Dyon Condensation at $\theta = 2\pi$ from Wilson–'t Hooft Loops in Yang–Mills Theory

Okuto Morikawa

with H. Watanabe, I. Kanamori, Y. Nagai, Y. Tanizaki, and A. Tomiya

iTHEMS, RIKEN

29 July 2026, Lattice 2026 @ University of Maryland

Based on arXiv:2606.13428

$\theta \to \theta + 2\pi$ shifts the confinement SPT

Witten effect

A magnetic object with charges (e, m) acquires

$$e_{\text{eff}} = e + \frac{\theta}{2\pi} m.$$

Thus a screened monopole becomes dyonic:

$$(e, m) = (0, 1) \quad \longrightarrow \quad (1, 1).$$

monopole $\longrightarrow$ dyon

[Witten '79]

1-form anomaly

For a $\mathbb{Z}_N^{[1]}$ background B,

$$\mathcal{Z}_{\theta+2\pi}[B] = \mathrm{e}^{-\mathrm{i} I_N[B]} \mathcal{Z}_\theta[B],$$

where

$$I_N[B] \equiv \frac{2\pi}{N} \int_{T^4} \frac{1}{2} B \cup B.$$

If the 1-form symmetry is unbroken, this factor shifts its discrete SPT label.

[Gaiotto et al. '17]

A phase transition must lie between $\theta = 0$ and 2π.

Line operators identify what condenses

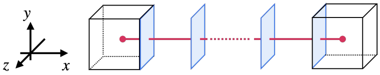
- $W(C)$ probes electric charge, $H(C,\Sigma)$ probes magnetic charge, and $HW(C,\Sigma)$ probes a dyon.
- Screening gives a perimeter law; an unscreened probe gives an area law.

Condensate	$\langle W\rangle$	$\langle H\rangle$	$\langle HW\rangle$	$\mathbb{Z}_2^{[1]}$ gap
Monopole	Area	Perimeter	Area	Trivial
Dyon	Area	Area*	Perimeter*	SPT

- Our target at $\theta = 2\pi$: test the two starred entries directly.
- The Wilson area law is already guaranteed by confinement.

['t Hooft '78, '79]

The 't Hooft loop is a magnetic lattice defect



- Couple the Wilson action to a background 2-form field $B_p \in \mathbb{Z}_2$:

$$S_{\mathrm{W}}[U, B] = -\frac{\beta}{2} \sum_p (-1)^{B_p} \operatorname{Re}\operatorname{tr} U_p.$$

- Choose an open dual surface $\widetilde{\Sigma}$:

$$B = \delta[\widetilde{\Sigma}], \qquad \mathrm{d}B = \delta[\widetilde{C}].$$

Here $\partial\widetilde{\Sigma} = \widetilde{C}$.

- The boundary of an $R \times N_{\mathrm{t}}$ sheet is a static monopole pair separated by R.

[arXiv:2606.13428]

At $\theta = 2\pi$, the hard part is a topological phase

For an electric observable $\mathcal{O}$, reweight the defect ensemble as

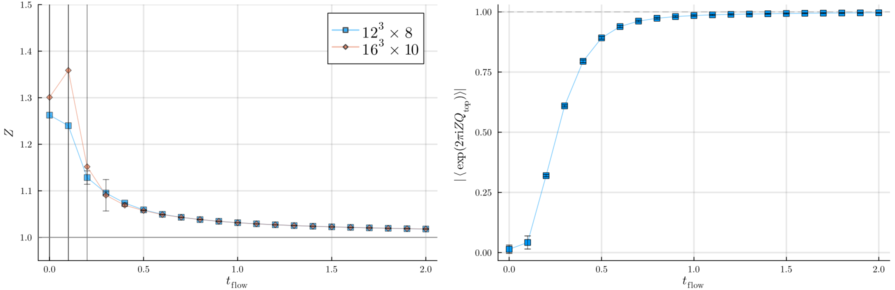
$$\langle H\mathcal{O}\rangle_\theta = \underbrace{\left(\frac{\mathcal{Z}_\theta}{\mathcal{Z}_0}\right)^{-1}}_{\text{global normalization}} \underbrace{\langle H\rangle_{\theta=0}}_{\text{known perimeter law [de Forcrand et al. '01]}} \times \underbrace{\left\langle \mathcal{O}\, \mathrm{e}^{\mathrm{i}\theta Z Q_{\mathrm{top}}[U,B]} \right\rangle_{S_{\mathrm{W}}[U,B]}}_{\text{determines the long-distance law at } \theta=2\pi} .$$

- Z rescales the lattice topological charge toward integer sectors when $B = 0$.
- The determining observables are

$$\begin{aligned} H: \quad \mathcal{O} &= 1, & \left\langle \mathrm{e}^{2\pi\mathrm{i}ZQ_{\mathrm{top}}} \right\rangle_B , \\ HW: \quad \mathcal{O} &= W, & \left\langle W\, \mathrm{e}^{2\pi\mathrm{i}ZQ_{\mathrm{top}}} \right\rangle_B . \end{aligned}$$

- With the defect, $Q_{\mathrm{top}}[U, B]$ is physically non-integer: rounding is not allowed.

