

SU(2): Tanizaki, Tomiya, Watanabe, arXiv:2411.14812
SU(3): this work, paper in preparation

Gradient-Flow Actions, Integrerness, and Gluonic–Fermionic Topology Matching in SU(3) Yang–Mills Theory

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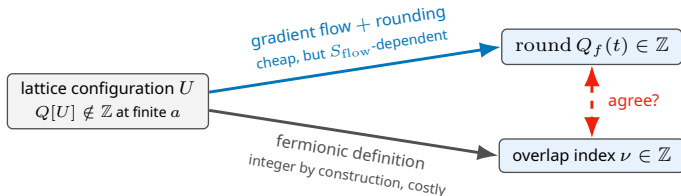
In collaboration with:

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1. Motivation
2. Gradient flow and flow actions
3. Diagnostics and ensembles
4. Topology readout across flow actions
5. Gluonic–fermionic matching
6. Continuum and infinite-volume limits
7. Summary

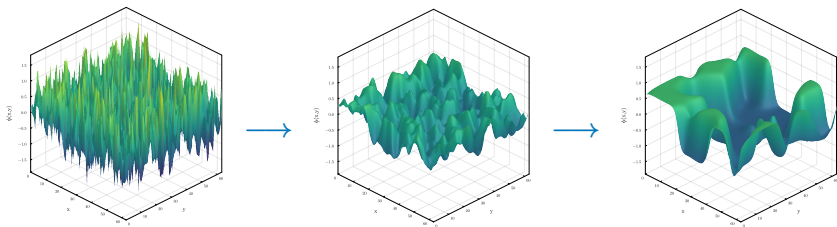
Which gradient flow should we trust for topology?



- On the lattice, the topological charge is **not an integer**; sectors are strictly defined only for **admissible** (smooth) fields.
- Gradient flow + rounding** makes it an integer, but the flow, and hence round $Q_f(t)$, **depends on the flow action**, and its **stability along t** is not automatic.
- The **overlap Dirac index** is an integer **by construction**, but **costly**.

This talk: which flow action gives an integer Q that **agrees with the overlap index**, is **stable** along the flow, and how do the differences approach the **continuum**? (SU(3), four flows, identical ensembles.)

After the flow, the topological charge approaches an integer



$t = 0$: raw MC configuration

flow time $t = 1$

flow time $t = 10$

- Gradient flow = steepest descent along the action, $\partial_t U^{f,t} = -\delta S_{\text{flow}}[U^{f,t}]/\delta U^{f,t}$.
- Acts like a **heat equation**: fluctuations below the growing smearing radius ($\propto \sqrt{t}$) are averaged out, **long-distance structure survives**.
- Observables built from the flowed field are finite (Lüscher-Weisz) $\Rightarrow Q_f(t)$ is well defined.

Key point: gradient flow is smoothing, but the flow is **defined through an action**, so its properties depend on the choice of S_{flow} : the four flow actions erase **different** short-distance lumps (quantified next).

Setup, and what SU(2) taught us

Flow-action family; the lesson of arXiv:2411.14812

Lattice gradient flow, one action family

$$\frac{d}{dt} U_{x,\mu}^{f,t} = -\frac{\delta}{\delta U_{x,\mu}^{f,t}} S_{\text{flow}}[U^{f,t}], \quad U^{f,0} = U,$$

$$S_{\text{flow}}[U] = c_0 S_{\text{plaq}}[U] + 2c_1 S_{\text{rect}}[U], \quad c_0 = 1 - 8c_1,$$

$$R = a\sqrt{8t}, \quad t^{(\text{Box})} = \frac{L^2}{32} (2R = La).$$

Flow-action coefficients

flow	c_0	c_1	$1+12c_1$
Wilson	1	0	1.0
Symanzik	5/3	-1/12	0.0
Iwasaki	3.648	-0.331	-2.97
DBW2	12.270	-1.4088	-15.91

SU(2) findings (Tanizaki–

Tomiya–Watanabe):

- single-instanton stability under the flow is classified by **1+12c₁**;
- **negative** values (Iwasaki, DBW2) remove **small dislocations**: $Q_f(t)$ locks onto an integer **early** and stays there;
- Wilson/Symanzik keep dislocations alive → late, unstable sector readout.

