

SU(2): Tanizaki, Tomiya, Watanabe, arXiv:2411.14812  
SU(3): this work, paper in preparation

# Gradient-Flow Actions, Integrerness, and Gluonic–Fermionic Topology Matching in SU(3) Yang–Mills Theory

**Akio Tomiya**

Associate Professor, Tokyo Woman's Christian University  
Specially Appointed Assoc. Prof., Kyoto Univ. (BOOST, concurrent)  
Visiting Researcher, RIKEN R-CCS (concurrent)  
Contact: [akio@yukawa.kyoto-u.ac.jp](mailto:akio@yukawa.kyoto-u.ac.jp)



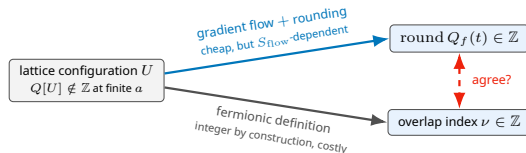
**In collaboration with:**

H. Watanabe<sup>1</sup>, I. Kanamori<sup>2</sup>, O. Morikawa<sup>3</sup>,  
Y. Nagai<sup>4</sup>, Y. Tanizaki<sup>5</sup>

<sup>1</sup> Keio U. <sup>2</sup> RIKEN R-CCS <sup>3</sup> RIKEN iTHEMS <sup>4</sup> U. Tokyo <sup>5</sup> YITP, Kyoto U.

1. Motivation
2. Gradient flow and flow actions
3. Diagnostics and ensembles
4. Topology readout across flow actions
5. Gluonic-fermionic matching
6. Continuum and infinite-volume limits
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## Which gradient flow should we trust for topology?

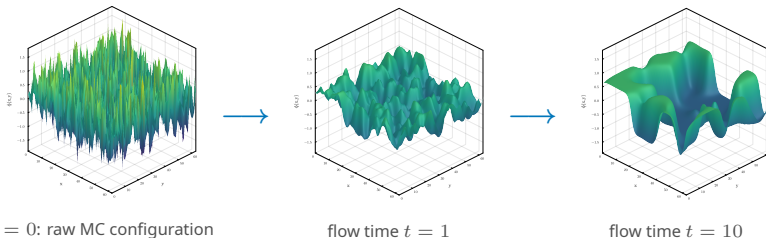


- On the lattice, the topological charge is **not an integer**; sectors are strictly defined only for **admissible** (smooth) fields.
- **Gradient flow + rounding** makes it an integer, but the flow, and hence round  $Q_f(t)$ , **depends on the flow action**, and its **stability along  $t$**  is not automatic.
- The **overlap Dirac index** is an integer **by construction**, but **costly**.

**This talk:** which flow action gives an integer  $Q$  that **agrees with the overlap index**, is **stable** along the flow, and how do the differences approach the **continuum**? (SU(3), four flows, identical ensembles.)

# Gradient flow = smoothing, defined by an action

After the flow, the topological charge approaches an integer



- Gradient flow = steepest descent along the action,  $\partial_t U^{f,t} = -\delta S_{\text{flow}}[U^{f,t}]/\delta U^{f,t}$ , the well-defined successor of historical **cooling** (1980s).
- Acts like a **heat equation**: fluctuations below the growing smearing radius ( $\propto \sqrt{t}$ ) are averaged out, **long-distance structure survives**.
- Observables built from the flowed field are finite (Lüscher–Weisz)  $\Rightarrow Q_f(t)$  is well defined.

**Key point:** gradient flow is smoothing, but the flow is **defined through an action**, so its properties depend on the choice of  $S_{\text{flow}}$ : the four flow actions erase **different** short-distance lumps (quantified next).



# Setup, and what SU(2) taught us

## Flow-action family; the lesson of arXiv:2411.14812

### Lattice gradient flow, one action family

$$\frac{d}{dt}U_{x,\mu}^{f,t} = -\frac{\delta}{\delta U_{x,\mu}^{f,t}}S_{\text{flow}}[U^{f,t}], \quad U^{f,0} = U,$$

$$S_{\text{flow}}[U] = c_0 S_{\text{plaq}}[U] + 2c_1 S_{\text{rect}}[U], \quad c_0 = 1 - 8c_1,$$

$$R = a\sqrt{8t}, \quad t^{(\text{Box})} = \frac{L^2}{32} (2R = La).$$

#### Flow-action coefficients

| flow     | $c_1$     | $1+12c_1$ | note                                |
|----------|-----------|-----------|-------------------------------------|
| Wilson   | 0         | 1.0       | simple (plaquette)                  |
| Symanzik | $-1/12$   | 0.0       | tree-level improved                 |
| Iwasaki  | $-0.331$  | $-2.97$   | RG improved                         |
| DBW2     | $-1.4088$ | $-15.91$  | two-coupling perfect-action approx. |

#### SU(2) findings (Tanizaki–Tomiya–Watanabe):

- single-instanton stability under the flow is classified by  $1+12c_1$ ;
- **negative** values (Iwasaki, DBW2) remove **small dislocations**:  $Q_f(t)$  locks onto an integer **early** and stays there;
- Wilson/Symanzik keep dislocations alive  $\rightarrow$  late, unstable sector readout.

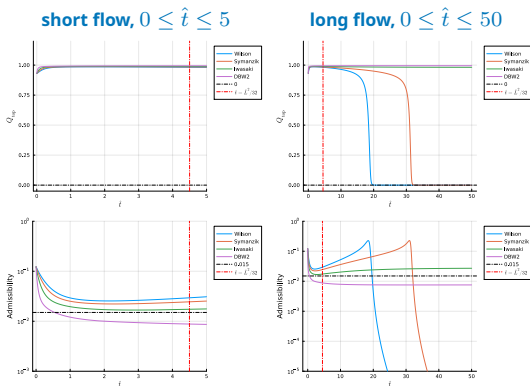
#### Open in SU(2), answered here for SU(3):

does the hierarchy hold against a **fermionic (overlap)** definition, toward the **continuum**, in **larger volumes**?

