
TOWARDS THE SMEARED R -RATIO WITH $N_f = 2 + 1 + 1$ HIGHLY IMPROVED STAGGERED QUARKS

Matteo Saccardi

C. DeTar, W.I. Jay, S. Lahert, R. Merino, J. Sitison, A.D.M. Valois

Fermilab Lattice and MILC Collaboration

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University of Maryland, College Park

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- ★ Introduction & Motivations
- ★ Numerical setup
- ★ Linear reconstruction methods
- ★ Non-linear reconstruction methods
- ★ Conclusions & Outlook

⇒ Introduction & Motivations

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THE (SMEARED) R -RATIO

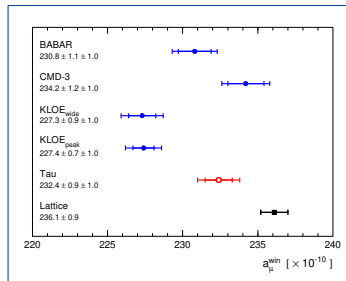
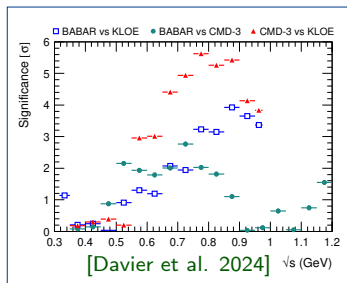
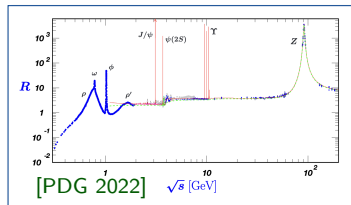
A prime example of (smeared) spectral density



Total cross-section $\sigma(e^+e^- \rightarrow \text{hadrons}) \propto R(s)$

[Cabibbo and Gatto 1961; Cabibbo et al. 1970]

$$R(q^2) \left(q_\mu q_\nu - q^2 \delta_{\mu\nu} \right) \\ \propto \langle 0 | V_\mu^{\text{EM}}(0) \delta^4(\mathbb{H} - q) V_\nu^{\text{EM}}(0) | 0 \rangle \\ V_\mu^{\text{EM}}(x) = \sum_f q_f \bar{\psi}(x) \gamma_\mu \psi(x)$$



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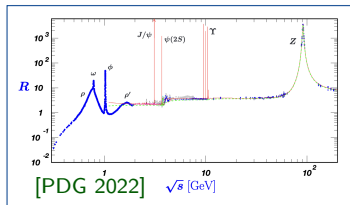
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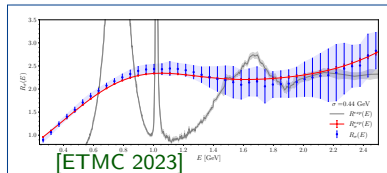
In Lattice QCD we only measure $C(t) = \int_0^\infty d\omega \frac{R(\omega)}{12\pi^2} \omega^2 e^{-\omega t}$

★ **Inverse problem** $\Leftarrow$

★ Only **smeared** ($\sigma > 0$) quantities

$$R_\sigma^g(E) = \int_0^\infty d\omega R(\omega) \delta_\sigma^g(\omega, E)$$

$$\delta_\sigma^g(\omega, E) = e^{-\frac{(\omega-E)^2}{2\sigma^2}} / \sqrt{2\pi\sigma^2}$$



[Margari Fri 16:50]

$$\begin{aligned} C(t) &= -\frac{1}{3} \sum_{k=1}^3 \int d^3\vec{x} \langle V_k^{\text{EM}}(t, \vec{x}) V_k^{\text{EM}}(0) \rangle = \int_0^\infty d\omega \frac{R(\omega)}{12\pi^2} \omega^2 e^{-\omega t} \\ &= \int_0^\infty d\omega e^{-\omega t} \rho(\omega) \quad \stackrel{?}{\implies} \quad \rho[\kappa] \equiv \int_0^\infty d\omega \rho(\omega) \kappa(\omega) \end{aligned}$$

Many approaches are currently viable, including

- ★ Backus-Gilbert, HLT, Chebyshev polynomials, Gaussian processes
[Hansen et al. 2017, HLT 2019, Bailas et al. 2020, Del Debbio et al. 2025]
- ★ Analyticity/causality-based [Fields, Abbott, Jay Mon 14:40–15:20]

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$$= \int_0^\infty d\omega e^{-\omega t} \rho(\omega) \stackrel{?}{\Rightarrow} \rho[\kappa] \equiv \int_0^\infty d\omega \rho(\omega) \kappa(\omega)$$

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Staggered correlation functions also admit *oscillating* contributions

$$C(na) = \int_0^\infty d\omega e^{-\omega na} \rho(\omega) - (-1)^n \int_0^\infty d\omega e^{-\omega na} \rho_{\text{osc}}(\omega)$$

Here:

1. Generalize methods to staggered correlation functions
2. Test reconstruction methods on mock data
⇒ reconstructions on real data with controlled systematic errors

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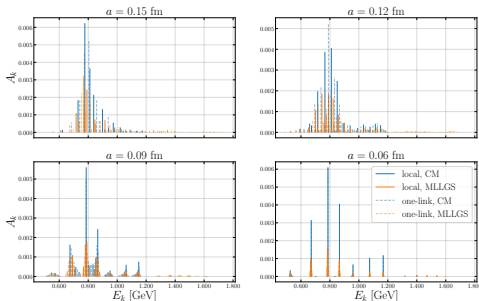
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$$= \sum_k A_k e^{-E_k na} - (-1)^n \sum_k A_{k,\text{osc}} e^{-E_{k,\text{osc}} na}$$

a [fm]	L [fm]	$N_s^3 \times N_t$
0.15	5.00	$32^3 \times 48$
0.12	5.95	$48^3 \times 64$
0.09	5.70	$64^3 \times 96$
0.06	5.48	$96^3 \times 128$