Open in SU(2), answered here for SU(3):

does the hierarchy hold against a **fermionic (overlap)** definition, toward the **continuum**, in **larger volumes**?

Charge operator Q_{imp} and flow conventions: backup.

Diagnostics for sector readout

A. Tomiya

Integerness, stability, admissibility, susceptibility monitor

$$I_{\text{phase}}(t) = |\langle e^{2\pi i Q_f(t)} \rangle|$$

phase integerness

$$d_{\mathbb{Z}} = \langle |Q_f - \text{round } Q_f| \rangle$$

distance to integers

$$F_0 = \text{zero-jump fraction of round } Q_f(t)$$

sector stability

$$P_{\text{adm}}[V] = \max_{x, \mu < \nu} \left| 1 - \frac{1}{3} \text{ReTr } U_{x, \mu \nu}[V] \right|$$

local roughness

$$\chi^{\text{gluonic}}(t) = \langle Q_f(t)^2 \rangle / (La)^4$$

susceptibility, $\chi^{1/4}$ in MeV

$$F_0 \text{ counts jumps per configuration in the window } 0.5 \leq t \leq \min(100, t^{(\text{Box})}).$$

Reading rule

$I_{\text{phase}} \rightarrow 1$ and $d_{\mathbb{Z}} \rightarrow 0$: charges concentrate on integers $\Rightarrow$ **clear sector readout**.

$F_0 \rightarrow 1$: readout is **stable along the flow**, not an accident of one chosen t .

Small P_{adm} : locally smooth field, close to the admissibility regime where sectors are well defined.

Also monitored: **BPS diagnostic** $g^2 S_W \geq 8\pi^2 |Q|$ (later); autocorrelations, P_{adm} trajectories, $d_{\mathbb{Z}}$ and F_0 results: backup.

Cross-check: **overlap Dirac index** on ensemble C (I. Kanamori) $\rightarrow$ gluonic-fermionic matching.

Statistical errors: 10-group block jackknife; window-based τ_{int} (Lüscher, App. E). Gap-aware blocks for LB.

Fixed physical volume A-D; matched- β larger volumes LA, LB, LC

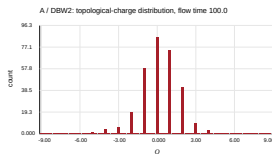
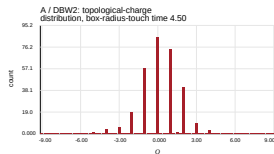
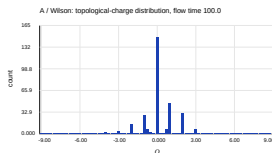
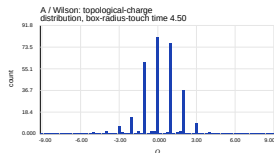
set	L	β	a [fm]	La [fm]	configs	role
A	12	5.95	0.102	1.224	301	fixed $La \approx 1.22$ fm, $a^2 \rightarrow 0$ series
B	14	6.04	0.087	1.218	301	
C	16	6.12	0.077	1.232	301	
D	20	6.27	0.061	1.220	301	
LA	16	5.95	0.102	1.632	62	same β as A/B/C, larger box
LB	18	6.04	0.087	1.566	76	
LC	20	6.12	0.077	1.540	177	

- Wilson plaquette ensembles, configurations separated by 20 updates; all four flows are measured on **identical configurations**, so every comparison is a change of the **readout operator** only.
- Statistical errors: 10-group block jackknife with window-based τ_{int} ; the finest lattice D carries long autocorrelations, so it is never used alone (details in backup).

Same configurations, different readout

A. Tomiya

Q distributions on A ($a \approx 0.10$ fm) at $t^{(\text{Box})}$ and late flow



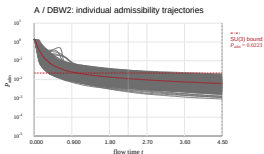
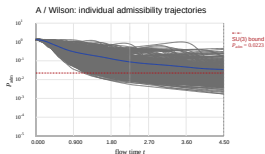
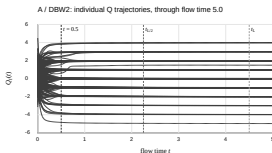
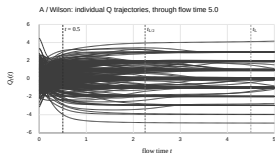
- Rows: **Wilson** (top), **DBW2** (bottom); columns: $t = t^{(\text{Box})}$, $t = 100$.
- **Iwasaki/DBW2**: sharp integer comb already at **early flow time**.
- **Wilson**: fractional parts stay broad, and at late t configurations **pile up at $Q = 0$** : the flow **destroys** topological structures.