Charge operator  $Q_{\text{imp}}$  and flow conventions: backup.

# Flows on a single instanton (SU(2) result)

One-instanton configuration with radius  $R = \rho/a = 3.0$  (arXiv:2411.14812, Fig. 2)



Top:  $Q_{\text{top}}(\hat{t})$ ; bottom:  $P_{\text{adm}}(\hat{t})$ , on the same one-instanton configuration.

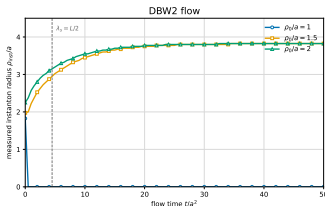
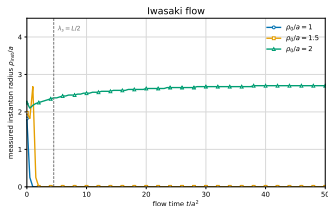
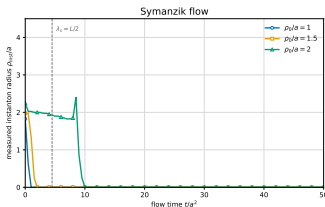
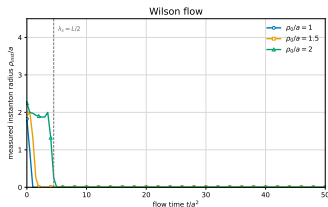
Y. Tanizaki, A. Tomiya, H. Watanabe, arXiv:2411.14812 (SU(2)); instanton construction: Fox et al. (1985).

- Classical **one-instanton** configuration with radius  $R = 3.0$  ( $L = 12$ , SU(2)); read  $Q_{\text{top}}$  and  $P_{\text{adm}}$  under the four flows.
- Short flow: **all four flows** detect  $Q_{\text{top}} \approx 1$ .
- Long flow: **Iwasaki/DBW2 keep the instanton**; **Wilson/Symanzik destroy it** ( $Q_{\text{top}}: 1 \rightarrow 0$  at  $\hat{t} \approx 18.5 / 31$ ).
- At  $R = 1.3$ , **none** of the flows detects the charge.
- A broad prior literature includes **improved cooling**; see the references in arXiv:2411.14812.

**SU(2) benchmark:** "DBW2 detects the smallest instantons with  $R \approx 1.5$ ."  $1+12c_1 < 0$  flows erase only **sub-threshold dislocations**, the origin of the clean sector readout.

# Instanton radius along the flow (SU(2) result)

Measured  $\rho_{\text{inst}}/a$  vs.  $\hat{t} = t/a^2$  for input  $\rho_0/a = 1, 1.5, 2$  on  $12^4$



- $\rho_{\text{inst}} = 0$ : the  $q^2$  profile has **no resolved instanton** in the AIC-selected fit.
- **Wilson/Symanzik**: all three inputs are unresolved by  $\hat{t} \simeq 10$ .
- **Iwasaki**:  $\rho_0/a = 2$  survives; the two smaller inputs disappear.
- **DBW2**:  $\rho_0/a = 1.5, 2$  survive and grow;  $\rho_0/a = 1$  disappears.

The **minimum resolved size** is **flow-action dependent**.

SU(2),  $L = 12$ ;  $\rho_{\text{inst}}$  from AIC-selected  $q^2$  fits. Dashed line:  $\lambda_s = L/2$  at  $\hat{t} = 4.5$ .

# Diagnostics for sector readout

## Integerness, stability, admissibility, susceptibility monitor

$$I_{\text{phase}}(t) = |\langle e^{2\pi i Q_f(t)} \rangle|$$

phase integerness

$$d_{\mathbb{Z}} = \langle |Q_f - \text{round } Q_f| \rangle$$

distance to integers

$$F_0 = \text{zero-jump fraction of round } Q_f(t)$$

sector stability

$$P_{\text{adm}}[V] = \max_{x, \mu < \nu} \left| 1 - \frac{1}{3} \text{ReTr } U_{x, \mu\nu}[V] \right|$$

local roughness

$$\chi^{\text{gluonic}}(t) = \langle Q_f(t)^2 \rangle / (La)^4$$

susceptibility,  $\chi^{1/4}$  in MeV

$F_0$  counts jumps per configuration in the window  $0.5 \leq t \leq t^{(\text{Box})}$ .

### Reading rule

$I_{\text{phase}} \rightarrow 1$  and  $d_{\mathbb{Z}} \rightarrow 0$ : charges concentrate on integers  $\Rightarrow$  **clear sector readout**.

$F_0 \rightarrow 1$ : **stable sector readout** along the flow.

Small  $P_{\text{adm}}$ : locally smooth, near the admissibility regime.

Also monitored: **BPS diagnostic**  $g^2 S_W \geq 8\pi^2 |Q|$ ,  $d_{\mathbb{Z}}$  and  **$F_0$  results** (later); autocorrelations,  $P_{\text{adm}}$  trajectories: backup.

Cross-check: **overlap Dirac index** on ensemble C (I. Kanamori)  $\rightarrow$  gluonic-fermionic matching.

Statistical errors: 10-group block jackknife; window-based  $\tau_{\text{int}}$  (Lüscher, App. E).

