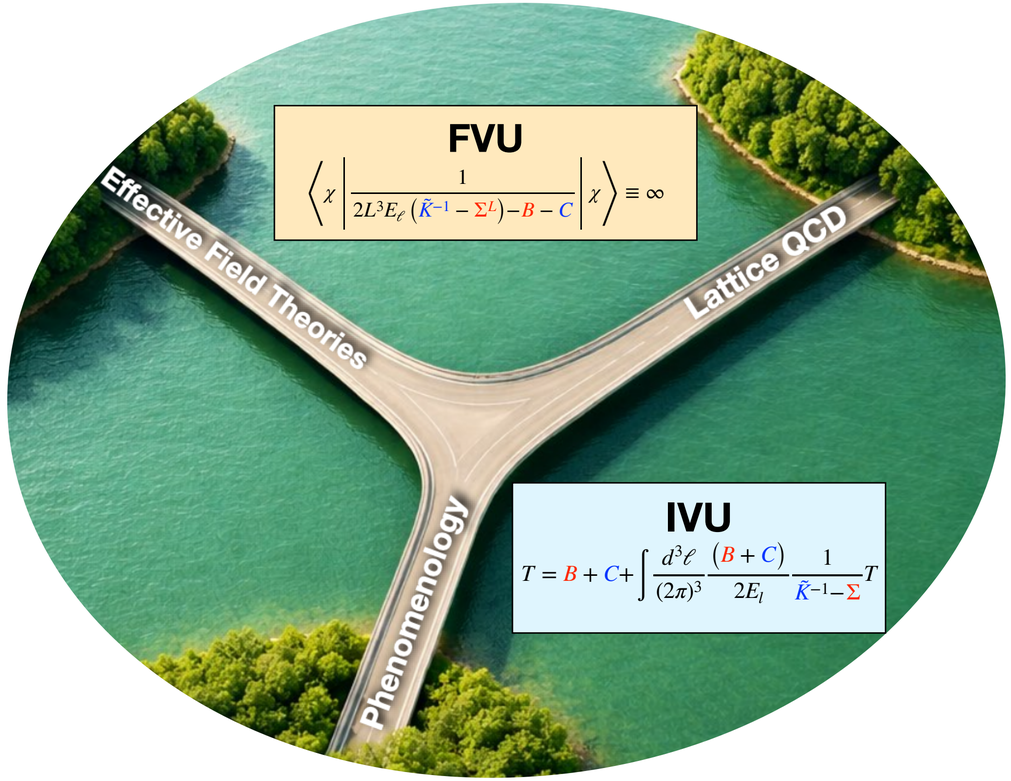
FVU 3-BODY QUANTIZATION CONDITION DEVELOPMENTS/APPLICATIONS

Lattice2026 (UMD) — July/August 2026

MAXIM MAI

UNIVERSITY OF BERN

THE GEORGE WASHINGTON UNIVERSITY



DFG: Heisenberg Programme (532635001)
TRR110: NSFC Grant No. 12070131001,
DFG Project-ID 196253076
DOE: DE-SC0016582, DE-AC05-06OR23177, DE-FG02-95ER40907
NSF: PHY-2012289

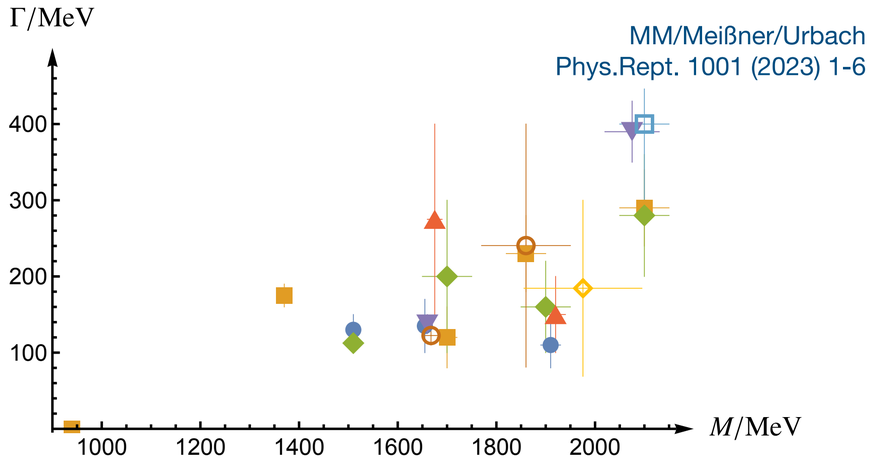
HADRON SPECTRUM

Experimental progress

- 70y research: $\Delta(1232), \rho(770), \omega(782), \ldots$

  ≈ 100 mesons + 50 baryons (****)

- ongoing experiments @CERN, GSI, JLAB, BES…



MM/Meißner/Urbach
Phys.Rept. 1001 (2023) 1-6

Theory tools (selection)

Lattice Gauge Theory (LQCD)

[+] non-perturbative access to QCD Green's fct

[–] discretized Euclidean space-time

[–] finite-volume

[~] unphysical quark masses

S-matrix

[+] analytical formulations

[–] agnostic to interaction

[+] constrains on-shell configurations

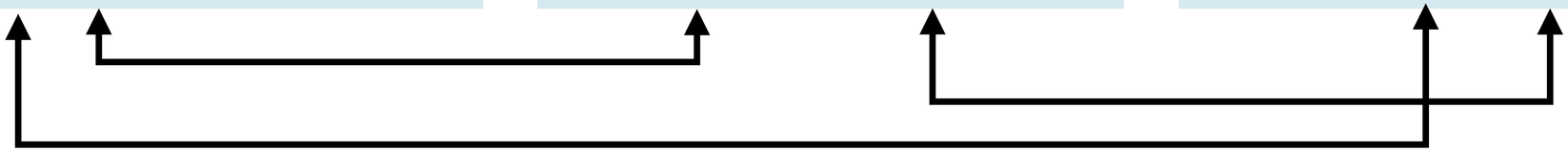
Effective Field Theory (EFT/CHPT)