Model light connected energies and amplitudes [Bazavov et al. 2025]

1. **finite-volume** effects
→ Chiral and MLLGS models
2. **taste-breaking**
→ local and one-link data
3. **one oscillating state**
with $E_{1,\text{osc}} = 1.170$ GeV



Finite-volume *unsmeared* R -ratio

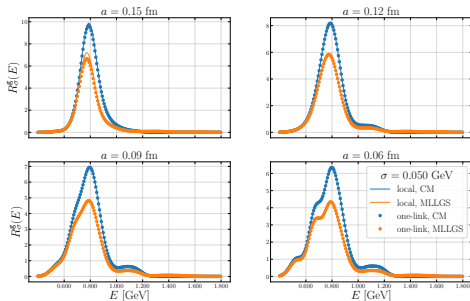
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1. Standard linear reconstruction inversion strategies solve for

$$C(na) = \int_0^\infty d\omega \rho(\omega) b_n(\omega) \implies \rho[\kappa] = \int_0^\infty d\omega \rho(\omega) \kappa(\omega)$$

- i. Approximated target smearing kernel: $\kappa(\omega) \approx \sum_n g_n b_n(\omega) = \bar{\kappa}(\omega)$
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2. Can be applied to **even-odd** staggered data [Stewart et al. Lattice 2024]

$$\begin{cases} C(2na) = \int_0^\infty d\omega \rho_{\text{even}}(\omega) b_{2n}(\omega), & \rho_{\text{even}}(\omega) = \rho(\omega) + \rho_{\text{osc}}(\omega) \\ C((2n+1)a) = \int_0^\infty d\omega \rho_{\text{odd}}(\omega) b_{2n+1}(\omega), & \rho_{\text{odd}}(\omega) = \rho(\omega) - \rho_{\text{osc}}(\omega) \end{cases}$$

$$\frac{1}{2} (\rho_{\text{even}}[\bar{\kappa}_{\text{even}}] + \rho_{\text{odd}}[\bar{\kappa}_{\text{odd}}]) = \sum_n \left[g_{2n}^{\text{even}} C(2na) + g_{2n+1}^{\text{odd}} C((2n+1)a) \right]$$

$$= \rho\left[\frac{1}{2}(\bar{\kappa}_{\text{even}} + \bar{\kappa}_{\text{odd}})\right] + \rho_{\text{osc}}\left[\frac{1}{2}(\bar{\kappa}_{\text{even}} - \bar{\kappa}_{\text{odd}})\right]$$

$$= \rho[\bar{\kappa}] + \rho_{\text{osc}}[\bar{\kappa}_{\text{osc}}], \quad \bar{\kappa}_{\text{osc}} \approx \kappa_{\text{osc}}(\omega) = 0$$

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3. Can we design strategies to directly target $\rho[\kappa]$ and $\rho_{\text{osc}}[\kappa_{\text{osc}} = 0]$?

1. Apply HLT inversion strategy [HLT 2019] to even-odd data

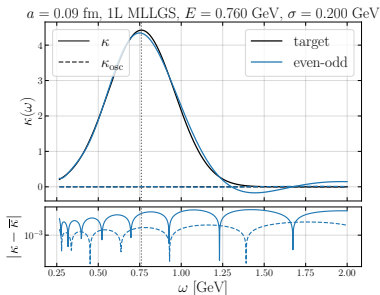
$$\kappa(\omega) \approx \sum_n g_n b_n(\omega) \equiv \bar{\kappa}(\omega)$$

$$\Rightarrow g_n \text{ minimize } A[g] + \lambda B[g]$$

$$A[g] = \int d\omega |\kappa(\omega) - \bar{\kappa}(\omega)|^2$$

$$B[g] = g^T B g, \quad B = \frac{A[0]}{C(a)^2} E^6 \text{Cov}$$

$$\Rightarrow \text{Tune } \lambda \text{ s.t. } A[g_\lambda] = 10B[g_\lambda]$$



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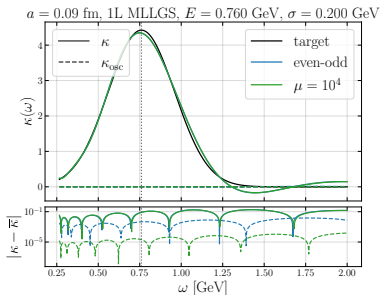
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2. Implement dedicated staggered HLT strategy

$$\kappa(\omega) \approx \sum_n g_n b_n(\omega) \equiv \bar{\kappa}(\omega), \quad \kappa_{\text{osc}}(\omega) \approx \sum_n g_n (-1)^n b_n(\omega) \equiv \bar{\kappa}_{\text{osc}}(\omega)$$

Define g_n to simultaneously target $\kappa(\omega)$ and $\kappa_{\text{osc}}(\omega) = 0$ by minimizing

$$A[g] + \mu A_{\text{osc}}[g] + \lambda B[g], \quad A_{\text{osc}}[g] = \int d\omega |\kappa_{\text{osc}}(\omega) - \bar{\kappa}_{\text{osc}}(\omega)|^2$$

1. Apply GP inversion strategy [Del Debbio et al. 2025] to even-odd data

$a = 0.09$ fm, 1L MLLGS, $E = 0.760$ GeV, $\sigma = 0.200$ GeV (odd)

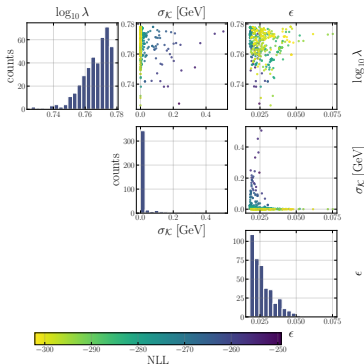
⇒ Take $\rho \sim \text{GP}(\rho_{\vartheta}^{\text{prior}}, \mathcal{K}_{\vartheta}^{\text{prior}})$