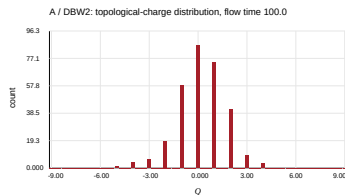
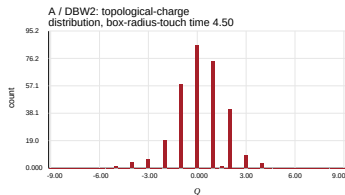
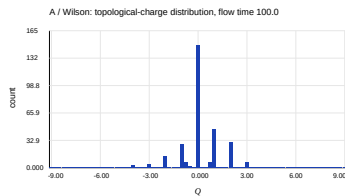
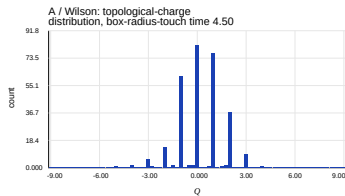
Fixed physical volume A-D; matched- $\beta$  larger volumes LA, LB, LC

| set | $L$ | $\beta$ | $a$ [fm] | $La$ [fm] | configs | role                                                      |
|-----|-----|---------|----------|-----------|---------|-----------------------------------------------------------|
| A   | 12  | 5.95    | 0.102    | 1.224     | 301     | fixed $La \approx 1.22$ fm,<br>$a^2 \rightarrow 0$ series |
| B   | 14  | 6.04    | 0.087    | 1.218     | 301     |                                                           |
| C   | 16  | 6.12    | 0.077    | 1.232     | 301     |                                                           |
| D   | 20  | 6.27    | 0.061    | 1.220     | 301     |                                                           |
| LA  | 16  | 5.95    | 0.102    | 1.632     | 62      | same $\beta$ as A/B/C,<br>larger box                      |
| LB  | 18  | 6.04    | 0.087    | 1.566     | 79      |                                                           |
| LC  | 20  | 6.12    | 0.077    | 1.540     | 177     |                                                           |

- **All ensembles are generated with the Wilson plaquette gauge action** (configurations separated by 20 updates); all four flows are measured on **identical configurations**, so every comparison is a change of the **readout operator** only.
- Statistical errors: 10-group block jackknife with window-based  $\tau_{\text{int}}$ ; the finest lattice D carries long autocorrelations, so it is never used alone (details in backup).

# Same configurations, different readout

$Q$  distributions on A ( $a \approx 0.10$  fm) at  $t^{(\text{Box})}$  and late flow

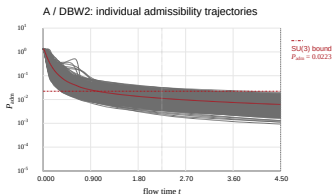
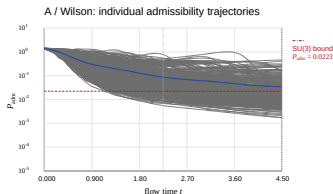
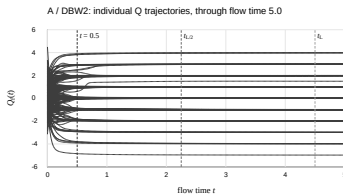
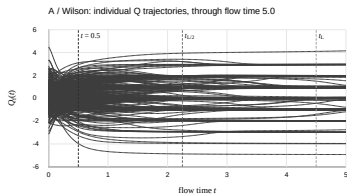


- Rows: **Wilson** (top), **DBW2** (bottom); columns:  $t = t^{(\text{Box})}$ ,  $t = 100$ . Here  $t^{(\text{Box})} = L^2/32$ , with  $R = a\sqrt{8t}$  and  $2R = La$ .
- **Iwasaki/DBW2**: sharp integer comb already at **early flow time**.
- **Wilson**: fractional parts stay broad, and at late  $t$  configurations **pile up at**  $Q = 0$ : the flow **destroys** topological structures.

One Markov chain, four operators: every difference here is a property of the **read-out**, not of the ensemble.

# Q and roughness along the flow

## Individual trajectories on A ( $a \approx 0.10$ fm): Wilson vs. DBW2

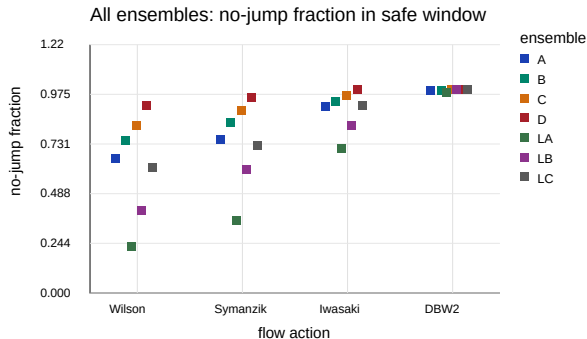


- Top:  $Q_f(t)$  of **each configuration**,  $0 \leq t \leq 5$ ; bottom: roughness  $P_{\text{adm}}(t)$ . Left: **Wilson**, right: **DBW2**.
- **Wilson**:  $Q$  keeps **wandering between sectors**;  $P_{\text{adm}}$  trajectories **bounce back up** (dislocations).
- **DBW2**:  $Q$  **locks onto integers early**,  $P_{\text{adm}}$  drops at once; mean at  $t=1$ : 0.270 vs. **0.023**.

The SU(2) picture (arXiv:2411.14812, Fig. 4) **carries over to SU(3)**. (All four flows: backup.)

# Integer distance $d_{\mathbb{Z}}$ and sector stability $F_0$

## Distance to integers and zero-jump fraction, all ensembles



$$d_{\mathbb{Z}} = \langle |Q_f - \text{round } Q_f| \rangle \text{ at } t = 5$$

| flow     | A     | C     | LA    |
|----------|-------|-------|-------|
| Wilson   | 0.025 | 0.010 | 0.097 |
| Symanzik | 0.020 | 0.009 | 0.046 |
| Iwasaki  | 0.015 | 0.007 | 0.029 |
| DBW2     | 0.008 | 0.003 | 0.014 |

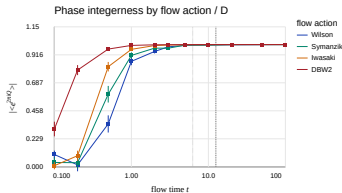
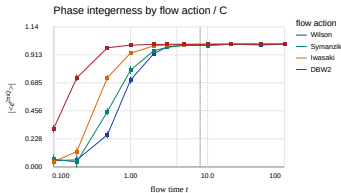
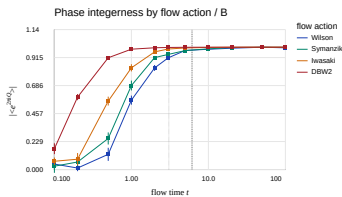
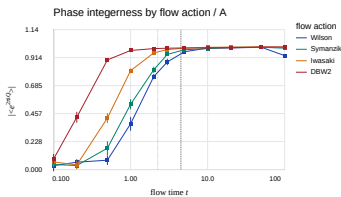
- $F_0$  rises monotonically Wilson  $\rightarrow$  DBW2 in **every ensemble**; A: 0.66  $\rightarrow$  0.99, LA: 0.23  $\rightarrow$  0.98.
- $d_{\mathbb{Z}}$  shows the same ordering; largest in the big box LA, **smallest for DBW2** everywhere.