[+] effective degrees of freedom

[+] perturbative expressions

[–] unknown low-energy constants

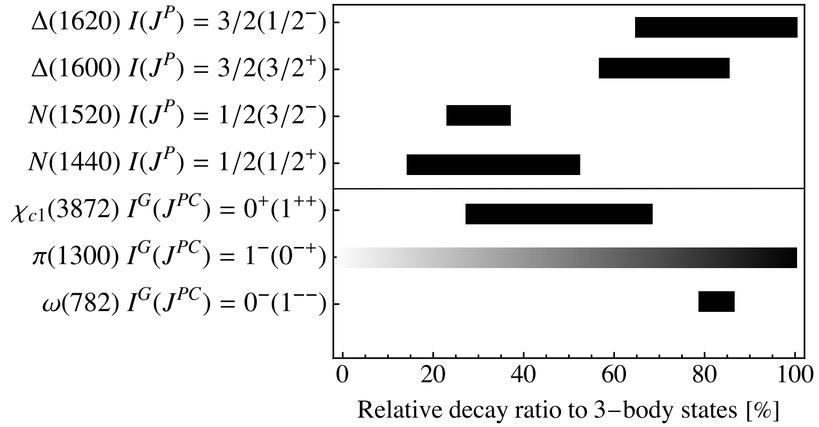
[+] quark mass dependence



COMPLICATION: 3-BODY EFFECTS

Right-hand cut $\sqrt{s} > M_1 + M_2 + M_3$

- infinite volume >> imaginary part of scattering amplitudes
- finite-volume >> new polynomial ($M_\pi L$) effects



new 3-body quantization conditions

NREFT / RFT / **FVU** / NPHF /…

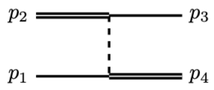
related talks: Yan/Jackura/Hansen/Wu/…

Poleyaeva/Rusetsky (2012), Hansen/Sharpe (2014), MM/Döring EPJA 53 (2017) 12, 240, Hammer+ JHEP 10 (2017) 115, Wunderlich+ JHEP 08 (2019); Jackura+ EPJC 79 (2019); Kang Yu+ 2603.07205 [hep-lat], Dawid+ PLB 864 (2025) 139442… **Reviews: Hansen/Sharpe: Ann.Rev.Nucl.Part.Sci. 69 (2019) 65-107 MM/Rusetsky/Doring Eur.Phys.J.ST 230 (2021) 6, 1623-1643**

Analytically connected regions ($s \leftrightarrow t \leftrightarrow u$)

- one-particle exchange induced left-hand cut

  Meng, Epelbaum, Du, Raposo, Vujmilovic, Sharpe, Filin, …

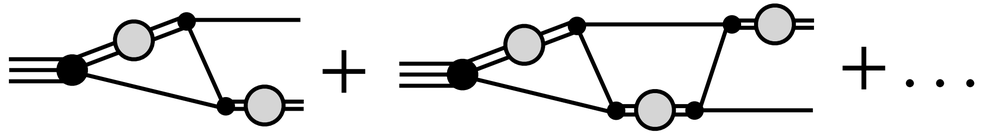


- Heavy-light systems, $T_{cc}, N^*(1440), NN, \ldots$

  related talks: Vujmilovic/Wu/Sharpe/Dawid/Qiu/Salg/Meng…

- triangle singularities vs. resonances

  Guo+ Prog.Part.Nucl.Phys. 112 (2020) 103757



3-BODY SYSTEMS: FVU

"Finite Volume Unitarity" (FVU)

$$\left\langle \chi \left| \frac{1}{2L^3 E_\ell \left(\tilde{K}^{-1} - \Sigma^L\right) - B - C} \right| \chi \right\rangle \equiv \infty$$

- only discrete momenta are possible $2\pi/L\,\mathbb{Z}^3$
- singularities/onshell states $(+O(e^{-ML}))$

MM/Döring Eur.Phys.J.A 53 (2017) 12, 240

on-shell configurations
— fixed —

offshell "2-/3-body force"
— to be determined —

"Infinite Volume Unitarity" (IVU)

$$T = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_\ell} \frac{1}{\tilde{K}^{-1} - \Sigma} T$$

- Three-body scattering amplitude
- 3b-Unitarity
- intergal equation (momentum sp.)

MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)

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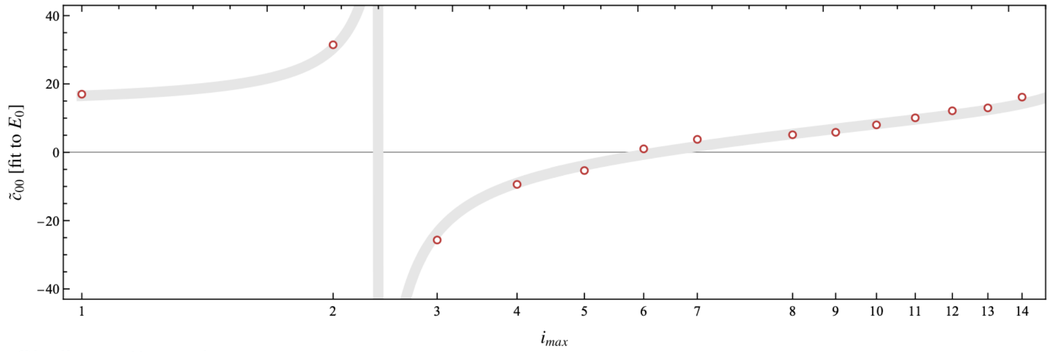
MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)

THEORETICAL UPDATE I (Cutoff dependence)

Feng+ 2601.16916 [hep-lat] [PRD in press]

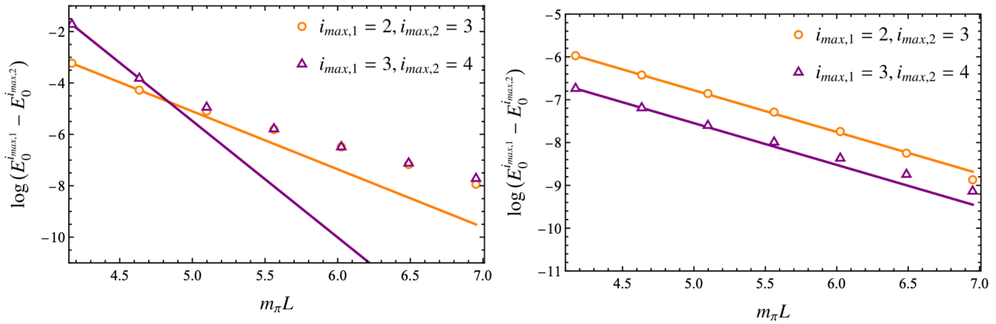
- inherit due to 2+1 formations
- 3-body force depends on cutoff/integration details



Similar patterns in
Bedaque, Hammer, van Kolck PRL 82, 463–467 (1999) Doring+ PRD 97, 114508 (2018) Fu+ JHEP 07, 081 (2025)