One Markov chain, four operators: every difference here is a property of the **readout**, not of the ensemble.

Q and roughness along the flow

A. Tomiya

Individual trajectories on A ($a \approx 0.10$ fm): Wilson vs. DBW2



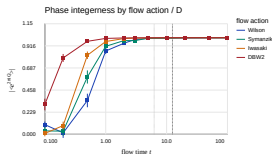
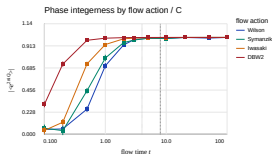
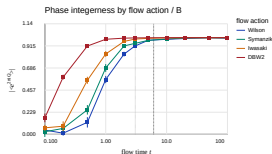
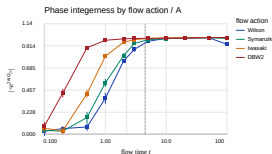
- Top: $Q_f(t)$ of **each configuration**, $0 \leq t \leq 5$;
bottom: roughness $P_{\text{adm}}(t)$.
Left: **Wilson**, right: **DBW2**.
- **Wilson**: Q keeps **wandering between sectors**; P_{adm} trajectories **bounce back up** (dislocations).
- **DBW2**: Q **locks onto integers early**, P_{adm} drops at once; mean at $t=1$: 0.270 vs. **0.023**.

The SU(2) picture
(arXiv:2411.14812, Fig. 4) **carries over to SU(3)**. (All four flows: backup.)

Phase integerness along the flow

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DBW2 $\rightarrow$ Iwasaki $\rightarrow$ Symanzik $\rightarrow$ Wilson, at every a

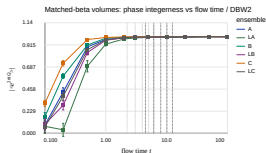
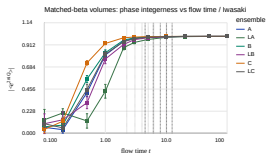
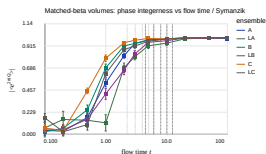
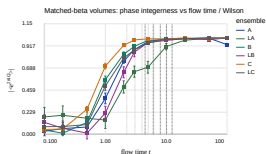


- $I_{\text{phase}}(t)$ on A-D ($a = 0.102 \rightarrow 0.061$ fm); dotted/dashed lines = $t^{(\text{Box})}/2, t^{(\text{Box})}$.
- Rise order follows the **rectangle coefficient**: DBW2 first, Wilson last, at every a .
- Coarsest A at $t=1$: 0.373(56) (Wilson) vs. **0.972(7)** (DBW2); at $t=5$: 0.961 vs. **0.992**.
- Finer lattices: all flows improve, differences **shrink**.

Consistent with SU(2): $1+12c_1 < 0$ flows **lock the sector early**.

Not a finite-volume touching effect

Nothing special happens when the smearing diameter $2R$ reaches La



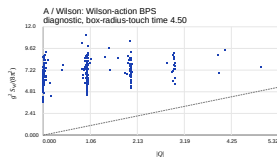
- Matched- β pairs A/LA, B/LB, C/LC: same a (0.102/0.087/0.077 fm), box $1.22 \rightarrow 1.54$ – 1.63 fm.
- $I_{\text{phase}}(t)$ crosses $t^{(\text{Box})}/2$ and $t^{(\text{Box})}$ **smoothly**: no kink or drop at the touching scale.
- Larger box at fixed t : Wilson readout gets **worse** (LA: $I_{\text{phase}} = 0.70$) while DBW2 stays clean (**0.994**), the gap **widens** with volume.

Large- t degradation is a **flow-action property**, **not** a $2R \gtrsim La$ finite-volume artifact.