$F_0$ : fraction of configurations whose round  $Q_f(t)$  **never jumps** in  $0.5 \leq t \leq t^{(\text{Box})}$ .



# Phase integrerness along the flow

DBW2  $\rightarrow$  Iwasaki  $\rightarrow$  Symanzik  $\rightarrow$  Wilson, at every  $a$

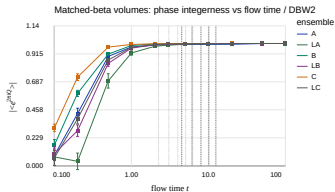
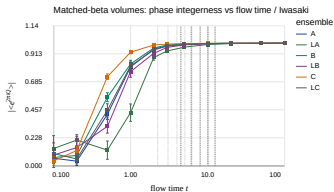
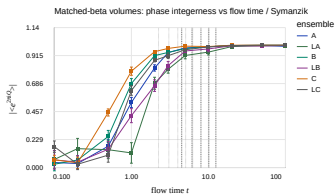
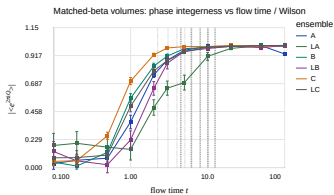


- $I_{\text{phase}}(t) = |\langle e^{2\pi i Q_f(t)} \rangle|$  on A-D ( $a = 0.102 \rightarrow 0.061$  fm); dotted/dashed lines =  $t^{(\text{Box})}/2$ ,  $t^{(\text{Box})}$ .
- Rise order follows the **rectangle coefficient**: DBW2 first, Wilson last, at every  $a$ .
- Coarsest A at  $t=1$ : 0.373(56) (Wilson) vs. **0.972(7)** (DBW2); at  $t=5$ : 0.961 vs. **0.992**.
- Finer lattices: all flows improve, differences **shrink**.

Consistent with SU(2):  $1+12c_1 < 0$  flows  
**lock the sector early.**

# Not a finite-volume touching effect

Nothing special happens when the smearing diameter  $2R$  reaches  $La$

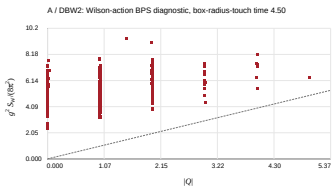
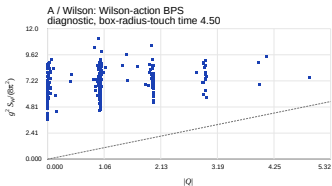


- Matched- $\beta$  pairs A/LA, B/LB, C/LC: same  $a$  (0.102/0.087/0.077 fm), box  $1.22 \rightarrow 1.54$ – $1.63$  fm.
- $I_{\text{phase}}(t) = |\langle e^{2\pi i Q_f(t)} \rangle|$  crosses  $t^{(\text{Box})}/2$  and  $t^{(\text{Box})}$  **smoothly**: no kink or drop at the touching scale.
- Larger box at fixed  $t$ : Wilson readout gets **worse** (LA:  $I_{\text{phase}} = 0.70$ ) while DBW2 stays clean (**0.994**), the gap **widens** with volume.

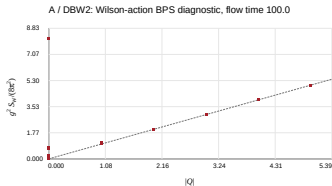
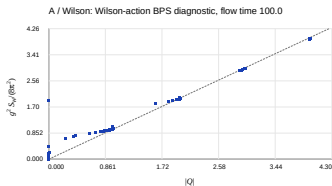
Large- $t$  degradation is a **flow-action property**, **not** a  $2R \gtrsim La$  finite-volume artifact.

$g^2 S_W / 8\pi^2$  vs.  $|Q|$  on A; dashed diagonal = BPS bound

$t = t^{(\text{Box})} = 4.5$



$t = 100$  (late flow)



Classical BPS bound:

$$g^2 S_W \geq 8\pi^2 |Q|,$$

with equality for (anti-)self-dual, i.e. **classical instanton-like** fields.

- At  $t = t^{(\text{Box})}$ : **DBW2**-flowed configurations sit **closest to the bound**; Wilson still far above (rough).
- At  $t = 100$ : all flows approach the diagonal, DBW2 **essentially on it**.

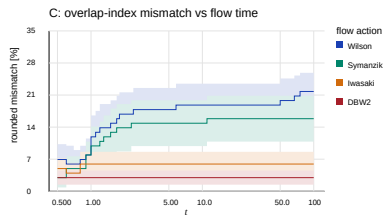
The semiclassical face of the same story:

**DBW2 makes the flowed field classical earliest.**

Top: Wilson; bottom: DBW2. Each point = one configuration.

# Gluonic–fermionic topology matching

round  $Q_f(t)$  vs. overlap Dirac index on ensemble C ( $a \approx 0.08$  fm)



Fraction of configurations with round  $Q_f(t) \neq$  index.