- smooth cutoffs/form-factors/sharp cutoff…
- over-subtracted OPE

$$\left[\tilde{B}(s)\right]_{(\boldsymbol{p}',j)(\boldsymbol{p},i)} = \tilde{I}_{ji} \frac{v_j^*(p, P_3 - p - p') v_i(p', P_3 - p - p')}{2E_{\boldsymbol{p}'+\boldsymbol{p}}(\sqrt{s} - E_{\boldsymbol{p}} - E_{\boldsymbol{p}'} - E_{\boldsymbol{p}'+\boldsymbol{p}})} \qquad \mapsto \tilde{B}\left(\frac{s}{s_{on}}\right)^n$$



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MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)

THEORETICAL UPDATE II (2/3 systems)

- many states decay to both channels (XYZ, N^*, …)
- two approaches [*]
  - **explicit stable states**
  - **isobars with a particle-pole**
    >> talk Jinzi Wu (Friday)

(*)
Jülich-Bonn model: e.g. PRC **62**, 025207 (2000), Eur. Phys. J. A 49, 44 (2013)
RFT: *PRD* 95 (2017) 7, *JHEP* 02 (2026) 221

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MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)

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  - **explicit stable states** — **XX / Y(YY)**
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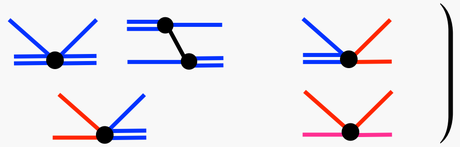
$$\left\langle \chi_\Gamma \left| \begin{pmatrix} \tau_{I1} \cdot 2L^3 E_{S1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} \tau_{I1} & 0 \\ 0 & 1 \end{pmatrix} \frac{\mathbf{1}}{\mathbf{1} - (B+C)\hat{\tau}} (B+C) \begin{pmatrix} \tau_{I1} & 0 \\ 0 & 1 \end{pmatrix} \right| \chi_\Gamma \right\rangle = \infty$$

$$\hat{\tau} = E_L^{-1}\tau = \begin{pmatrix} \frac{1}{2L^3 E_{S1}} \frac{1}{K_{I1}^{-1} - \Sigma^L} & 0 \\ 0 & \frac{1}{2L^3 E_{I2}} \frac{1}{\sigma_{S2} - M_{I2}^2} \end{pmatrix}$$

<< ensures 3-body unitarity

<< ensures 2-body unitarity

$$B + C = \begin{pmatrix} & \\ & \end{pmatrix}$$



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MM/Döring Eur.Phys.J.A 53 (2017) 12, 240

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MM/Hu/Döring/Pilloni/Szczepaniak Eur.Phys.J.A 53 (2017)

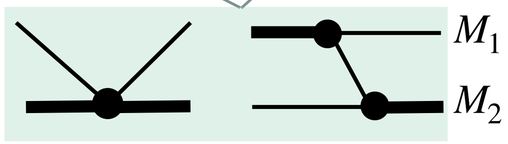
THEORETICAL UPDATE III (Left-hand cuts)

- OPE part of FVU through 3-body unitarity
- External states onshell >> left-hand cut

connects smoothly to the 3-body systems

>> talk: Ivan Vujmilovic (Friday)

$$\left\langle \chi_\Gamma \left| \frac{\mathbf{1}}{\mathbf{1} - (B+C)\frac{1}{2L^3E_1}\frac{1}{\sigma - M_2^2}} (B+C) \right| \chi_\Gamma \right\rangle = \infty$$



IVU/FVU APPLICATIONS

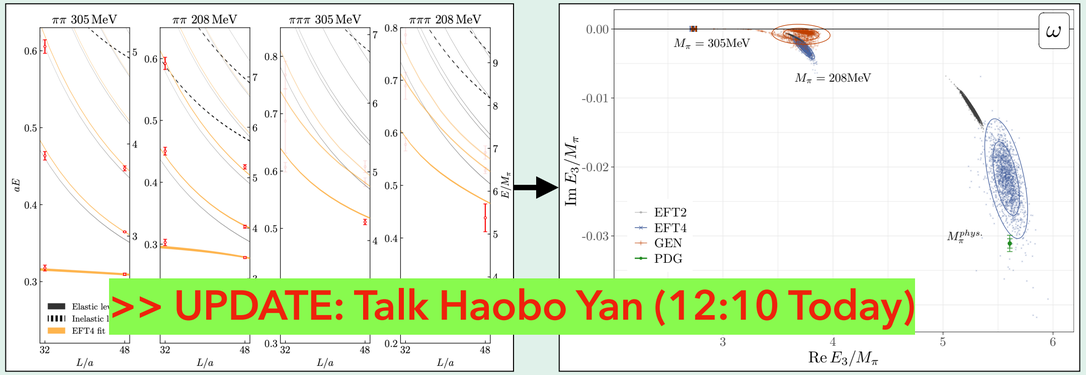
- generalized to all systems $(\pi, K, \ldots)$
- $(3\pi)^{I=3}$, $(3K)^{I=3/2}$, $a_1(1260)$, $K(1460)$

Feng+ *Phys.Rev.D* 110 (2024) 094002

PRL 122 (2019) 6, 062503 — *PRD* 102 (2020) 11, 114523 — *PRD* 101 (2020) 9, 094018— *PRD* 101 (2020) 11, 114507— *PRD* 101 (2020) 5, 054510 — *PRL* 127 (2021) 22, 222001 — *PRD* 105 (2022) 5, 054020— *PRD* 113 (2026) 3, 034032

