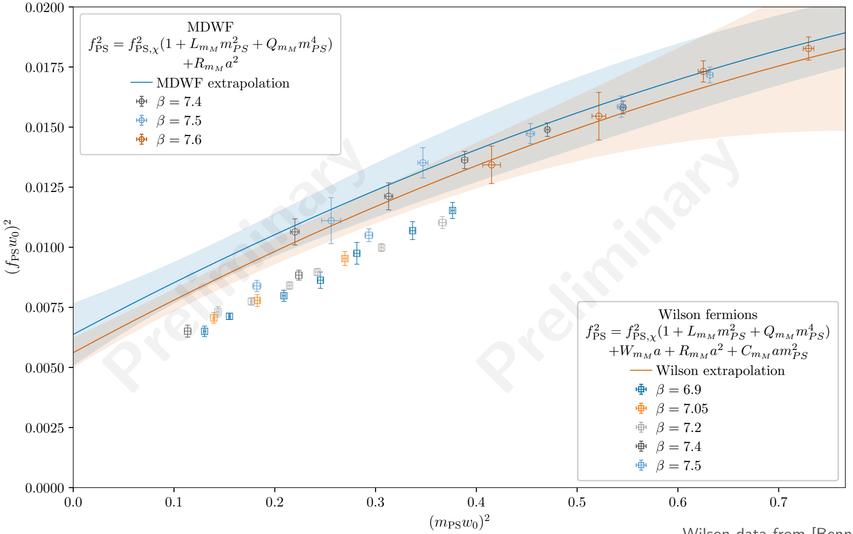
Wilson data from [Bennett et al., 2606.14546]

f_{PS} extrapolation



Wilson data from [Bennett et al., 2606.14546]

$f_{\rm PS}$ extrapolation



Wilson data from [Bennett et al., 2606.14546]

Conclusions

- **Summary**
  - First implementation of MDWF for $Sp(4)$ with two fundamental fermions: mapped the working (β, m_0) region and tuned Möbius parameters to suppress m_{res}
  - obtained first improved spectroscopy results for m_{V} and f_{PS}, to be compared with previous Wilson-fermion works

Conclusions

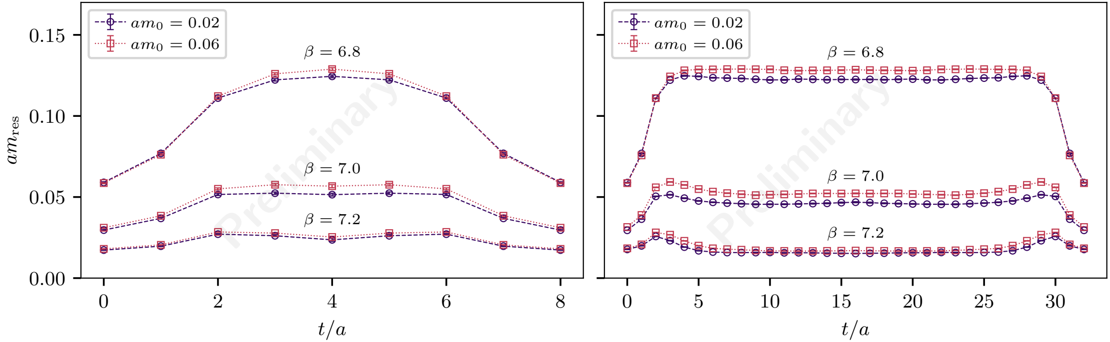
- **Summary**
  - First implementation of MDWF for $Sp(4)$ with two fundamental fermions: mapped the working (β, m_0) region and tuned Möbius parameters to suppress m_{res}
  - obtained first improved spectroscopy results for m_{V} and f_{PS}, to be compared with previous Wilson-fermion works
- **Outlook**
  - complete the m_{V} and f_{PS} extrapolations, as these results are still work in progress
  - complete the spectroscopy program and use MDWF to refine past measurements, such as scattering observables