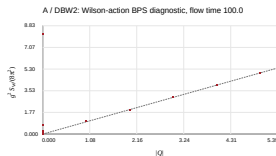
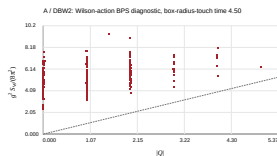
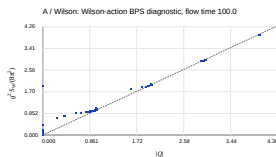
BPS diagnostic: how classical is the flowed field?

$g^2 S_W / 8\pi^2$ vs. $|Q|$ on A; dashed diagonal = BPS bound

$t = t^{(\text{Box})} = 4.5$



$t = 100$ (late flow)



Top: Wilson; bottom: DBW2. Each point = one configuration.

Classical BPS bound:

$$g^2 S_W \geq 8\pi^2 |Q|,$$

with equality for (anti)-self-dual, i.e. **classical instanton-like** fields.

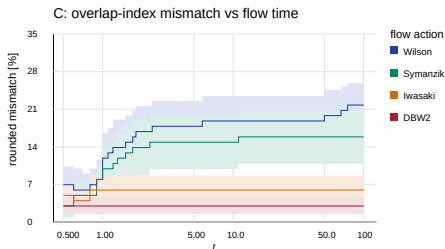
- At $t = t^{(\text{Box})}$: **DBW2**-flowed configurations sit **closest to the bound**; Wilson still far above (rough).
- At $t = 100$: all flows approach the diagonal, DBW2 **essentially on it**.

The semiclassical face of the same story: **DBW2 makes the flowed field classical earliest**.

Gluonic-fermionic topology matching

A. Tomiya

round $Q_f(t)$ vs. overlap Dirac index on ensemble C ($a \approx 0.08$ fm)



Fraction of configurations with round $Q_f(t) \neq$ index.

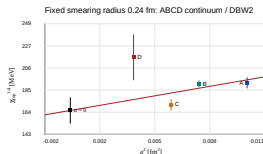
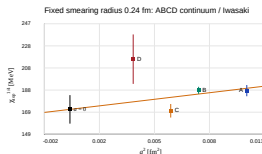
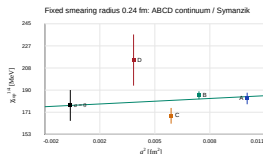
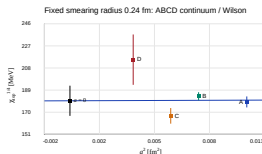
flow	disagreement at $t = 5$	
Wilson	18 %	grows with t
Symanzik	15 %	grows with t
Iwasaki	5.9 %	stable
DBW2	3.0 %	stable from $t=0.5$

- Overlap index = independent **fermionic** integer definition of the sector, on the **same configurations**.
- **Iwasaki/DBW2**: agree with the index at the **few-% level** (5.9 %, 3.0 %), **flat in t** .
- **Wilson/Symanzik**: ~ 15 –20 %, and the disagreement **grows** with flow time.

The flows that look integer-like by the **gluonic** diagnostics are the same ones that agree with the **fermionic** sector assignment.

(Errors, correlations, per-configuration checks: backup.)

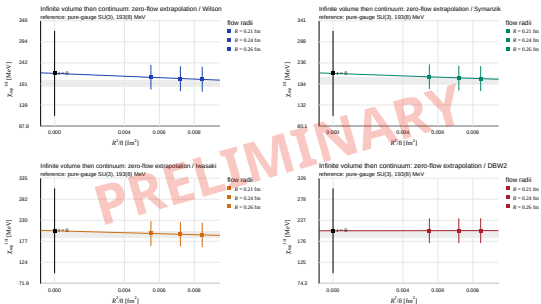
$[\chi^{\text{gluonic}}]^{1/4}$ at $R = 0.24 \text{ fm}$ vs. a^2



- Fixed $R = a\sqrt{8t}$ removes the smearing-scale mismatch; weighted linear fit in a^2 .
- Flow-action spread **shrinks toward** $a = 0$: ABCD intercepts
180/177/171/166 $\pm$ 13 MeV (W/S/I/D).
- C** ($a=0.077 \text{ fm}$) **sits low**, **D** ($a=0.061 \text{ fm}$) **has large bars**: dropping C (ABD) gives 197–205 MeV, $\chi^2/\text{dof} \approx 1$.