$\omega(782)$ from Lattice QCD

[agemo] Yan+ *PRL*133 (2024) 21, 211906



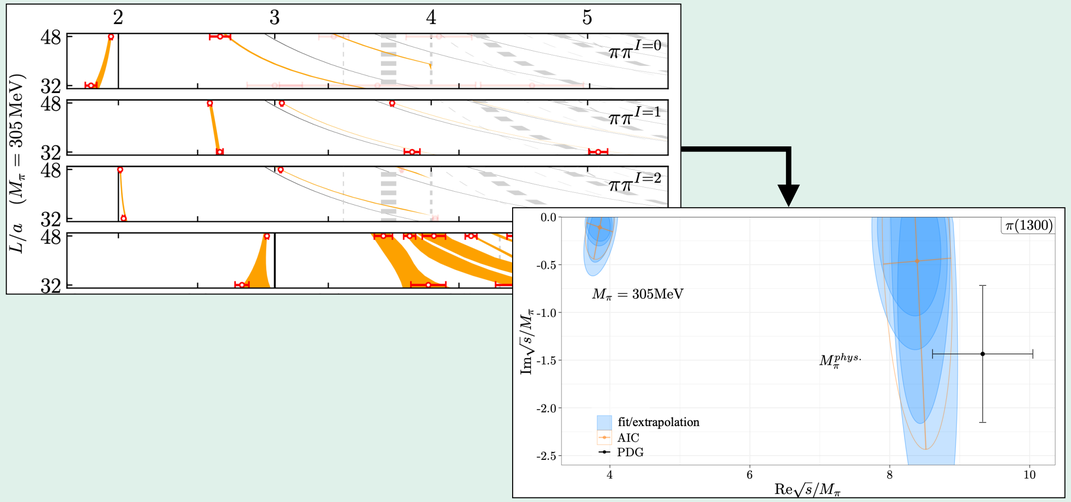
>> UPDATE: Talk Haobo Yan (12:10 Today)

- reanalyzed in non-perturbative Hamiltonian formalism (NPHF)

2603.07205 [hep-lat]

$\pi(1300)$ from Lattice QCD

[agemo] Yan+ PRL136 (2026) no.14, 141901



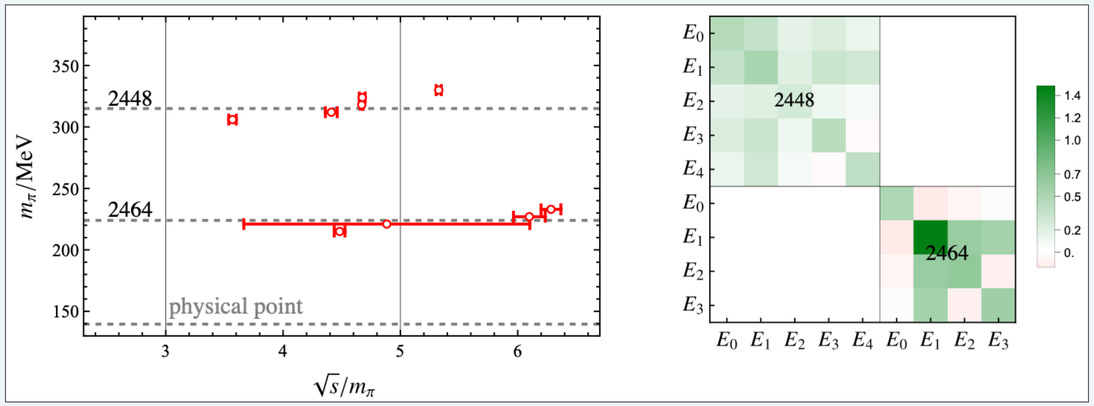
A.D. 2026 IVU/FVU APPLICATIONS: I

Coupled-channel approach to $(3\pi)\quad I^G(J^{PC}) = 2^-(1^{+-})$

- attractive-repulsive mix $\pi\rho/\pi G$ (``*isograviton*'' $G = (\pi\pi)^{I=2}$)

  [GWQCD] Feng/Culver/MM/Doring/Alexandru/Lee 2601.16916 [hep-lat] [PRD in press]

- $M_\pi \approx 224, 315$ MeV
- 2- and 3-body operators



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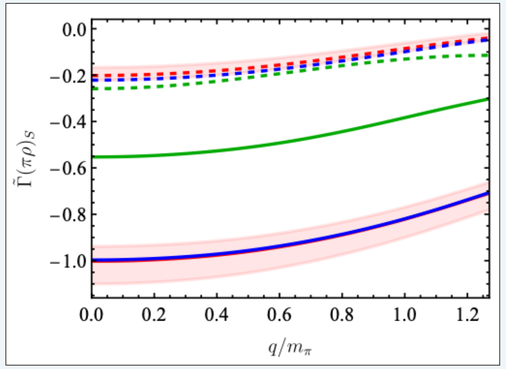
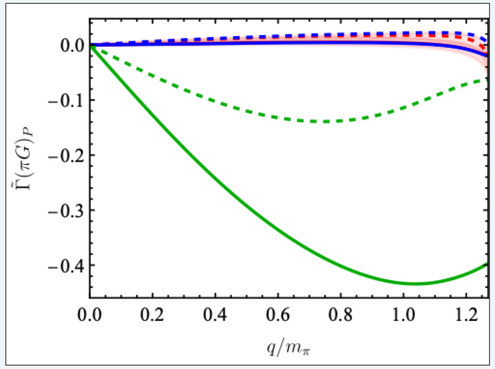
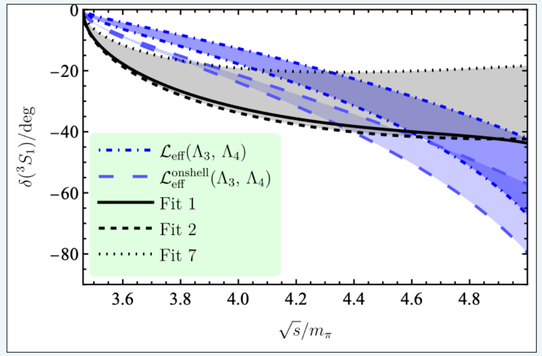
$$T = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_\ell} \frac{1}{\tilde{K}^{-1} - \Sigma} T$$

- infinite volume production amplitude
- systematics (parametrization) are crucial







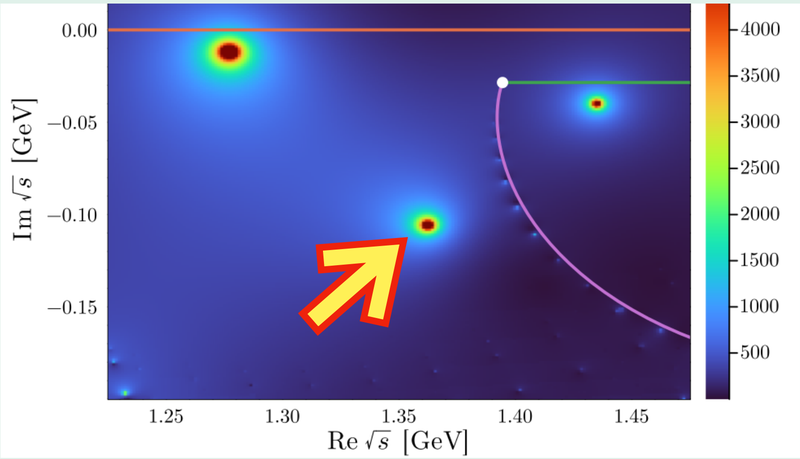
Talk: Jackura — related study in RFT: HadSpec Briceno/Hansen/Jackura/Edwards/Thomas 2510.24894 [hep-lat]

A.D. 2026 IVU/FVU APPLICATIONS: II

Coupled channel $\pi a_0 / K\bar{\kappa} / K\bar{K}^*$

Tao-Ran Hu et al. 2606.19040

- $f_1(1285)$
- $f_1(1420)$ dynamically generated (hadr. molecule)
- new third state with little imprint in the physical world >> Lattice QCD?