⇒ Infer $\rho \sim \text{GP}(\rho_{\vartheta}^{\text{post}}, \mathcal{K}_{\vartheta}^{\text{post}})$

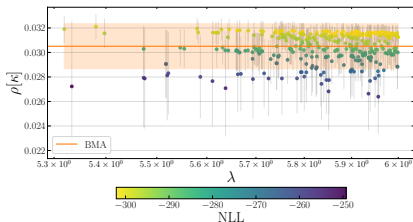
e.g. $\mathcal{K}^{\text{prior}}(\omega, \omega') = \frac{e^{\alpha\omega'}}{\lambda} \delta_{\sigma\mathcal{K}}^g(\omega, \omega')$

a. Fix ϑ by minimizing NLL

b. Marginalize (MC) w.r.t. $\vartheta \Leftarrow$



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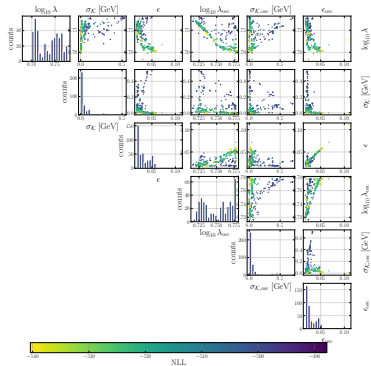
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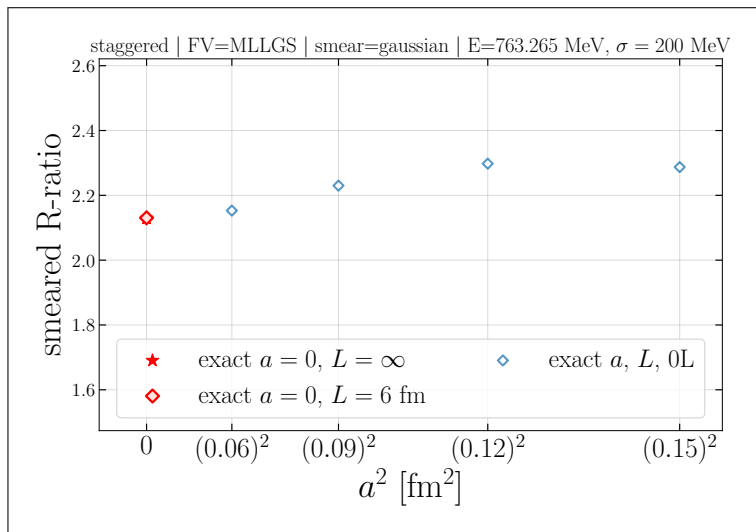
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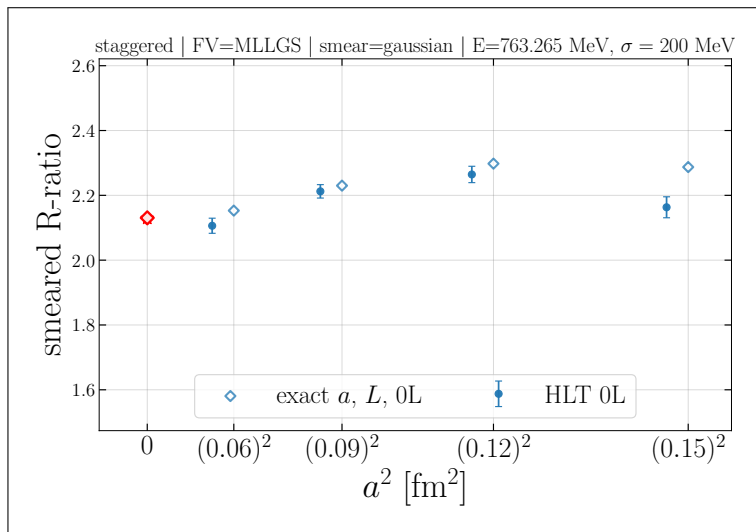


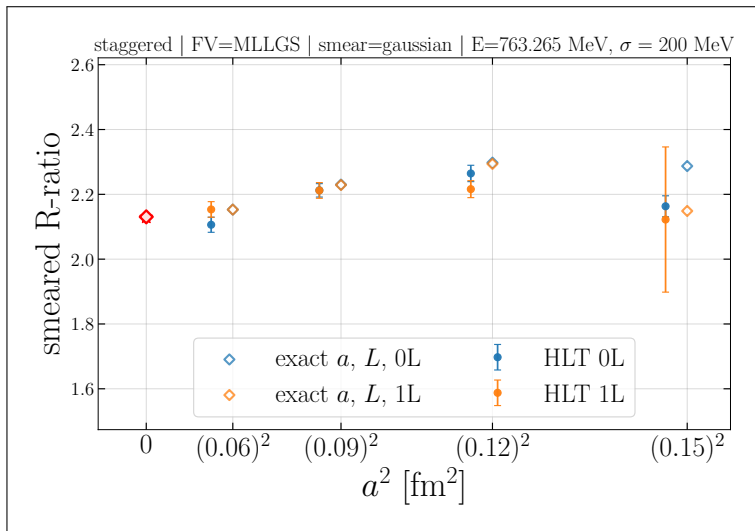
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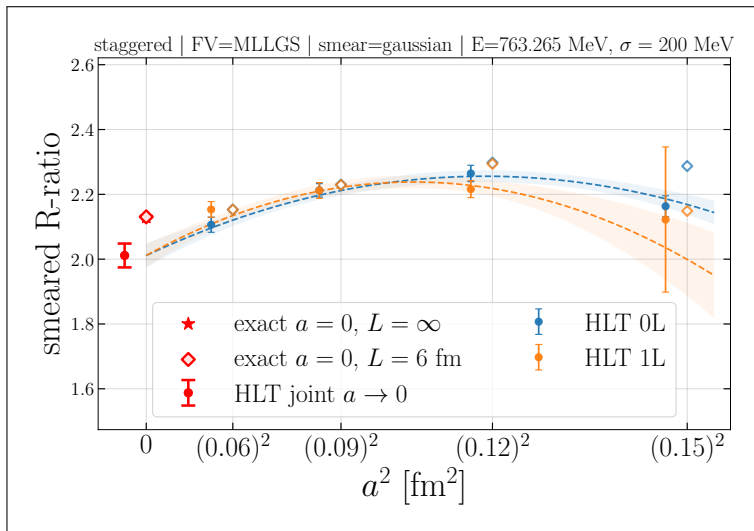
⇒ Take $\rho \sim \text{GP}(\rho_{\vartheta}^{\text{prior}}, \mathcal{K}_{\vartheta}^{\text{prior}})$ and $\rho_{\text{osc}} \sim \text{GP}(\rho_{\text{osc}, \vartheta}^{\text{prior}}, \mathcal{K}_{\text{osc}, \vartheta}^{\text{prior}})$