All four sets ABC/BCD/ABD/ABCD reported (backup): the spread is **C/D sensitivity**, not statistics.

All four flows agree at every extrapolation stage



Per flow: two-volume $V^{-1} \rightarrow 0$ at each β , continuum $a^2 \rightarrow 0$ at fixed $R = 0.21, 0.24, 0.26$ fm, then linear in $R^2/8$ down to $t = 0$ (leftmost points).

- Continuum at $R = 0.24$ fm:
204/199/195/202 ± 30 MeV
(W/S/I/D), $\chi^2/\text{dof} \approx 0.02$; all on
Dürr et al. 193(1)(8) MeV
(fermionic 191(5) MeV).
- $t \rightarrow 0$: $\chi_0^{1/4} =$
217/211/204/202 $\pm \sim 100$
MeV, **flow-independent within**
errors (preliminary).

Key consequence: for **every** flow action, $V \rightarrow \infty$, $a \rightarrow 0$, $t \rightarrow 0$ give a **consistent** susceptibility, the flow dependence is purely a finite- a readout effect.

Iwasaki/DBW2 flows give the cleanest SU(3) topology readout

- Four flows on identical Wilson-action SU(3) ensembles: at finite a , flow choice = **choice of the charge operator**.
- **Iwasaki/DBW2**: integers earlier, stable sectors ($F_0 \rightarrow 1$), admissible, and **match the overlap index** (few-%); Wilson/Symanzik degrade at large t .
- No feature at $2R = La$: the large- t behavior is a **flow-action property**, not box touching.
- **DBW2 reads the sector already at $t \approx 1$** on the coarsest lattice ($a \approx 0.1$ fm) $\rightarrow$ fewer flow steps (rectangle-term cost: benchmark pending).
- $V \rightarrow \infty$, $a \rightarrow 0$, $t \rightarrow 0$: **consistent for all four flows**, and with $\chi_{\text{top}}^{1/4} \approx 193$ MeV.

Take-away

If you round a gluonic $Q(t)$ to name a sector, **the flow action matters**: Iwasaki/DBW2 give an integer-like, volume-robust, fermion-consistent readout already at moderate flow time.

SU(2): arXiv:2411.14812. SU(3): paper in preparation by Tomiya, Watanabe, Kanamori, Morikawa, Nagai, Tanizaki.

Outlook: longer D statistics; third volume per β ; large box at the finest a ; resolve the low-C tension.

Thank you!



Errors, correlations, and per-configuration checks

flow	$t = 5$		$t = 100$	
	mismatch	r	mismatch	r
Wilson	18(5) %	0.925(21)	22(4) %	0.906(22)
Symanzik	15(5) %	0.940(21)	16(5) %	0.936(22)
Iwasaki	5.9(27) %	0.963(22)	5.9(27) %	0.963(22)
DBW2	3.0(15) %	0.96(4)	3.0(15) %	0.96(4)

Single-configuration effect: conf 2680 (index 0, rounded charge -3) carries 9/11 of DBW2's squared sector residual; excluding it diagnostically, r : Iwasaki 0.963 $\rightarrow$ 0.979, DBW2 0.957 $\rightarrow$ 0.992.

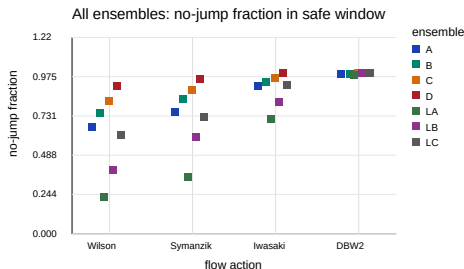
- $n = 101$ configurations of C; sign convention $+\nu$; 10-block jackknife errors with common delete groups.
- Mismatch **stable** from $t=1$ (Iwasaki) and $t=0.5$ (DBW2); **Wilson/Symanzik grow** with flow time.
- Iwasaki vs. DBW2, paired jackknife: mismatch 3.0(21) %pt, r 0.007(18), **indistinguishable** at $n = 101$.

Sector comparison only; no fermionic renormalization is transferred to the gluonic susceptibility.