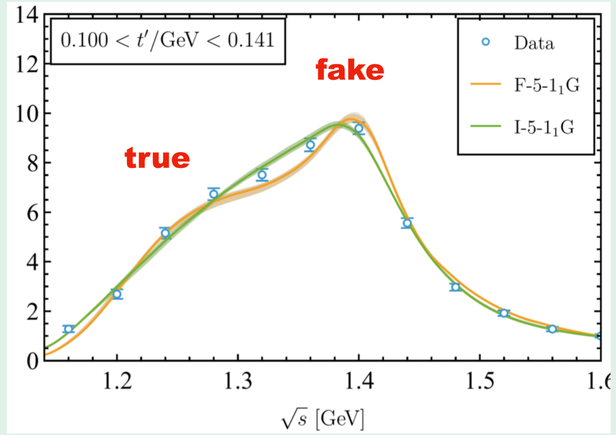
Can we use Lattice QCD to discover states not accessible through experiment?

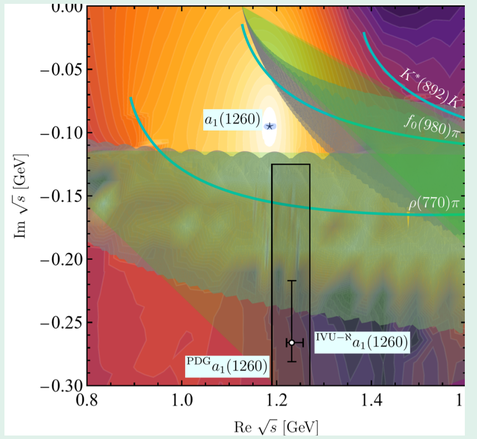
$a_1(1420)$ as triangle singularity

Ketzer/Mikhashenko/Aceti/Dai/Oset/Bayar/Guo…
Prog.Part.Nucl.Phys. 112 (2020) 103757

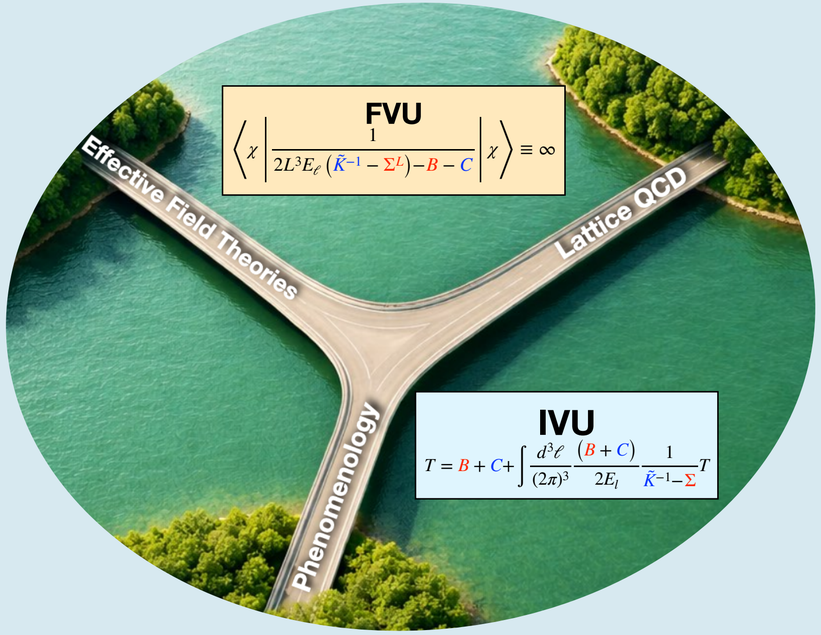
Sakthivasan/MM/Feng/Doring 2606.24709

- first unitary formalism with TS
- no state needed wrt IVU+COMPASS
- sensitive to quark masses >> Lattice QCD?

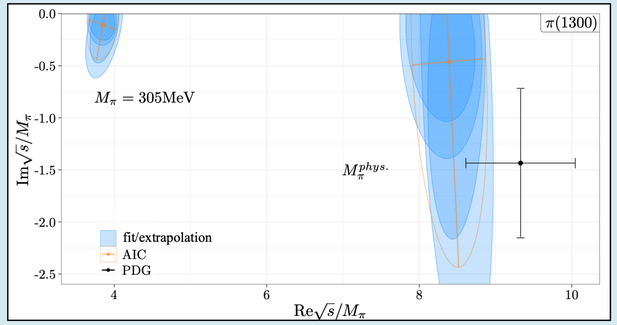


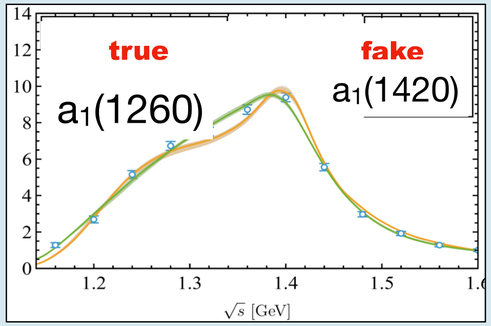


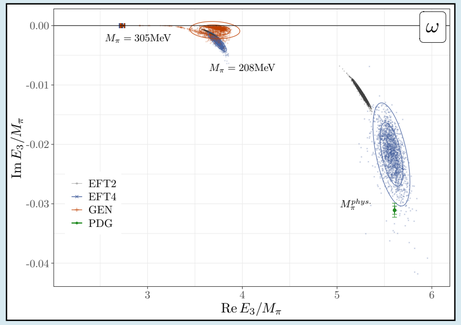
SUMMARY



- QCD input from Lattice QCD
- quark-mass extrapolations through EFT
- On/off-shell configurations (S-matrix)







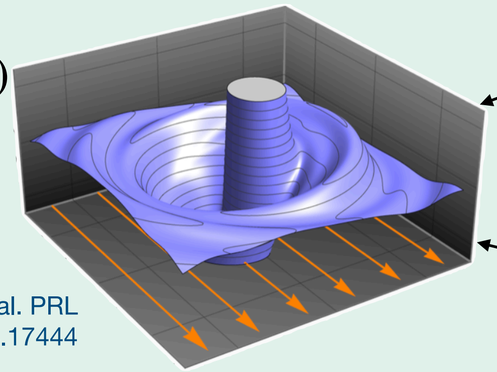
ONGOING

- ✳ heavy-light systems. Left-hand cut — talk: Ivan Vujmilovic (Friday)
- ✳ 2/3 systems $\omega/\phi(3\pi/K\bar{K})$ — talk: Haobo Yan (in a few mins)
- ✳ $N^*(1440)$ — talk: Jinzi Wu – 14:40 Friday

time=mass!