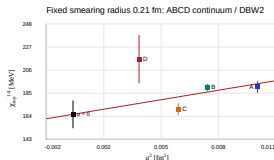
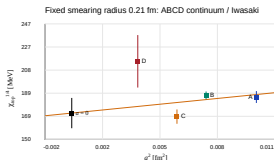
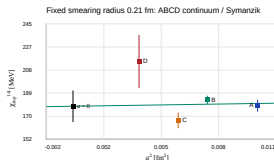
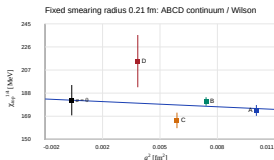
| flow     | disagreement at $t = 5$ |                     |
|----------|-------------------------|---------------------|
| Wilson   | 18 %                    | grows with $t$      |
| Symanzik | 15 %                    | grows with $t$      |
| Iwasaki  | 5.9 %                   | stable              |
| DBW2     | 3.0 %                   | stable from $t=0.5$ |

- Overlap index = independent **fermionic** integer definition of the sector, on the **same configurations**.
- **Iwasaki/DBW2**: agree with the index at the **few-% level** (5.9 %, 3.0 %), **flat in  $t$** .
- **Wilson/Symanzik**:  $\sim 15$ –20 %, and the disagreement **grows** with flow time.

The flows that look integer-like by the **gluonic** diagnostics are the same ones that agree with the **fermionic** sector assignment.

(Errors, correlations, per-configuration checks: backup.)

## $[\chi^{\text{gluonic}}]^{1/4}$ at $R = 0.21 \text{ fm}$ vs. $a^2$

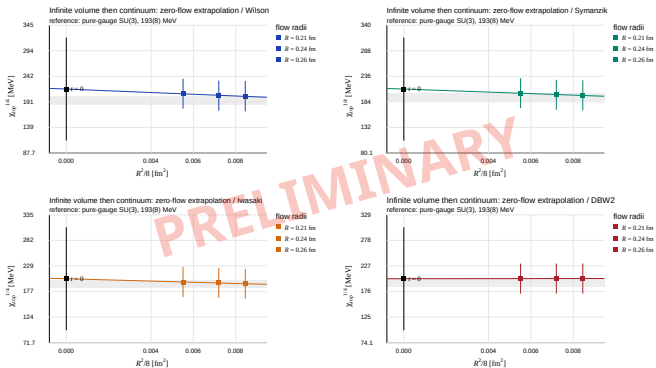


- Fixed  $R = a\sqrt{8t}$  (no smearing-scale mismatch); weighted linear fit in  $a^2$ .
- Spread **shrinks toward**  $a = 0$ : ABCD intercepts  $182/178/172/166 \pm 13 \text{ MeV}$  (W/S/I/D).
- C** ( $a=0.077 \text{ fm}$ ) **sits low**, **D** ( $a=0.061 \text{ fm}$ ) **noisy**: ABD gives  $198\text{--}208 \text{ MeV}$ ,  $\chi^2/\text{dof} \approx 1$ .

All four sets ABC/BCD/ABD/ABCD reported (backup): the spread is **C/D sensitivity**, not statistics. **Precision**  $\chi_{\text{top}}$  **is not the goal here**:  $\chi^{1/4}$  is a **monitor** for comparing flows.

# Infinite volume, continuum, and $t \rightarrow 0$

## All four flows agree at every extrapolation stage



Per flow: two-volume  $V^{-1} \rightarrow 0$  at each  $\beta$ , continuum  $a^2 \rightarrow 0$  at fixed  $R = 0.21, 0.24, 0.26$  fm, then linear in  $R^2/8$  down to  $t = 0$  (leftmost points).

Reference: Dür, Fodor, Hoelbling, Kurth, JHEP 04 (2007) 055; Del Debbio, Giusti, Pica, PRL 94 (2005) 032003.

- Continuum at  $R = 0.24$  fm:  
 $204/199/195/202 \pm 30$  MeV (W/S/I/D),  
 $\chi^2/\text{dof} \approx 0.02$ ; all on Dür et al.  
 $193(1)(8)$  MeV (fermionic  $191(5)$  MeV).
- $t \rightarrow 0$ :  
 $\chi_0^{1/4} = 217/211/204/202 \pm \sim 100$  MeV,  
**flow-independent within errors**  
 (preliminary).

**Key consequence:** for **every** flow action,  $V \rightarrow \infty$ ,  $a \rightarrow 0$ ,  $t \rightarrow 0$  give a **consistent** susceptibility, the flow dependence is purely a finite- $a$  readout effect. The  $\chi_{top}$  values are a **sanity check** against known results, not a precision determination.

## Iwasaki/DBW2 flows give the cleanest SU(3) topology readout

- **Setup:** four flows on identical Wilson-action SU(3) ensembles; at finite  $a$ , flow choice = **choice of the charge operator**.
- **Readout:** **Iwasaki/DBW2** reach integers earlier, keep sectors stable ( $F_0 \rightarrow 1$ ), stay admissible, and **match the overlap index** (few-%); Wilson/Symanzik degrade at large  $t$ .
- **Volume:** no feature at  $2R = La$ , the large- $t$  behavior is a **flow-action property**, not box touching.
- **Practical:** **DBW2 reads the sector already at  $t \approx 1$**  on the coarsest lattice ( $a \approx 0.1$  fm)  $\rightarrow$  fewer flow steps.
- **Limits:**  $V \rightarrow \infty, a \rightarrow 0, t \rightarrow 0$  **consistent for all four flows**, and with  $\chi_{\text{top}}^{1/4} \approx 193$  MeV (precision  $\chi_{\text{top}}$  is **not the goal here**).

### Take-away

Rounding a gluonic  $Q(t)$  to name a sector: **the flow action matters**. Iwasaki/DBW2 give an integer-like, volume-robust, fermion-consistent readout at moderate flow time.

**Outlook:** longer D statistics · third volume per  $\beta$  · large box at the finest  $a$  · **energy-density measurements**.

SU(2): arXiv:2411.14812. SU(3) in preparation:  
Tomiya, Watanabe, Kanamori, Morikawa, Nagai,  
Tanizaki.