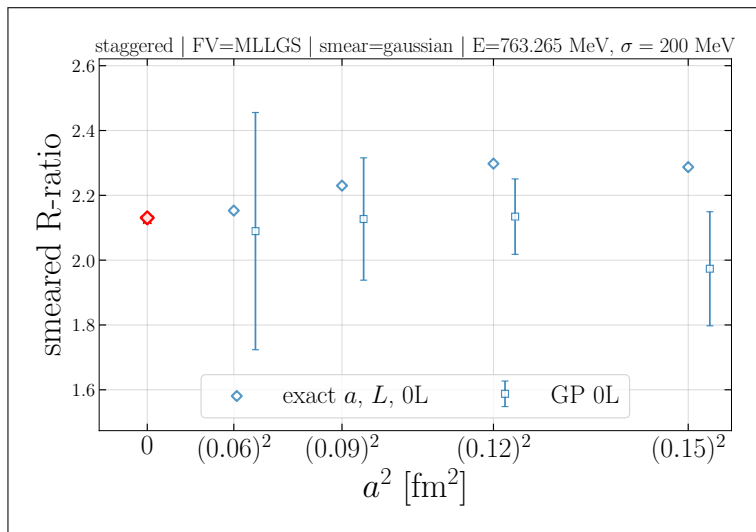
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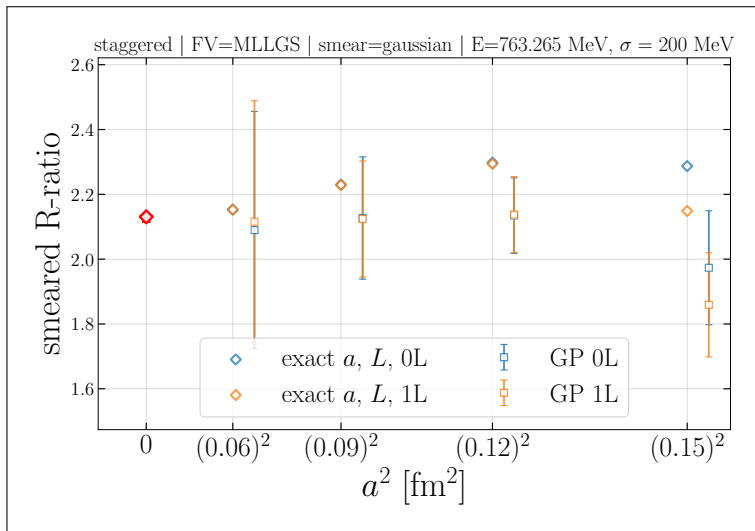


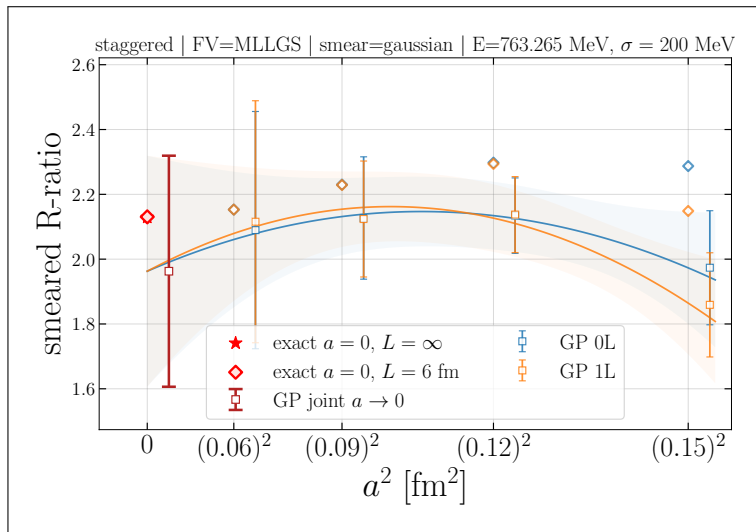


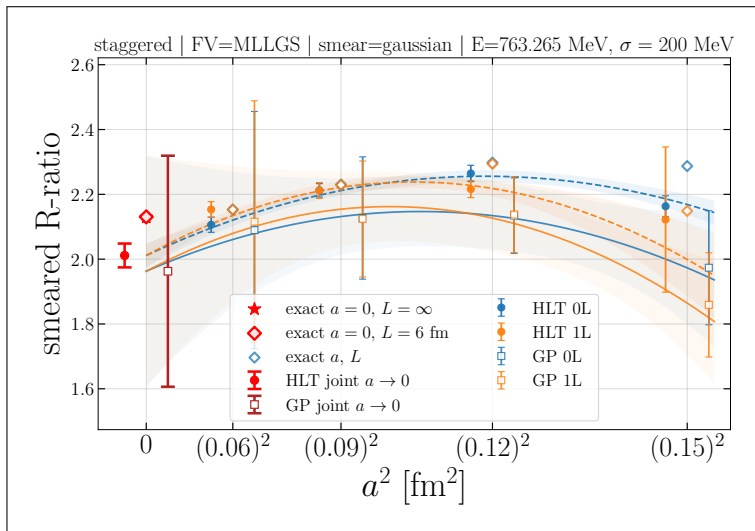












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1. Apply CB inversion strategy to **even-odd** data