DBW2 flow makes $\theta = 2\pi$ reweighting tractable



- Lattice definition: evaluate the tree-level-improved gluonic charge on links evolved by the 1-form-covariant DBW2 flow, $c_1 = -1.4088$.
- For $B = 0$, ZQ_{top} rapidly becomes almost integer and

$$\left| \frac{\mathcal{Z}_{2\pi}}{\mathcal{Z}_0} \right| = \left| \left\langle \mathrm{e}^{2\pi \mathrm{i} Z Q_{\mathrm{top}}} \right\rangle_{B=0} \right| \longrightarrow 1.$$

[A. Tomiya, preceding talk in this session] [arXiv:2606.13428]

Two confined ensembles probe long distances

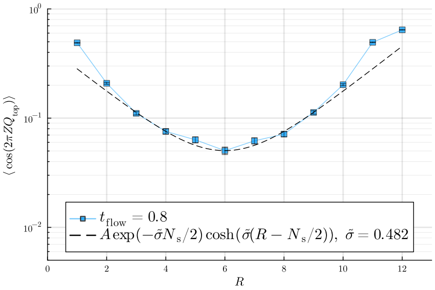
- Wilson plaquette action at $\beta = 2.45$; heat-bath updates; implementation built on JuliaQCD.
- $12^3 \times 8$: $R = 1, \dots, 12$, 40,000 configurations per R, $t_{\mathrm{flow}} = 0.8$.
- $16^3 \times 10$: $R = 6, \dots, 10$, 80,000 configurations per R, $t_{\mathrm{flow}} = 1.0$.
- Benchmark: the known perimeter law of H at $\theta = 0$ is reproduced.

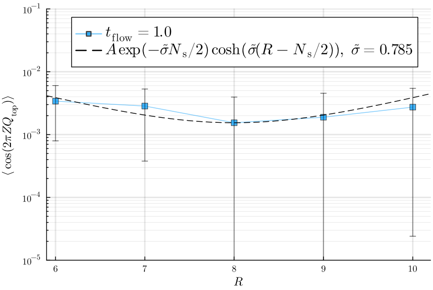
With diffusion length $d = \sqrt{8t_{\mathrm{flow}}}$, the test requires non-overlapping smeared probes:

$$2d \lesssim R \lesssim N_{\mathrm{s}} - 2d.$$

Thus the larger volume is essential: it supplies the useful window $6 \leq R \leq 10$.

The 't Hooft signal is consistent with the area law

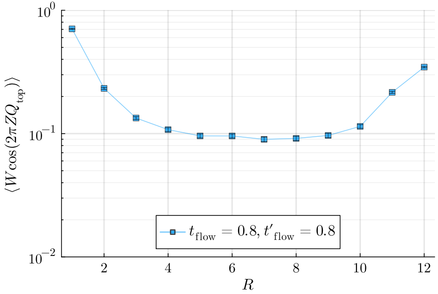


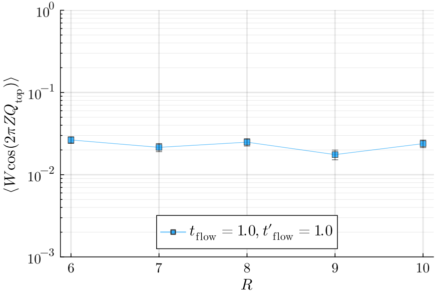


- Dashed curves: area-law eye guides using the Polyakov-loop mass gap; not fitted to these data.
- On $16^3 \times 10$, the area-law scale is 1–3 $\times 10^{-3}$, comparable to the statistical floor $N_{\text{conf}}^{-1/2} \simeq 3.5 \times 10^{-3}$.
- The observed smallness and R dependence are consistent with the area law.

[arXiv:2606.13428]

A clear dyonic plateau signals a perimeter law





- $12^3 \times 8$: weak R dependence for $4 \lesssim R \lesssim 10$.
- $16^3 \times 10$: a plateau throughout the analyzed region $6 \leq R \leq 10$.
- The dyonic signal remains well above the statistical floor: *HW* obeys the perimeter law at $\theta = 2\pi$.

[arXiv:2606.13428]

The screened line is dyonic at $\theta = 2\pi$

At $\theta = 0$		At $\theta = 2\pi$
H: perimeter law	$\Longrightarrow$	HW: perimeter law
screened $(e, m) = (0, 1)$		screened $(e, m) = (1, 1)$
monopole condensation		dyon condensation

- In a gapped Lorentz-invariant $SU(2)$ vacuum, only one of W, H, and HW can obey a perimeter law.
- The clear HW plateau therefore also supports the area law for H, even though the direct H signal is noise-limited.
- The screened line changes its $\mathbb{Z}_2$ electric N-ality: this is the $\mathbb{Z}_2^{[1]}$ SPT label.