**Thank you!**





## Errors, correlations, and per-configuration checks

| flow     | $t = 5$   |           | $t = 100$ |           |
|----------|-----------|-----------|-----------|-----------|
|          | mismatch  | $r$       | mismatch  | $r$       |
| Wilson   | 18(5) %   | 0.925(21) | 22(4) %   | 0.906(22) |
| Symanzik | 15(5) %   | 0.940(21) | 16(5) %   | 0.936(22) |
| Iwasaki  | 5.9(27) % | 0.963(22) | 5.9(27) % | 0.963(22) |
| DBW2     | 3.0(15) % | 0.96(4)   | 3.0(15) % | 0.96(4)   |

Single-configuration effect: conf 2680 ( $\nu = 0$ , rounded charge  $-3$ ) carries 9/11 of DBW2's squared sector residual; excluding it diagnostically,  $r$ : Iwasaki 0.963  $\rightarrow$  0.979, DBW2 0.957  $\rightarrow$  0.992.

- $n = 101$  configurations of C; sign convention  $+\nu$ ; 10-block jackknife errors with common delete groups.
- Mismatch **stable** from  $t=1$  (Iwasaki) and  $t=0.5$  (DBW2); **Wilson/Symanzik grow** with flow time.
- Iwasaki vs. DBW2, paired jackknife: mismatch 3.0(21) %pt,  $r$  0.007(18), **indistinguishable** at  $n = 101$ .

Sector comparison only; no fermionic renormalization is transferred to the gluonic susceptibility.

# Backup: what limits the current precision

## Honest error budget before quoting final numbers

- **D (finest  $a$ ):**  $\tau_{\text{int}} \simeq 25 \Rightarrow N_{\text{eff}} \simeq 6$  of 301; jackknife block length  $< 2\tau_{\text{int}}$ , so D-including fits may **underestimate errors**.
- **C:** raw flow data independently re-verified (**no corruption**, max. deviation  $4 \times 10^{-13}$ ); window/block scans suggest reading its error  **$\times 1.3$ – $1.5$**  larger; C still sits low at  $1.9$ – $2.4 \sigma$  vs. ABD, cause not identified.
- **Volumes:** two boxes per  $\beta$  only, so the  $V^{-1}$  functional form is untested; no large box at the finest  $a$ .

### Error policy

- common fixed 10-block jackknife in all figures;
- conservative C factors and D warnings quoted **alongside**, never silently absorbed;
- ABC/BCD/ABD/ABCD all reported, none promoted to a final value;
- $V \rightarrow \infty$  and  $t \rightarrow 0$  labeled **preliminary** (linear forms, no covariance across  $R$ ).

None of these caveats touches the **qualitative** flow-action ordering: it is visible **ensemble by ensemble**, before any extrapolation.

# Backup: charge operator and conventions

## Clover-rectangle improved $Q$ ; flow integration

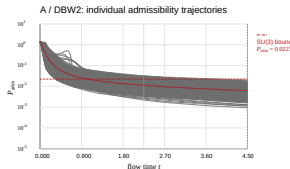
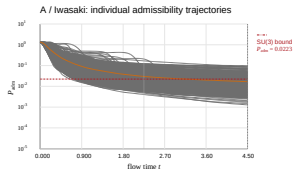
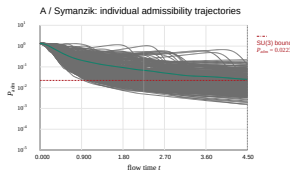
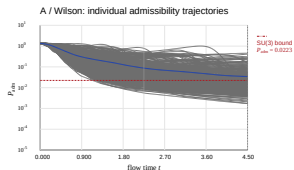
- Measured charge: clover-rectangle improved combination

$$Q_{\text{imp}}[V] = \frac{5}{3} Q_{\text{clov}}[V] - \frac{1}{12} (2Q_{\text{rect}}[V]), \quad Q_f(t) = Q_{\text{imp}}[U^{f,t}],$$

where  $Q_X$  uses the traceless anti-Hermitian part of the corresponding loop sum ( $n_{\text{clov}} = 4, n_{\text{rect}} = 8$ ).

- Changing the flow action at finite  $a$  is a change of the **topological-charge operator**,  $Q[U] \rightarrow Q[U^{f,t}]$ ; the generated Markov chain is untouched.
- Flow integrated to  $t = 100$  with  $dt = 0.01$  (10001 rows per configuration and flow);  $t^{(\text{Box})}/2$  (dotted) and  $t^{(\text{Box})}$  (dashed) marked in all  $t$ -dependence plots.
- LB: 79 adopted configurations; all four flow actions are measured on the same configuration labels.

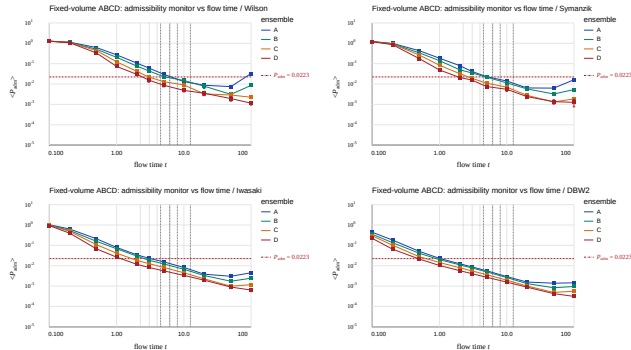
## Per-configuration roughness on A ( $a \approx 0.10$ fm)



Gray:  $P_{\text{adm}}(t)$  of each configuration up to  $t^{(\text{Box})}$ ; colored: mean. Wilson/Symanzik trajectories keep bouncing back up (dislocations re-roughen the field); Iwasaki suppresses them; DBW2 shows almost no revivals.