$Sp(2N)$ gauge groups

$$Sp(2N) = \left\{ U \in \mathrm{SU}(2N) \,\middle|\, U\,\Omega\,U^T = \Omega \right\}, \qquad \Omega = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$$

- $Sp(2N)$ is the subgroup of $\mathrm{SU}(2N)$ preserving the symplectic form Ω
- rank$(Sp(2N)) = N$ and $\dim(Sp(2N)) = N(2N+1)$
- the fundamental representation is pseudoreal, so for N_f Dirac flavours the global symmetry is enhanced to $\mathrm{SU}(2N_\mathrm{f})$
- for $Sp(4)$: $N = 2$, the group has dimension 10 and $Sp(2) \simeq \mathrm{SU}(2)$

Residual Mass at coarse lattice



Backup slides

Axial-vector and pseudoscalar operators

$$
\begin{aligned}
\mathcal{O}_\mu^{\mathrm{AV},\,1} &= \overline{\psi^{1a}}\gamma_\mu\gamma_5\psi^{2a} + \overline{\psi^{2a}}\gamma_\mu\gamma_5\psi^{1a},\\
\mathcal{O}_\mu^{\mathrm{AV},\,2} &= -i\overline{\psi^{1a}}\gamma_\mu\gamma_5\psi^{2a} + i\overline{\psi^{2a}}\gamma_\mu\gamma_5\psi^{1a},\\
\mathcal{O}_\mu^{\mathrm{AV},\,3} &= \overline{\psi^{1a}}\gamma_\mu\gamma_5\psi^{1a} - \overline{\psi^{2a}}\gamma_\mu\gamma_5\psi^{2a},\\
\mathcal{O}_\mu^{\mathrm{AV},\,4} &= -i\overline{\psi^{1}}\gamma_\mu\psi_C^{2} + i\overline{\psi_C^{2}}\gamma_\mu\psi^{1},\\
\mathcal{O}_\mu^{\mathrm{AV},\,5} &= \overline{\psi^{1}}\gamma_\mu\psi_C^{2} + \overline{\psi_C^{2}}\gamma_\mu\psi^{1}.
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{O}_{\mathrm{PS}}^{1} &= \overline{\psi^{1a}}\gamma_5\psi^{2a} + \overline{\psi^{2a}}\gamma_5\psi^{1a},\\
\mathcal{O}_{\mathrm{PS}}^{2} &= -i\overline{\psi^{1a}}\gamma_5\psi^{2a} + i\overline{\psi^{2a}}\gamma_5\psi^{1a},\\
\mathcal{O}_{\mathrm{PS}}^{3} &= \overline{\psi^{1a}}\gamma_5\psi^{1a} - \overline{\psi^{2a}}\gamma_5\psi^{2a},\\
\mathcal{O}_{\mathrm{PS}}^{4} &= -i\left(\overline{\psi^{1a}}\psi_C^{2a} + \overline{\psi_C^{2a}}\psi^{1a}\right),\\
\mathcal{O}_{\mathrm{PS}}^{5} &= i\left(-i\overline{\psi^{1a}}\psi_C^{2a} + i\overline{\psi_C^{2a}}\psi^{1a}\right).
\end{aligned}
$$

$\psi_C^{i\,a} \equiv -\gamma_5\Omega^{ab}C(\overline{\psi^i}^{\,T})_b,$ (spinor indices are suppressed)

Residual-mass correlators

Five-dimensional local current

$$j_\mu^1(x,s) = \frac{1}{\sqrt{2}}\Big[\bar{\Psi}(x+\hat{\mu},s)(1+\gamma_\mu)U_\mu^\dagger(x)T^1\Psi(x,s) - \bar{\Psi}(x,s)(1-\gamma_\mu)U_\mu(x)T^1\Psi(x+\hat{\mu},s)\Big].$$