OUTLOOK

- ✳ spin-exotics, $DD\pi, \ldots$
- ✳ $N^*(1440)$



EXP (10^5 data)

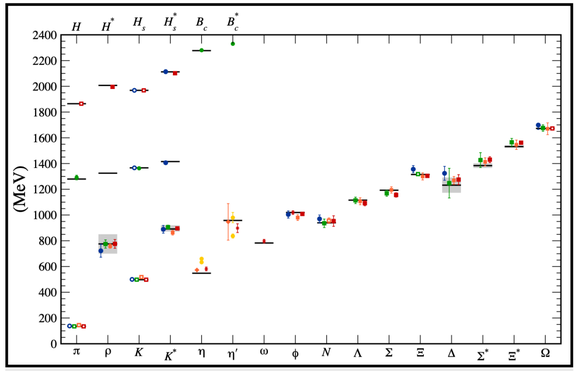
LQCD?

[JBW] Wang et al. PRL 2404.17444

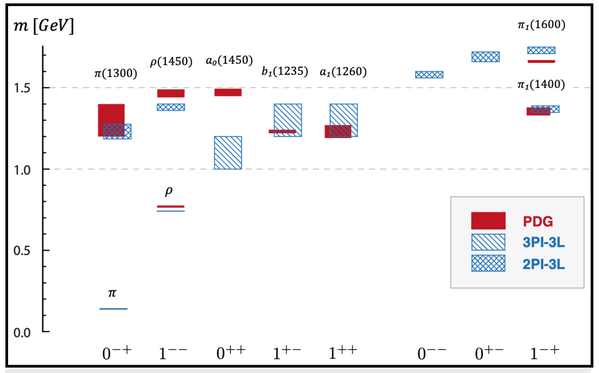
STATE OF THE ART

Theory

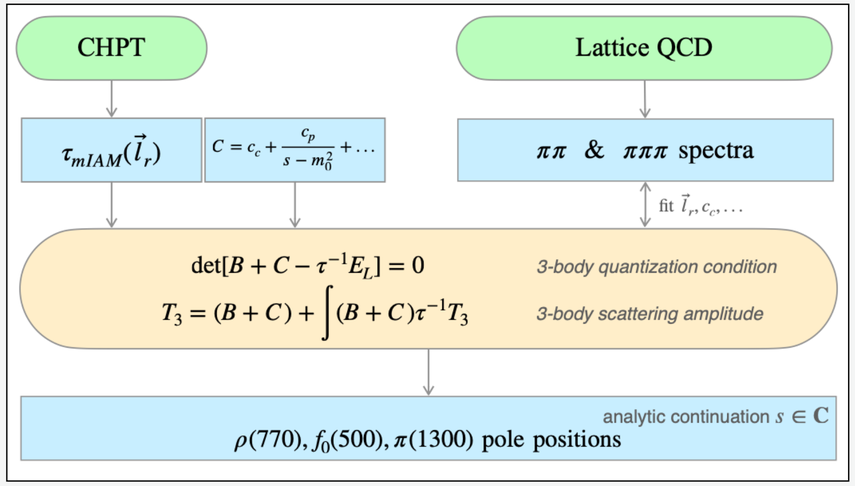
- Constituent quark model — Wang/Ping Chin. Phys. Lett. 24, 1195 (2007)
- DSE beyond rainbow-ladder cont. — Williams/Fischer Phys.Rev.D 93 (2016) 3, 034026
- Unitary models $\pi K \bar{K}$ — Phys.Rev.D 84 (2011) 074027
- Pioneering Lattice QCD (spectrum) calculations — Dudek+ (HadSpec) PRD 82, 034508 (2010); McNeile/Michael (UKQCD) PLB 642, 244 (2006)



Kronfeld Ann. Rev. Nucl.Part. Sci. 62, 265 (2012)



Eichmann+ Prog.Part.Nucl.Phys. 91 (2016)