Backup: $d_{\mathbb{Z}}$ and sector stability F_0

A. Tomiya

Distance to integers and zero-jump fraction, all ensembles



F_0 : fraction of configurations whose round $Q_f(t)$ **never** jumps in $0.5 \leq t \leq \min(100, t^{(\text{Box})})$.

$$d_{\mathbb{Z}} = \langle |Q_f - \text{round } Q_f| \rangle \text{ at } t = 5$$

flow	A	C	LA
Wilson	0.025	0.010	0.097
Symanzik	0.020	0.009	0.046
Iwasaki	0.015	0.007	0.029
DBW2	0.008	0.003	0.014

- F_0 rises monotonically Wilson $\rightarrow$ DBW2 in **every ensemble**; A: 0.66 $\rightarrow$ 0.99, LA: 0.23 $\rightarrow$ 0.98.
- $d_{\mathbb{Z}}$ shows the same ordering; largest in the big box LA, **smallest for DBW2** everywhere.

Honest error budget before quoting final numbers

- **D (finest a):** $\tau_{\text{int}} \simeq 25 \Rightarrow N_{\text{eff}} \simeq 6$ of 301; jackknife block length $< 2\tau_{\text{int}}$, so D-including fits may **underestimate errors**.
- **C:** raw flow data independently re-verified (**no corruption**, max. deviation 4×10^{-13}); window/block scans suggest reading its error $\times 1.3\text{--}1.5$ larger; C still sits low at $1.9\text{--}2.4 \sigma$ vs. ABD, cause not identified.
- **Volumes:** two boxes per β only, so the V^{-1} functional form is untested; no large box at the finest a .

Error policy

- common fixed 10-block jackknife in all figures;
- conservative C factors and D warnings quoted **alongside**, never silently absorbed;
- ABC/BCD/ABD/ABCD all reported, none promoted to a final value;
- $V \rightarrow \infty$ and $t \rightarrow 0$ labeled **preliminary** (linear forms, no covariance across R).

None of these caveats touches the **qualitative** flow-action ordering: it is visible **ensemble by ensemble**, before any extrapolation.

Clover-rectangle improved Q ; flow integration

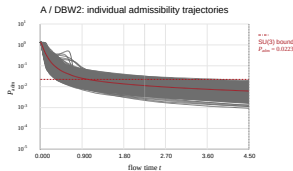
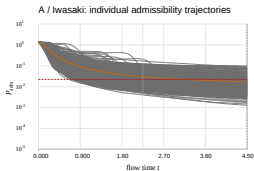
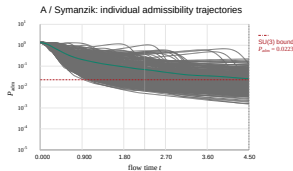
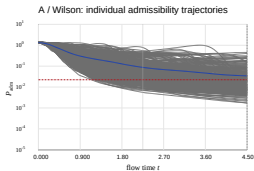
- Measured charge: clover-rectangle improved combination

$$Q_{\text{imp}}[V] = \frac{5}{3} Q_{\text{clov}}[V] - \frac{1}{12} (2 Q_{\text{rect}}[V]), \quad Q_f(t) = Q_{\text{imp}}[U^{f,t}],$$

where Q_X uses the traceless anti-Hermitian part of the corresponding loop sum ($n_{\text{clov}} = 4$, $n_{\text{rect}} = 8$).

- Changing the flow action at finite a is a change of the **topological-charge operator**, $Q[U] \rightarrow Q[U^{f,t}]$; the generated Markov chain is untouched.
- Flow integrated to $t = 100$ with $dt = 0.01$ (10001 rows per configuration and flow); $t^{(\text{Box})}/2$ (dotted) and $t^{(\text{Box})}$ (dashed) marked in all t -dependence plots.
- LB: 76 unique configurations in 11 contiguous label segments; autocorrelation and jackknife are **gap-aware** (no pair crosses a segment boundary; balanced delete groups of 7-8).