Fails if not modified, as $\rho_{\text{odd}} = \rho - \rho_{\text{osc}}$ is not necessarily positive

2. Implement dedicated **staggered** strategies

Primal problem i.e. computing upper/lower bounds on $\rho[\kappa]$

$$\begin{aligned} \max/\min_{\rho, \rho_{\text{osc}}} \int_0^\infty d\omega \rho(\omega) \kappa(\omega) \\ \text{s.t. } \rho(\omega) \geq 0, \rho_{\text{osc}}(\omega) \geq 0 \\ \text{and } (C[\rho, \rho_{\text{osc}}] - C)^T \text{Cov}^{-1}(C[\rho, \rho_{\text{osc}}] - C) \leq \chi_0^2 \end{aligned}$$

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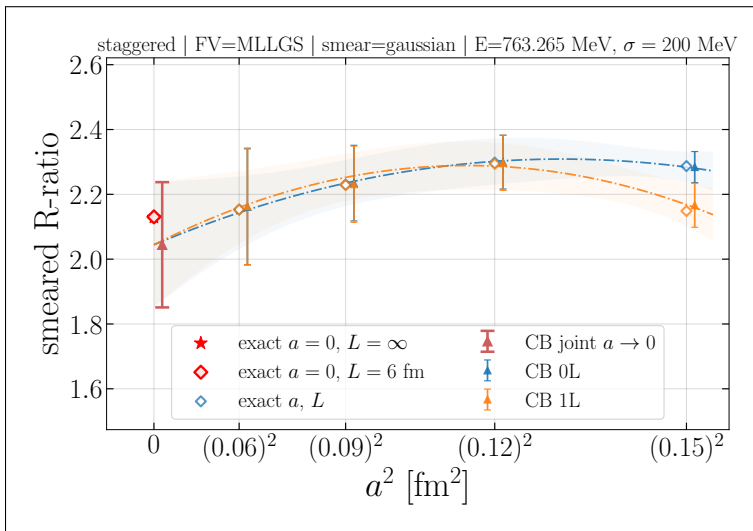
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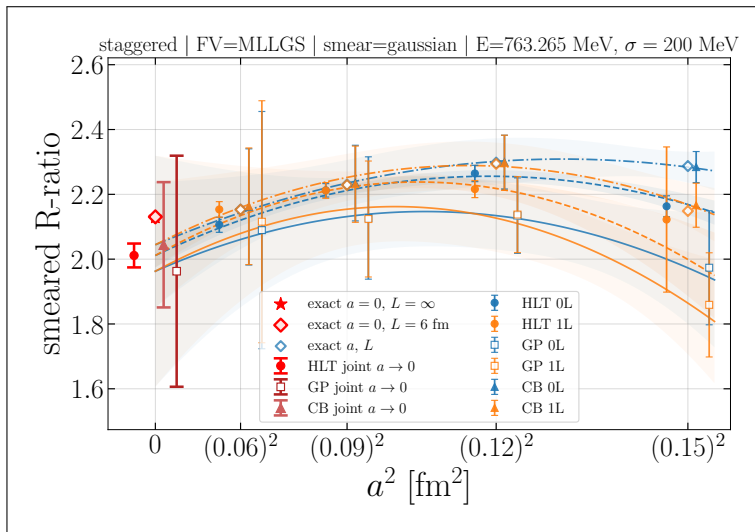
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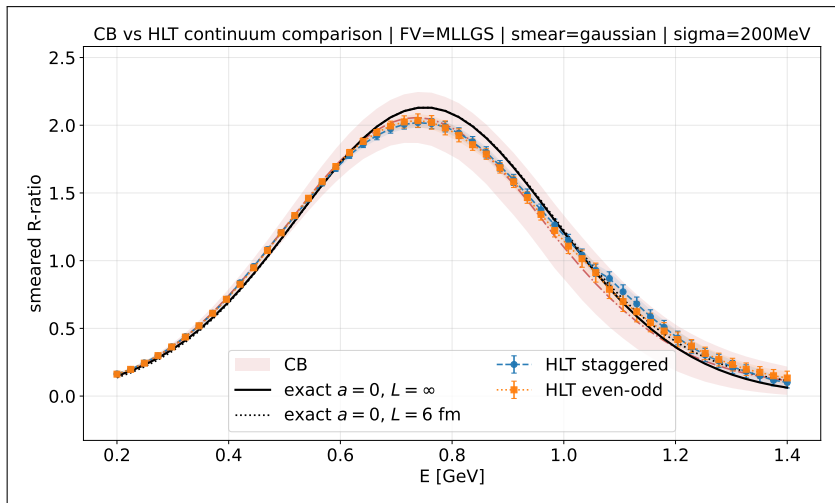
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Dual problem i.e. computing upper/lower bounds on κ and $\kappa_{\text{osc}} = 0$

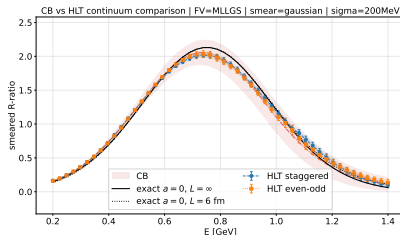
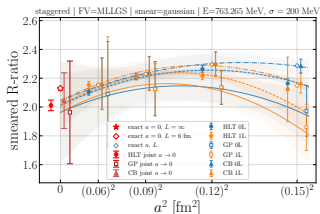
$$\begin{aligned} \max/\min_g g^T C - \chi_0 \sqrt{g^T \text{Cov} g} \\ \text{s.t. } \kappa(\omega) - \sum_n g_n b_n(\omega) \leq 0 \\ \text{and } \kappa_{\text{osc}}(\omega) - \sum_n g_n (-1)^n b_n(\omega) \leq 0 \end{aligned}$$