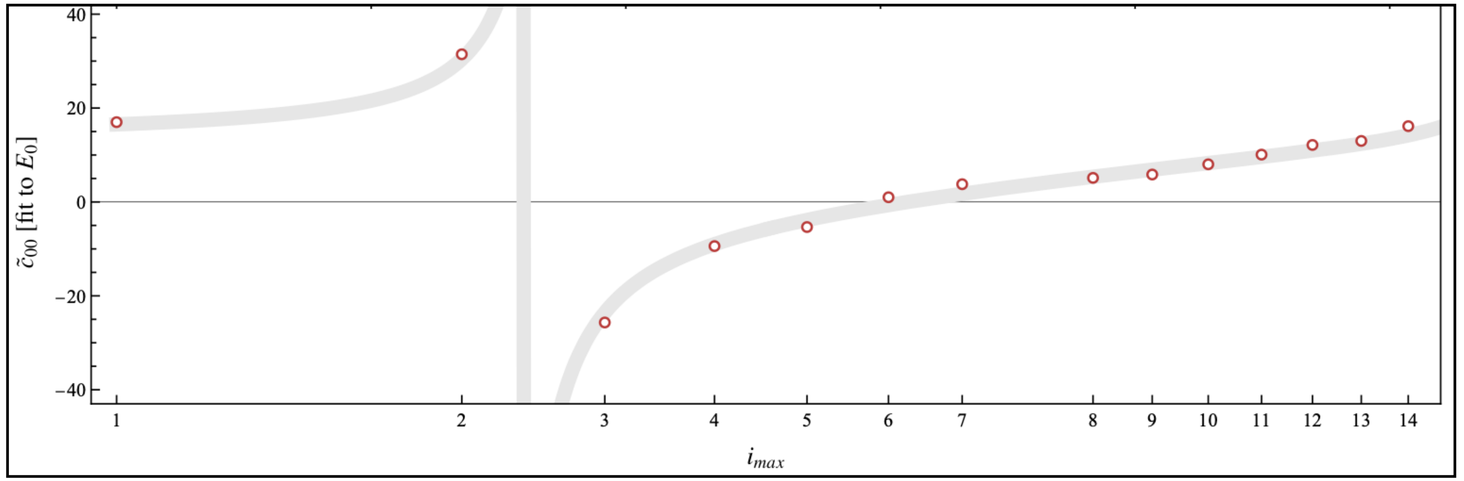


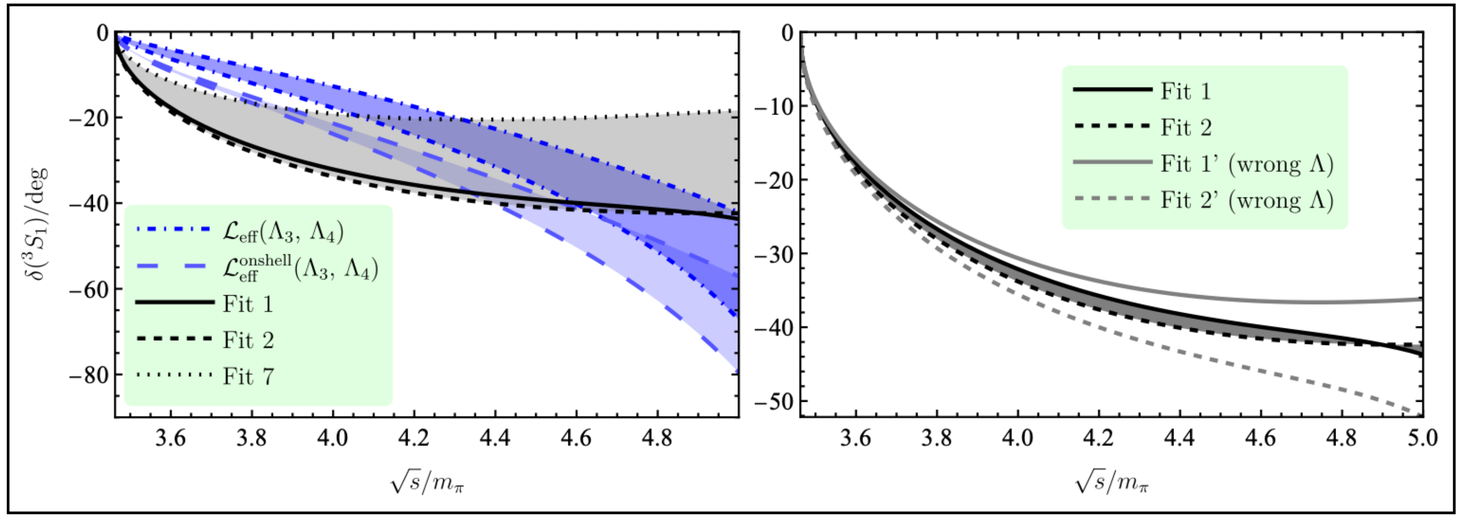
This work — workflow

- Three-body operators/scattering states lattice QCD calc.
- Three-body finite-volume formalisms
- Pole positions
- Chiral extrapolations (CHPT) to the physical point

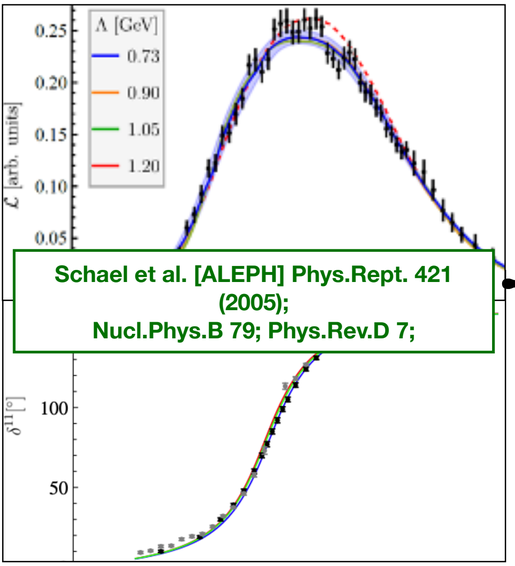
CUTOFF DEPENDENCE

cutoff dependence of the 3-body force





AXIAL-VECTOR MESON



Schael et al. [ALEPH] Phys.Rept. 421 (2005);
Nucl.Phys.B 79; Phys.Rev.D 7;

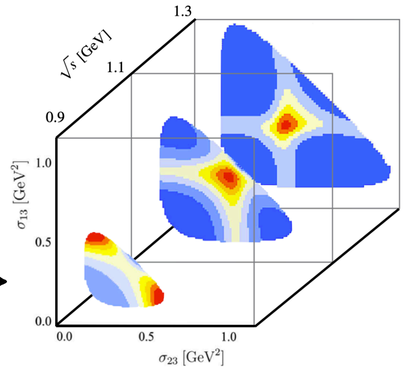
INPUT
EXPERIMENT

IVU

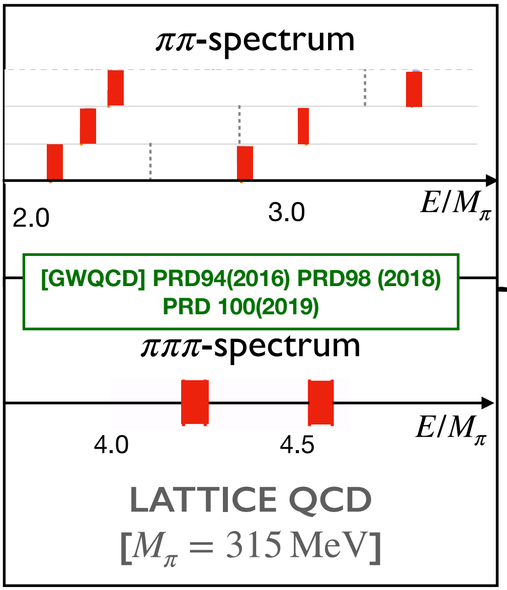
$$T^c = B + C + \int \frac{d^3\ell}{(2\pi)^3} \frac{(B+C)}{2E_l} \frac{1}{\tilde{K}^{-1} - \Sigma} T^c$$