# Backup: admissibility monitor on A-D

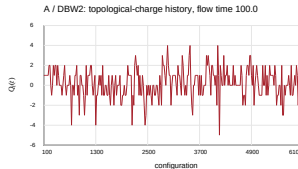
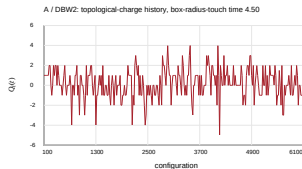
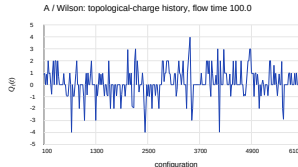
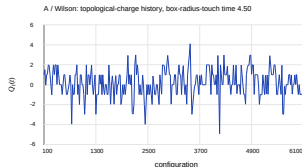
Ensemble means,  $a = 0.102 \rightarrow 0.061$  fm



Mean  $P_{\text{adm}}$  on A-D; all panels use the same logarithmic vertical range,  $10^{-5} \leq P_{\text{adm}} \leq 10^1$ . Max  $P_{\text{adm}}$  at  $t=5$  on A: 0.64 (Wilson) vs. 0.051 (DBW2). Same flow ordering as the integerness diagnostics at every lattice spacing.

# Backup: $Q$ histories on A ( $a \approx 0.10$ fm)

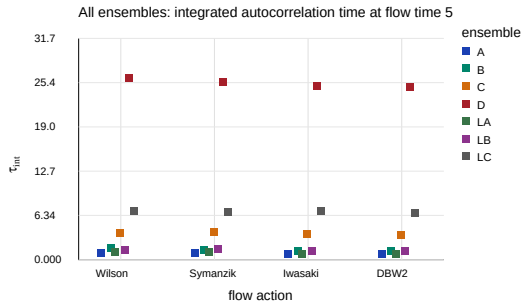
## Configuration-by-configuration readout at three flow times



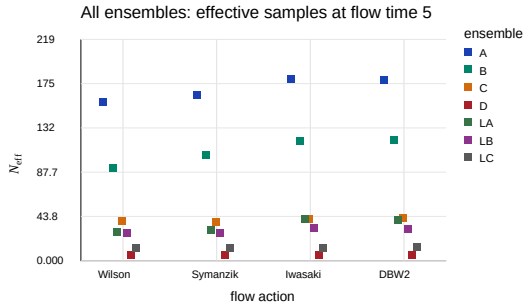
Rows: Wilson (top), DBW2 (bottom); columns:  $t = 4.5$  ( $t^{\text{Box}}$ ) and  $t = 100$ . Same Markov chain in every panel; other flows and times in the full report.

# Backup: autocorrelation at $t = 5$

Why D (finest,  $a = 0.061$  fm) is the statistical bottleneck

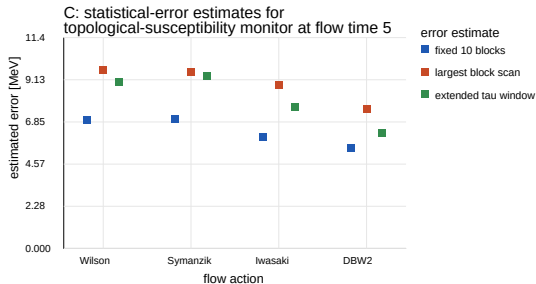


$\tau_{\text{int}}$  of  $Q_f(t=5)$ :  $D \simeq 25$ ,  $LC \simeq 7$ ,  $C \simeq 3.8$ .



Effective sample counts; D drops to  $N_{\text{eff}} \simeq 6$ .

## Fixed 10-block vs. block-width scan vs. extended window

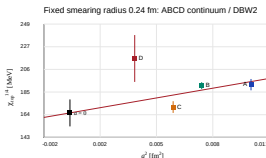
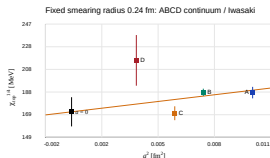
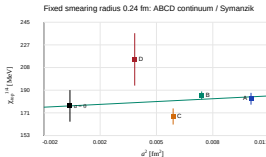
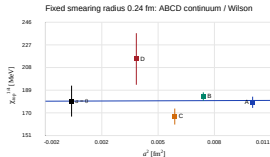


Max/fixed error ratio 1.36–1.46 (block-width scan) and 1.14–1.33 (window extension). With these factors the C–ABD tension drops from  $2.3\text{--}2.9\sigma$  to  $1.9\text{--}2.4\sigma$ ; the low central value of C remains. Short-time DBW2 raw data independently re-verified: max. abs. deviation  $3.6 \times 10^{-13}$  over all 301 configurations.



# Backup: continuum at fixed $R = 0.24$ fm

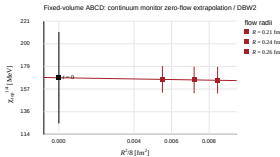
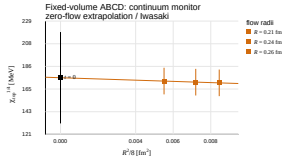
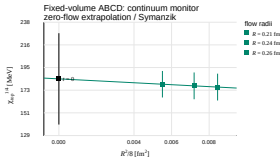
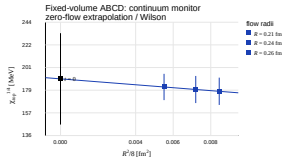
$[\chi^{\text{gluonic}}]^{1/4}$  at  $R = 0.24$  fm vs.  $a^2$



ABCD intercepts at  $R = 0.24$  fm:  $180/177/171/166 \pm 13$  MeV (Wilson/Symanzik/Iwasaki/DBW2); dropping C (ABD) gives 197–205 MeV with  $\chi^2/\text{dof} \approx 1$ . Same pattern as  $R = 0.21$  fm in the main part.