The four-dimensional lattice axial-vector current

$$\mathcal{A}_\mu^1(x) = \sum_{s=0}^{L_s-1} \operatorname{sgn}\left(s - \frac{L_s-1}{2}\right) j_\mu^1(x,s)\,.$$

$$\mathcal{O}_{\mathrm{PS}}^1(x) = \sqrt{2}\left[\bar{\Psi}(x,0)P_L T^1\Psi(x,L_s-1) - \bar{\Psi}(x,L_s-1)P_R T^1\Psi(x,0)\right]\,.$$

Midpoint current

$$\mathcal{O}_{\mathsf{PS},\,q}^1(x) = \sqrt{2}\Big[\bar{\Psi}(x,L_s/2)P_L T^1\Psi(x,L_s/2-1) - \bar{\Psi}(x,L_s/2-1)P_R T^1\Psi(x,L_s/2)\Big].$$

Backup slides

Wilson kernel: $$D_{\mathrm{W}}\psi(x) \equiv \left(\frac{4}{a} - m_5\right)\psi(x) - \frac{1}{2a}\sum_{\mu}\Big[(1-\gamma_\mu)U_\mu(x)\psi(x+\hat{\mu}) + (1+\gamma_\mu)U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu})\Big].$$

The link variables $U_\mu(x)$ and $U_\mu^\dagger(x-\hat{\mu})$ are the same as in the gauge action.

Shamir kernel: $D_{\mathrm{S}}(x,s,y,r) \equiv \delta_{s,r}D_{\mathrm{W}}(x,y) + \delta_{x,y}D_5^{\mathrm{S}}(s,r)$,

$$D_5^{\mathrm{S}}(s,r) \equiv \delta_{s,r} - (1-\delta_{s,N_s-1})\,P_R\delta_{s+1,r} - (1-\delta_{s,0})\,P_L\delta_{s-1,r} + m_0\left(P_R\delta_{s,N_s-1}\delta_{0,r} + P_L\delta_{s,0}\delta_{N_s-1,r}\right).$$

$$D_{\mathrm{S}} = \frac{a_5 D_W}{2 + a_5 D_W}, \qquad a_5 \equiv (b-c)a > 0, \qquad D^{\mathrm{M}} = \alpha D_{\mathrm{S}}, \qquad \alpha = \frac{(b+c)a}{a_5}.$$

Spectrum details

$$C_{\mathrm{PS,PS}}(t) \xrightarrow[t\to\infty]{} \frac{|\langle 0|O_{\mathrm{PS}}|\mathrm{PS}\rangle|^2}{2m_{\mathrm{PS}}} \left(e^{-m_{\mathrm{PS}}t} + e^{-m_{\mathrm{PS}}(T-t)}\right),$$

$$C_{\mathrm{AV,PS}}(t) \xrightarrow[t\to\infty]{} \frac{f_{\mathrm{PS}}\langle 0|O_{\mathrm{PS}}|\mathrm{PS}\rangle^*}{2} \left(e^{-m_{\mathrm{PS}}t} - e^{-m_{\mathrm{PS}}(T-t)}\right),$$

$$C_{\mathrm{V},V}(t) \xrightarrow[t\to\infty]{} \frac{|\langle 0|O_V|V\rangle|^2}{2m_V} \left(e^{-m_V t} + e^{-m_V(T-t)}\right).$$

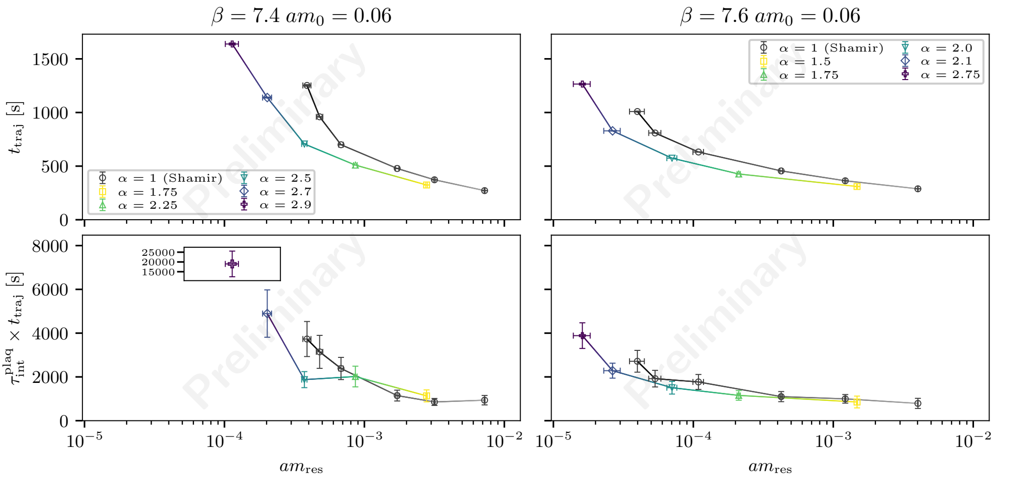
$$f_{\mathrm{PS}}^{\mathrm{ren}} = Z_A f_{\mathrm{PS}}$$