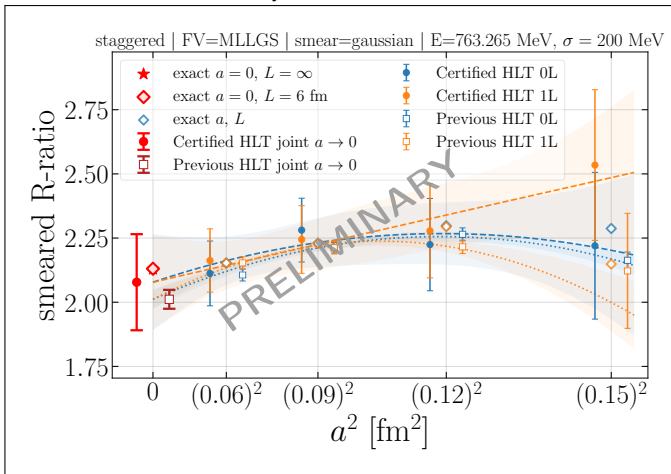


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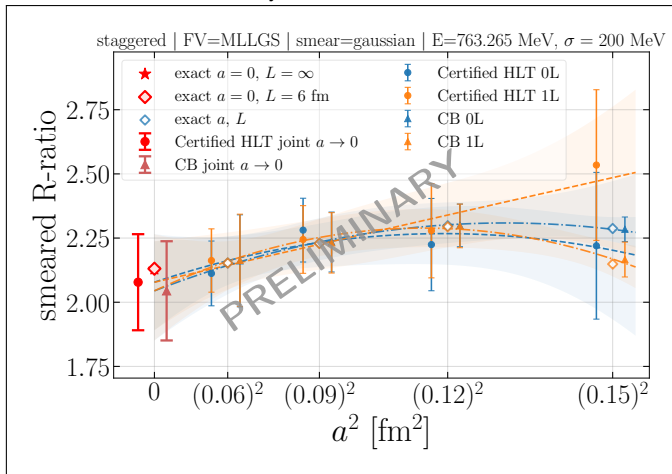


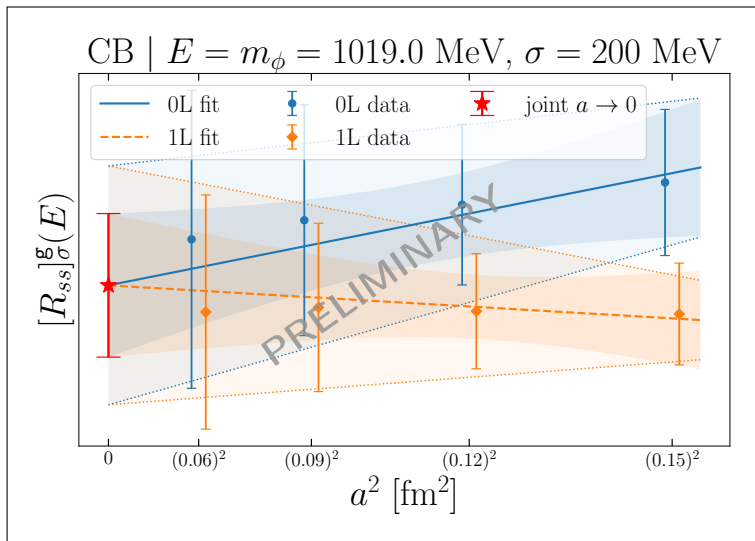
- ★ Staggered correlation functions require dedicated methods
 - Linear methods: HLT and GP
 - Causal bootstrap: certified bounds for $\rho(\omega) \geq 0$
- ★ Mock data with realistic FV and TB effects $\Rightarrow$ closure tests
- ★ Interesting questions remain open
 - Matrix-valued $C(t)$ (variational, e.g. $\pi\pi$) [Abbott Mon 15:00]
 - Choice of smearing kernels [Jay Mon 15:20]
 - Finite-volume corrections [Bresciani Tue 15:00]
 - Continuum limit: at fixed $t_{\min}$ [MS and Toniolo Tue 14:00–14:20]

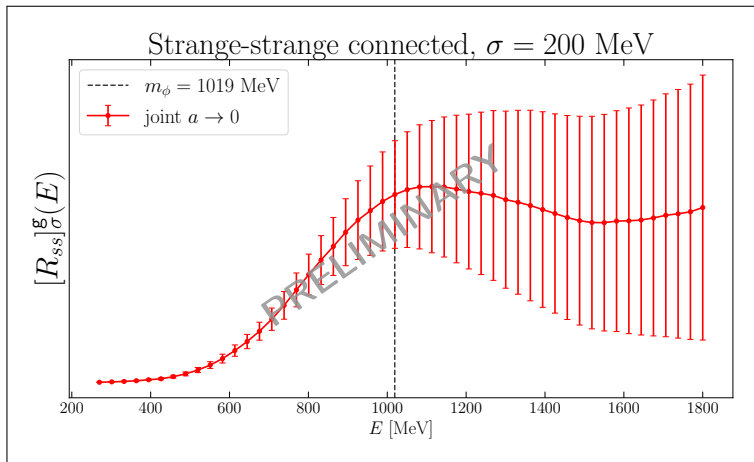
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2. Bound $\rho[\kappa] - \rho[\bar{\kappa}] \leq \bar{\sigma}_{\text{syst}}[\rho[\kappa]]$ from $C(t)$ [Jay and MS 2026]
3. Tune λ balancing e.g. $\bar{\sigma}_{\text{syst}}[\rho[\kappa]] = \sigma_{\text{stat}}[\rho[\bar{\kappa}]]$



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Thank you for your attention!