OUTPUT

Sadasivan/MM/… Phys.Rev.D 101 (2020);



$C, \tilde{K}$
volume independent



[GWQCD] PRD94(2016) PRD98 (2018)
PRD 100(2019)

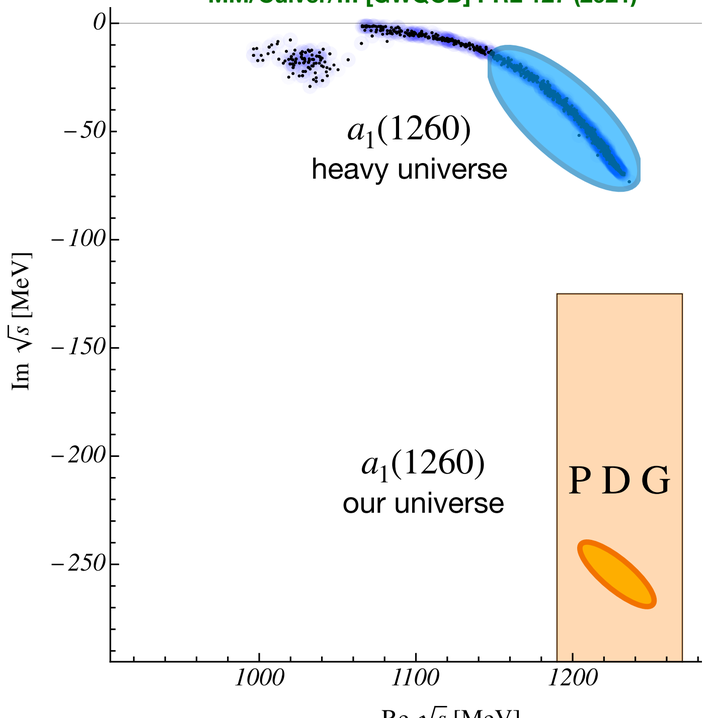
INPUT
Lattice QCD

FVU

$$\det\left[\, 2L^3 E_{\mathbf{p}} \left(\tilde{K}^{-1} - \Sigma^L\right) - B - C \,\right]^{\Lambda} \equiv 0$$

OUTPUT

MM/Culver/… [GWQCD] PRL 127 (2021)



… similar program carried out for $\omega(782)$
Yan/MM+ PRL133 (2024)

AXIAL-VECTOR MESON*

Excited axial-vector meson: $a_1(1420)$

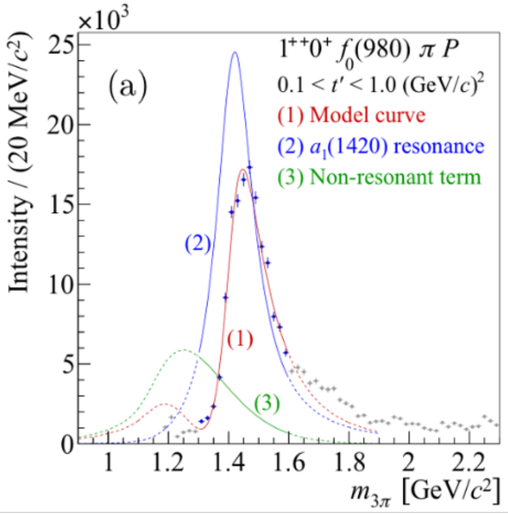
Observed by COMPASS/Belle in $\pi^-\pi^+\pi^-$ final state

COMPASS:2015kdx, Rabusov:2023tna

Creation mechanisms:

- Excited state of $a_1(1260)$ COMPASS:2020yhb
- "Triangle singularity" $K^*(892)\bar{K} \xrightarrow{K} \pi f_0(980)$

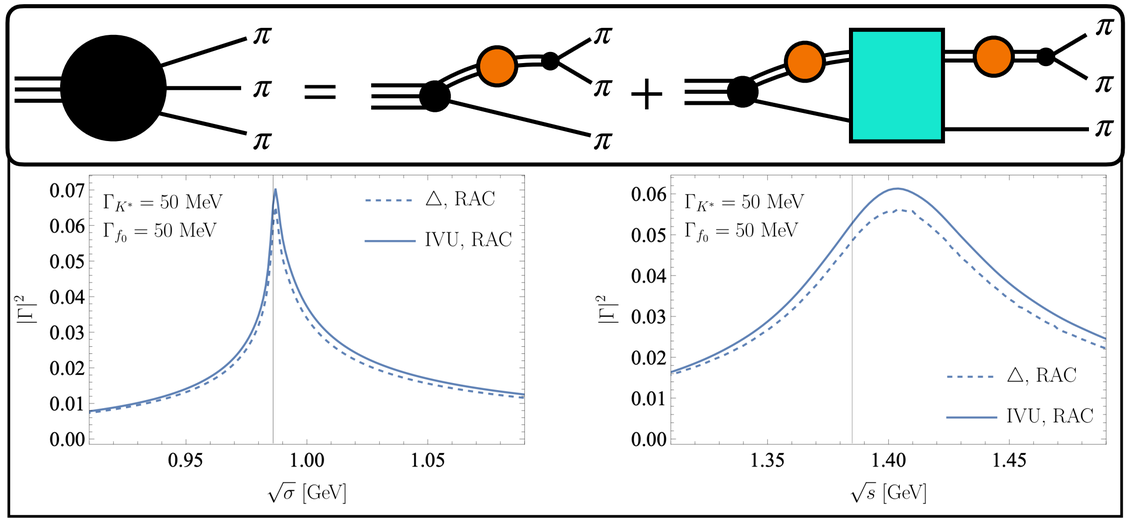
Mikhasenko:2015oxp Review: Guo:2019twa
Related: Dai:2018hqb, Dai:2018rra, Liang:2019jtr, Jing:2019cbw, Du:2021zdg, Duan:2023dky, Wang:2016dtb, Nakamura:2023obk, Zhang:2024dth, Achasov:2022onn, Nakamura:2023hbt, arXiv:1609.04133 [hep-ph].
Talks: J.J.Wu — Z.Zhang



3b unitary formalism IVU

Ajay Sakthivasan/MM/Döring/Rusetsky 2407.17969 [hep-ph]

- formalism to incorporate both hypothesis
- full 2- and 3-body re-scattering
- **for now:** only kinematic/analytical properties (no spin)



➡ Effect is small but distinguishable

➡ Add spin, fit to the line-shapes … in progress