$$Z_A(t) = \frac{1}{2}\left[\frac{R(t+a/2)+R(t-a/2)}{2L(t)} + \frac{2R(t+a/2)}{L(t)+L(t+a)}\right]$$

where

$$R\left(t+\tfrac{1}{2}\right) \equiv \sum_{\vec{x},\vec{y}} \langle \mathcal{A}_0^A(\vec{x},t) J_5^A(\vec{y},0)\rangle, \qquad L(t) \equiv \sum_{\vec{x}} \langle A_0^A(\vec{x},t) J_5^A(\vec{y},0)\rangle.$$

Computational costs



Continuum massless extrapolation

$$X_M^2 = X_{M,\chi}^2 \left(1 + L_X^M m_{\mathrm{PS}}^2 + Q_X^M m_{\mathrm{PS}}^4\right) + W_X^M a + R_X^M a^2 + C_X^M a m_{\mathrm{PS}}^2 \,.$$

$$X_M = m_V,\ f_{\mathrm{PS}} \qquad \text{MDWF:} \qquad W_X^M = C_X^M = 0 \,.$$

- At leading order, $m_{\mathrm{PS}}^2 \propto m_q$, so an analytic expansion in the fermion mass is naturally written as a polynomial in m_{PS}^2.
- The a, a^2, and $a\, m_{\mathrm{PS}}^2$ terms parametrise discretisation effects motivated by the Symanzik expansion.
- The m_{PS}^4 term accounts for the curvature observed at larger masses.

From lattice chiral symmetry to m_{res}

$$\gamma_5 D + D\gamma_5 = a\, D\gamma_5 D \qquad \Longrightarrow \qquad D_{\text{ov}}(m_0) = \frac{1+m_0}{2} + \frac{1-m_0}{2}\gamma_5\, \text{sgn}(H_M)$$

- D_{ov} satisfies the Ginsparg–Wilson relation, realizing exact chiral symmetry on the lattice.
- Möbius domain wall fermions approximate D_{ov} through a finite fifth dimension:

$$\varepsilon_{L_s}(H_M) = \frac{(1+H_M)^{L_s} - (1-H_M)^{L_s}}{(1+H_M)^{L_s} + (1-H_M)^{L_s}} \equiv \frac{1-T(H_M)^{L_s}}{1+T(H_M)^{L_s}} \qquad T(H_M) = \frac{1-H_M}{1+H_M}.$$

For Möbius,

$$\varepsilon_{L_s}(H_M) = \varepsilon_{L_s/\alpha}(\alpha H_M).$$

- At finite L_s, $\varepsilon_{L_s}(H_M) \neq \text{sgn}(H_M)$, so chiral symmetry is only approximate:

$$m_{\text{res}} \simeq \sum_\lambda w_\pi(\lambda)\, \Delta_{L_s}(\lambda) \qquad \Delta_{L_s}(\lambda) = \left[\text{sgn}(H_M) - \varepsilon_{L_s}(H_M)\right](\lambda) \qquad (\alpha = 1)$$

where $w_\pi(\lambda)$ represents the spectral weight associated with a given eigenmode λ of the kernel