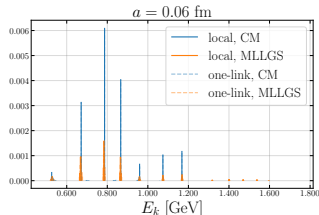
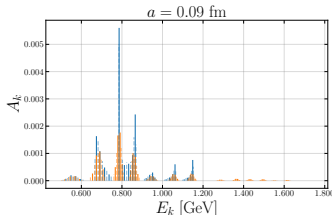
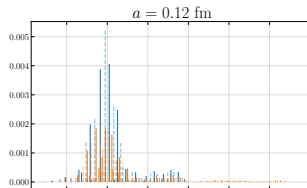
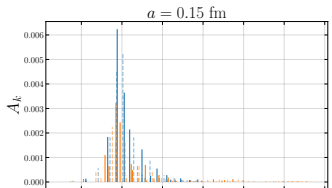
Backup slides

CLOSURE TESTS

Ensembles



a [fm]	L [fm]	$N_s^3 \times N_t$	am	aM_Ω
0.15	5.00	$32^3 \times 48$	0.002426	1.3246(26)
0.12	5.95	$48^3 \times 64$	0.0001907	1.0494(17)
0.09	5.70	$64^3 \times 96$	0.0001326	0.75372(97)
0.06	5.48	$96^3 \times 128$	0.0008	0.4834(11)



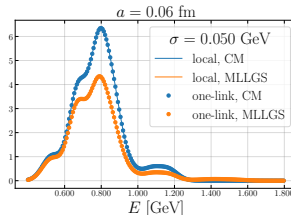
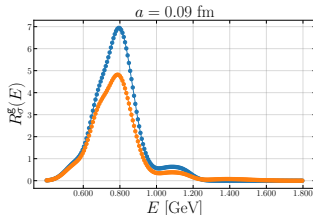
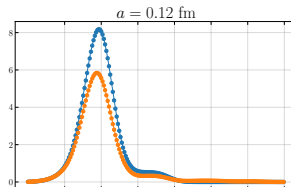
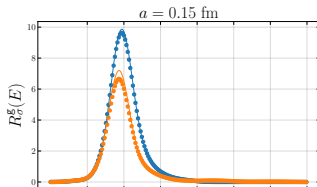
Finite-volume *unsmeared* R -ratio

CLOSURE TESTS

Ensembles



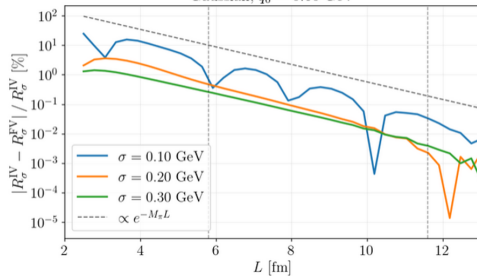
a [fm]	L [fm]	$N_s^3 \times N_t$	am	aM_Ω
0.15	5.00	$32^3 \times 48$	0.002426	1.3246(26)
0.12	5.95	$48^3 \times 64$	0.0001907	1.0494(17)
0.09	5.70	$64^3 \times 96$	0.0001326	0.75372(97)
0.06	5.48	$96^3 \times 128$	0.0008	0.4834(11)



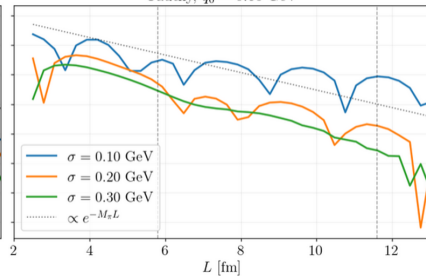
Finite-volume *smeared* R -ratio

CM model: relative FV correction $|R_\sigma^{\text{IV}} - R_\sigma^{\text{FV}}| / R_\sigma^{\text{IV}}$ vs L

Gaussian, $q_0 = 0.60$ GeV

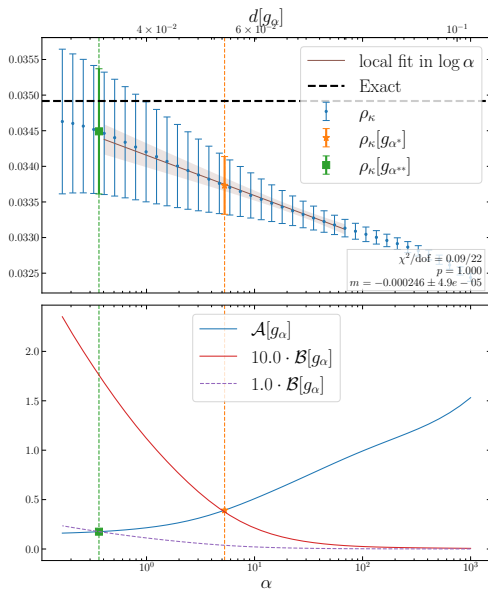


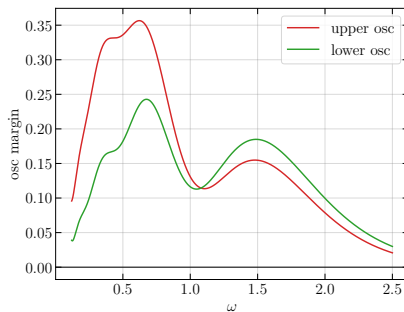
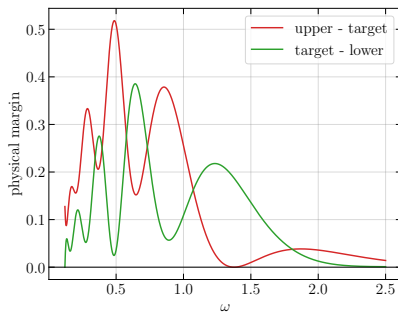
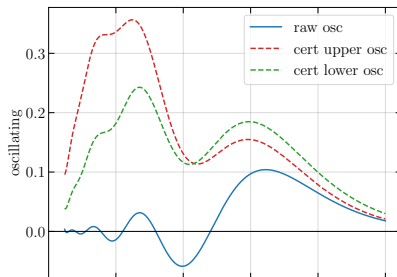
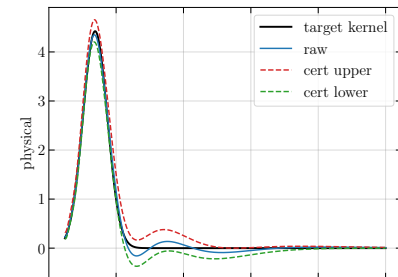
Cauchy, $q_0 = 0.60$ GeV