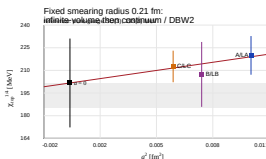
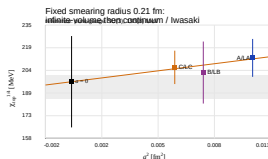
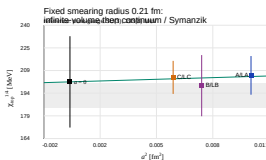
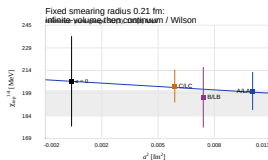
# Backup: $\chi_{\text{top}}^{1/4}$ vs. $R$ (fixed volume)

Continuum intercepts at each  $R$ , extrapolated linearly to  $t = 0$



Fixed-volume ABCD intercepts at  $R = 0.21, 0.24, 0.26$  fm vs.  $R^2/8$ ; leftmost point =  $t \rightarrow 0$ : 190/184/175/168  $\pm \sim 44$  MeV (Wilson/Symanzik/Iwasaki/DBW2), preliminary (three-radius linear form).

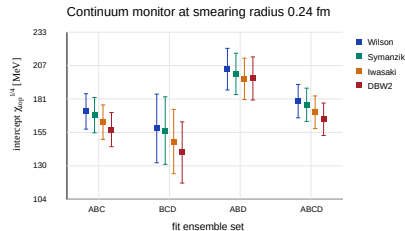
$A/LA, B/LB, C/LC \rightarrow V^{-1} \rightarrow 0$  per  $\beta$ , then  $a^2 \rightarrow 0$



Two-volume extrapolation at each  $\beta$ , then continuum at fixed  $R = 0.21$  fm: intercepts 208/203/197/202  $\pm$  30 MeV (Wilson/Symanzik/Iwasaki/DBW2),  $\chi^2/\text{dof} \approx 0.02$ . Consistent with the  $R = 0.24$  fm result and with Dürr et al. 193(1)(8) MeV.

# Backup: continuum-set comparison

## ABC / BCD / ABD / ABCD intercepts at $R = 0.24$ fm

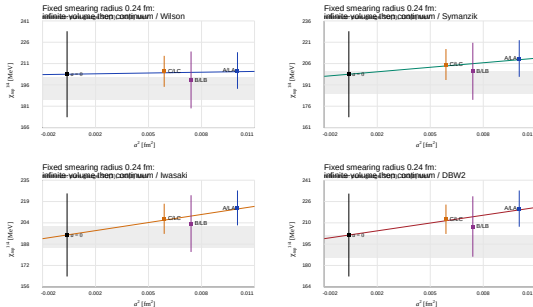


| flow     | ABC         | BCD         | ABD         | ABCD        |
|----------|-------------|-------------|-------------|-------------|
| Wilson   | 171.7(13.7) | 158.5(26.6) | 204.5(16.1) | 179.7(12.9) |
| Symanzik | 168.9(13.7) | 156.9(26.2) | 200.8(16.0) | 176.9(12.9) |
| Iwasaki  | 163.4(13.3) | 148.4(25.0) | 197.0(16.0) | 171.1(12.7) |
| DBW2     | 157.7(13.2) | 139.9(23.7) | 197.3(16.7) | 165.7(12.6) |

Intercepts in MeV. ABD:  $\chi^2/\text{dof} \approx 1.1\text{--}1.3$ ; C-including sets 3.8–7.9 (C/D sensitivity).

# Backup: infinite volume, then continuum

Two-volume extrapolation at each  $\beta$ , then  $a^2 \rightarrow 0$  at  $R = 0.24$  fm



A/LA, B/LB, C/LC  $\rightarrow V^{-1} \rightarrow 0$  per  $\beta$ , then continuum: intercepts 204/199/195/202  $\pm$  30 MeV (Wilson/Symanzik/Iwasaki/DBW2),  $\chi^2/\text{dof} \approx 0.02$ . Gray band: Dürre et al. quenched SU(3)  $\chi_{\text{top}}^{1/4} = 193(1)(8)$  MeV. Fixed-volume ABCD  $t \rightarrow 0$  side: 190(44)-168(43) MeV.

## Rounded-sector mismatch and Pearson correlation between flows

| set | W-DBW2 mismatch | $r(W, Iwa)$ | $r(W, DBW2)$ | $r(Iwa, DBW2)$ |
|-----|-----------------|-------------|--------------|----------------|
| A   | 0.309           | 0.934       | 0.909        | 0.969          |
| B   | 0.236           | 0.954       | 0.934        | 0.979          |
| C   | 0.143           | 0.962       | 0.942        | 0.984          |
| D   | 0.056           | 0.992       | 0.989        | 0.997          |
| LA  | 0.500           | 0.973       | 0.948        | 0.979          |
| LB  | 0.434           | 0.959       | 0.953        | 0.987          |
| LC  | 0.260           | 0.981       | 0.968        | 0.986          |

Rounded-sector mismatch decreases with finer  $a$  ( $A \rightarrow D$ ,  $0.102 \rightarrow 0.061$  fm) and increases with volume at fixed  $a$  ( $A \rightarrow LA$ ), the same pattern as the integerness diagnostics. The high Pearson  $r$  shows the charge *values* stay linearly related even when rounded sectors differ.

# Backup: why fixed $R$ , and renormalization

## No $Z(\beta)$ , $M(\beta)$ needed at positive flow time

- The unflowed lattice susceptibility needs multiplicative and additive renormalization,  $\chi_L = Z(\beta)^2 a^4 \chi + M(\beta)$  (D'Elia 2003).
- At positive physical smearing radius  $R > 0$ , gauge-invariant flowed fields need no extra renormalization (Lüscher-Weisz 2011); the contact-term region is smoothed on the scale  $R$ .
- Hence all cross-lattice comparisons are made at **fixed**  $R = a\sqrt{8t} = 0.21, 0.24, 0.26$  fm, and  $t \rightarrow 0$  is taken only *after* the continuum limit, linearly in  $R^2/8$ .
- Dürer et al. instead fix a nonperturbative  $Z$  from the integer-comb positions of a HYP-smeared charge; their 193(1)(8) MeV is used as an external reference here.