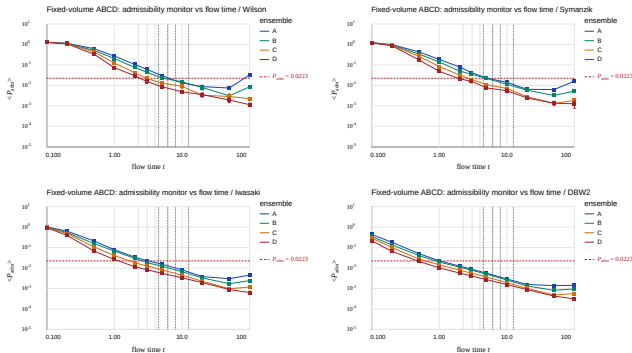
Per-configuration roughness on A ($a \approx 0.10$ fm)



Gray: $P_{\text{adm}}(t)$ of each configuration up to $t^{(\text{Box})}$; colored: mean. Wilson/Symanzik trajectories keep bouncing back up (dislocations re-roughen the field); Iwasaki suppresses them; DBW2 shows almost no revivals.

Backup: admissibility monitor on A-D

Ensemble means, $a = 0.102 \rightarrow 0.061$ fm

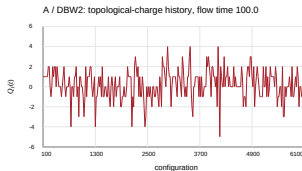
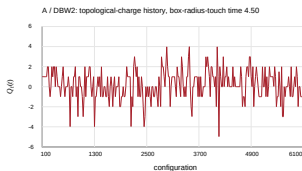
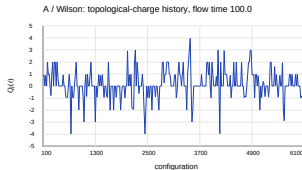
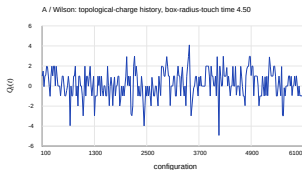


Mean P_{adm} on A-D; note the panel scales: DBW2 starts an order of magnitude lower. Max P_{adm} at $t=5$ on A: 0.64 (Wilson) vs. 0.051 (DBW2). Same flow ordering as the integerness diagnostics at every lattice spacing.

Backup: Q histories on A ($a \approx 0.10$ fm)

A. Tomiya

Configuration-by-configuration readout at three flow times

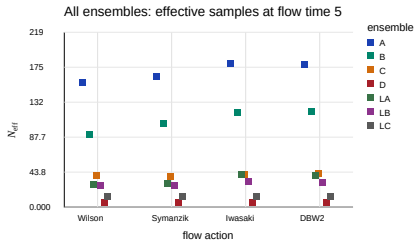
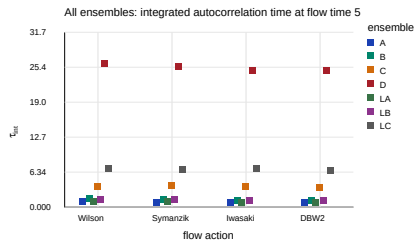


Rows: Wilson (top), DBW2 (bottom); columns: $t = 4.5$ (t^{Box}) and $t = 100$. Same Markov chain in every panel; other flows and times in the full report.

Backup: autocorrelation at $t = 5$

A. Tomiya

Why D (finest, $a = 0.061$ fm) is the statistical bottleneck

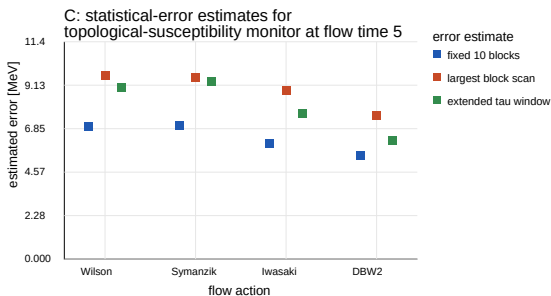


τ_{int} of $Q_f(t=5)$: $D \simeq 25$, $LC \simeq 7$, $C \simeq 3.8$.

Effective sample counts; D drops to $N_{\text{eff}} \simeq 6$.

LB autocorrelation pairs never cross segment boundaries (gap-aware); balanced delete-group jackknife with groups of 7-8 configurations, zero discarded rows.