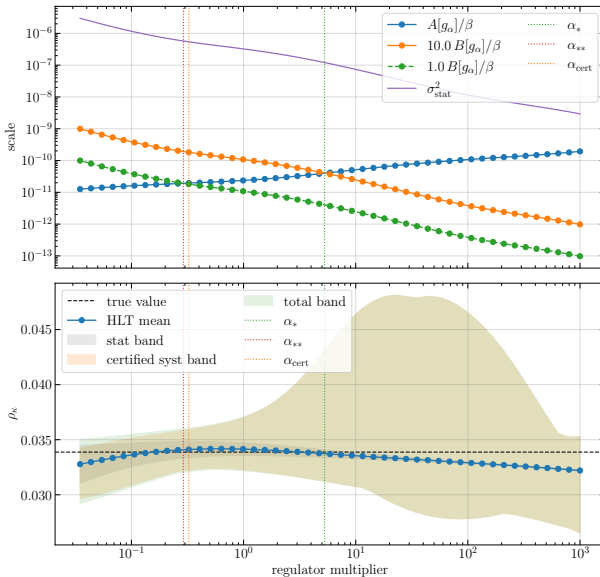
HLT STABILITY ANALYSIS

Tuning the regulator parameter

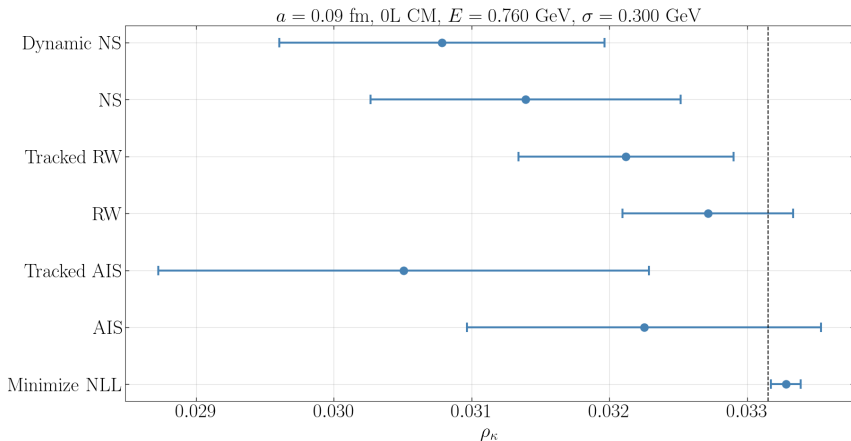




Certified staggered HLT: MLLGS 1L $a=0.09$ fm

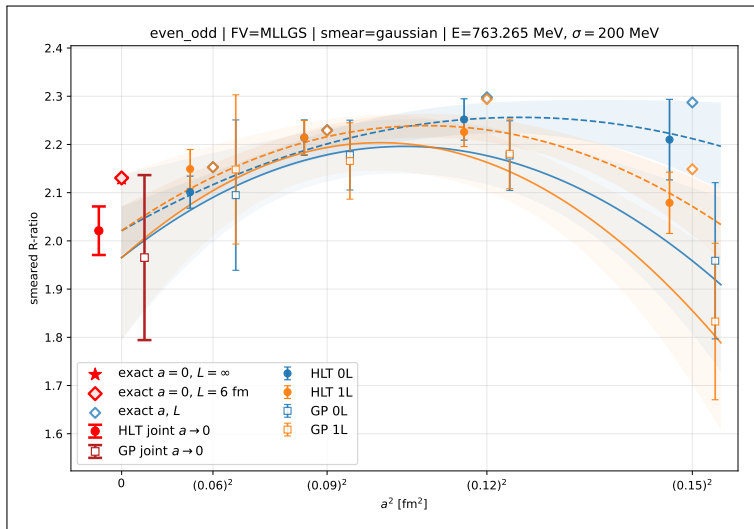


- ★ Nested Sampling (NS) [Skilling 2004; Skilling 2006]
- ★ Random walk (RW) MCMC
- ★ Affine-Invariant Sampling (AIS) [Goodman and Weare 2010]



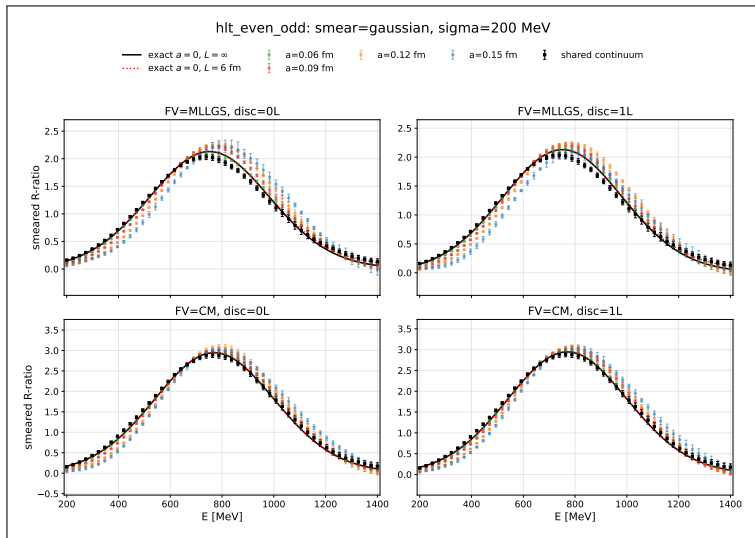
ANALYSIS PIPELINE

Pointwise extraction (even-odd)



ANALYSIS PIPELINE

HLT energy scan (even-odd)



ANALYSIS PIPELINE

CB energy scan