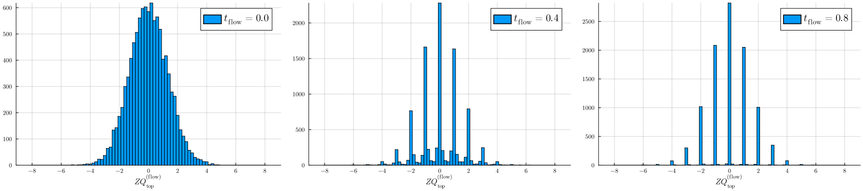
The data select dyon condensation at $\theta = 2\pi$

- A 1-form-covariant DBW2 flow makes the global $\theta = 2\pi$ reweighting factor numerically benign, without rounding Q_{top} in the defect ensemble.
- The dyonic loop *HW* shows a clear perimeter law on both volumes; the direct *H* signal lies at the expected area-law scale.
- The result gives direct numerical evidence that the confining vacua at $\theta = 0$ and 2π carry different $\mathbb{Z}_2^{[1]}$ SPT labels.

Toward further signal refinement

- Use geometries with a wider clean window and milder suppression, e.g. an anisotropic $16 \times 12^2 \times 8$ lattice.
- Define dyonic lines with Polyakov-loop correlators: no *R*-dependent renormalization factor from a spatial backtracking line.
- Extend to spatial loops and domain-wall tensions in the deconfined phase.

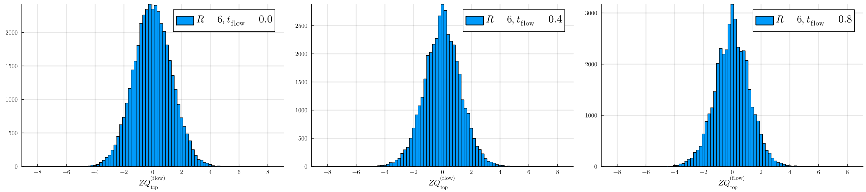
Backup: DBW2 flow resolves integer sectors



- $12^3 \times 8$ at $t_{\mathrm{flow}} = 0$, 0.4, and 0.8.
- The over-improved DBW2 choice satisfies $1 + 12c_1 < 0$ and stabilizes instantons against lattice-scale collapse.
- Nearly integer ZQ_{top} implies $\mathrm{e}^{2\pi \mathrm{i} ZQ_{\mathrm{top}}} \simeq 1$ for the normalization ensemble.

[arXiv:2606.13428]

Backup: the defect prevents integer quantization

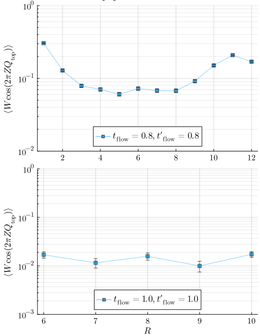


- $12^3 \times 8$ with an $R = 6$ 't Hooft loop at $t_{\mathrm{flow}} = 0$, 0.4, and 0.8.
- The non-flat background satisfies $\mathrm{d}B = \delta[\widetilde{C}] \neq 0$; no integer quantization is expected.
- The continuous phase fluctuations of $\mathrm{e}^{2\pi\mathrm{i}ZQ_{\mathrm{top}}[U,B]}$ are the physical discriminator between H and HW.

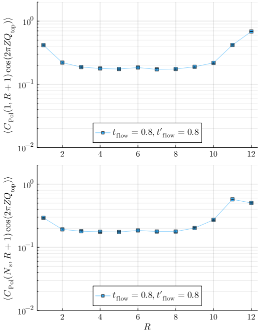
[arXiv:2606.13428]

Backup: operator checks of the dyonic plateau

Modified rectangular Wilson contour

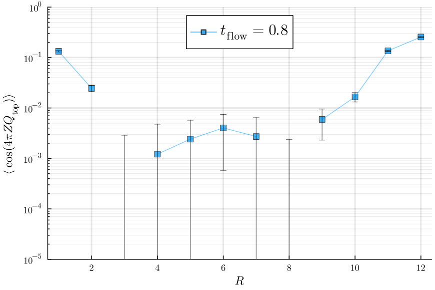


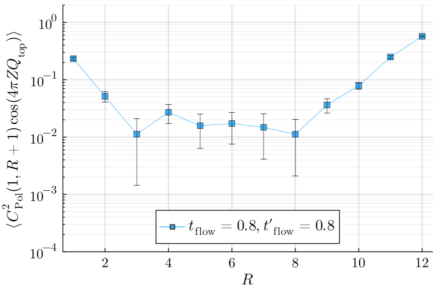
Fundamental Polyakov-loop pair



All definitions show a plateau in the long-distance window.

Backup: $\theta = 4\pi$ returns to the trivial SPT class





- For $SU(2)$, $\theta = 4\pi$ and 0 have the same $\mathbb{Z}_2^{[1]}$ SPT label.
- The bare H signal is again tiny; the Witten effect induces adjoint electric charge.
- After adjoint Wilson-line dressing, the middle-R data indicate the expected perimeter law.

[arXiv:2606.13428]