Fixed 10-block vs. block-width scan vs. extended window

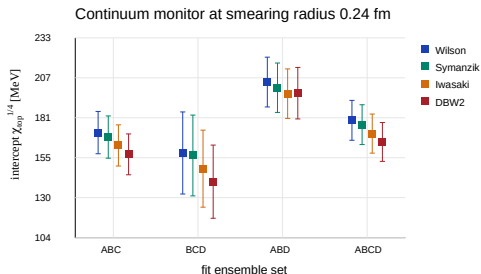


Max/fixed error ratio 1.36–1.46 (block-width scan) and 1.14–1.33 (window extension). With these factors the C-ABD tension drops from $2.3\text{--}2.9\sigma$ to $1.9\text{--}2.4\sigma$; the low central value of C remains. Short-time DBW2 raw data independently re-verified: max. abs. deviation 3.6×10^{-13} over all 301 configurations.

Backup: continuum-set comparison

A. Tomiya

ABC / BCD / ABD / ABCD intercepts at $R = 0.24$ fm

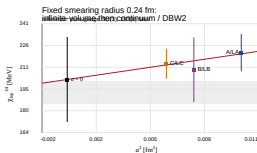
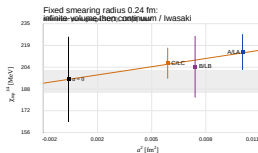
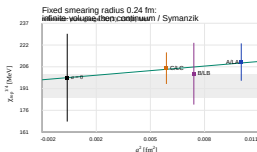
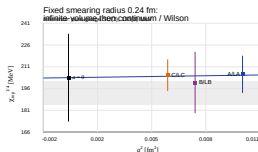


flow	ABC	BCD	ABD	ABCD
Wilson	171.7(13.7)	158.5(26.6)	204.5(16.1)	179.7(12.9)
Symanzik	168.9(13.7)	156.9(26.2)	200.8(16.0)	176.9(12.9)
Iwasaki	163.4(13.3)	148.4(25.0)	197.0(16.0)	171.1(12.7)
DBW2	157.7(13.2)	139.9(23.7)	197.3(16.7)	165.7(12.6)

Intercepts in MeV. ABD: $\chi^2/\text{dof} \approx 1.1\text{--}1.3$; C-including sets 3.8–7.9 (C/D sensitivity).

Backup: infinite volume, then continuum

Two-volume extrapolation at each β , then $a^2 \rightarrow 0$ at $R = 0.24$ fm



A/LA, B/LB, C/LC $\rightarrow V^{-1} \rightarrow 0$ per β , then continuum: intercepts 204/199/195/202 $\pm$ 30 MeV (Wilson/Symanzik/Iwasaki/DBW2), $\chi^2/\text{dof} \approx 0.02$. Gray band: Dürr et al. quenched SU(3) $\chi_{\text{top}}^{1/4} = 193(1)(8)$ MeV. Fixed-volume ABCD $t \rightarrow 0$ side: 190(44)-168(43) MeV.

Rounded-sector mismatch and Pearson correlation between flows

set	W-DBW2 mismatch	$r(W, Iwa)$	$r(W, DBW2)$	$r(Iwa, DBW2)$
A	0.309	0.934	0.909	0.969
B	0.236	0.954	0.934	0.979
C	0.143	0.962	0.942	0.984
D	0.056	0.992	0.989	0.997
LA	0.500	0.973	0.948	0.979
LB	0.434	0.959	0.953	0.987
LC	0.260	0.981	0.968	0.986

Rounded-sector mismatch decreases with finer a ($A \rightarrow D$, $0.102 \rightarrow 0.061$ fm) and increases with volume at fixed a ($A \rightarrow LA$), the same pattern as the integerness diagnostics. The high Pearson r shows the charge *values* stay linearly related even when rounded sectors differ.

No $Z(\beta)$, $M(\beta)$ needed at positive flow time

- The unflowed lattice susceptibility needs multiplicative and additive renormalization, $\chi_L = Z(\beta)^2 a^4 \chi + M(\beta)$ (D'Elia 2003).
- At positive physical smearing radius $R > 0$, gauge-invariant flowed fields need no extra renormalization (Lüscher-Weisz 2011); the contact-term region is smoothed on the scale R .
- Hence all cross-lattice comparisons are made at **fixed** $R = a\sqrt{8t} = 0.21, 0.24, 0.26$ fm, and $t \rightarrow 0$ is taken only *after* the continuum limit, linearly in $R^2/8$.
- Dürr et al. instead fix a nonperturbative Z from the integer-comb positions of a HYP-smeared charge; their 193(1)(8) MeV is used as an external reference here.