# Reconstructing spectral densities from integral transforms I

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Based on [Bruno, Giusti and MS PRD **111**, 094515 (2025), Giusti, MS and Toniolo arXiv:2606.28167 and in prep]

# Outline

- ⋆ **Introduction & Motivations**
- ⋆ **Spectral densities from integral transforms**
- ⋆ **The *continuum* case**
- ⋆ **The *discrete* case**
- ⋆ **Conclusions & Outlook**

$\Rightarrow$ **Introduction & Motivations**

$\star$ Spectral densities from integral transforms

$\star$ The *continuum* case

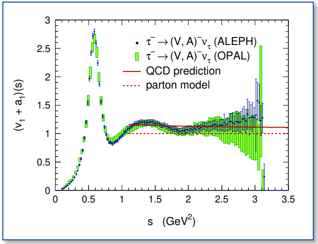
$\star$ The *discrete* case

$\star$ Conclusions & Outlook

# (SMEARED) SPECTRAL DENSITIES
## A non-perturbative probe of QCD dynamics

(Smeared) spectral densities play a pivotal role in particle physics

- ⋆ Inclusive hadronic cross sections
  e.g. $R = \frac{\sigma^{(0)}(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$
- ⋆ Semileptonic decays
  e.g. $\Gamma(\tau \to \bar{\nu}_\tau X)$
- ⋆ QGP, $(g-2)_\mu^{\text{HVP}}$, …

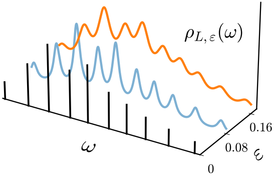



[Davier et al. 2006]

In Lattice QCD we measure $C(t) = \int_0^\infty d\omega \, e^{-\omega t} \rho(\omega)$

- ⋆ **Inverse problem** $\Longleftarrow$
- ⋆ Only smeared ($\varepsilon > 0$) quantities
  [Hansen, Meyer and Robaina 2017]

$$\rho_\varepsilon(E) = \int_0^\infty d\omega \, \delta_\varepsilon(\omega, E) \rho(\omega)$$
$$\delta_\varepsilon(\omega, E) = \frac{\varepsilon/\pi}{(\omega - E)^2 + \varepsilon^2}$$



$$C(t) = \int_0^\infty d\omega\, e^{-\omega t} \rho(\omega) \quad \overset{?}{\Longrightarrow} \quad \rho(\omega)\,,\ \rho_\kappa \equiv \int_0^\infty d\omega\, \rho(\omega) \kappa(\omega)$$

Many lattice-native approaches

- ⋆ Backus-Gilbert [Hansen, Meyer and Robaina 2017]
- ⋆ HLT [Hansen, Lupo and Tantalo 2019], Chebyshev [Bailas et al. 2020], ...
- ⋆ Analyticity/causality–based [Fields, Abbott, Jay Mon 14:40–15:20]

Here:

1. Target continuum definition of $\rho(\omega)$ from $C(t)$, $t \in [t_0, \infty)$
2. Discretized solutions from $C(t_0 + an)$, $n \in \mathbb{N}$
3. Analytic control of convergence rate to the continuum limit

- $\star$ Introduction & Motivations
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$$C(t) = \int_0^\infty d\omega\, e^{-\omega t} \rho(\omega)$$

- Define

$$\overline{\mathrm{C}}_s \equiv \int dt\, C(t) \overline{\mathrm{u}}_s(t)\,, \quad \hat{\rho}_s \equiv \int_0^\infty d\omega\, \rho(\omega) \hat{\mathrm{u}}_s(\omega)$$

- If bases functions are complete, orthonormal and satisfy

$$\lambda_s \hat{\mathrm{u}}_s(\omega) = \int dt\, \overline{\mathrm{u}}_s(t) e^{-\omega t}\,, \quad \int dt\, \frac{1}{t+t'} \overline{\mathrm{u}}_s(t) = |\lambda_s|^2 \overline{\mathrm{u}}_s(t')$$

then

$$\boxed{\overline{\mathrm{C}}_s \;=\; \lambda_s \hat{\rho}_s}$$

$\Longrightarrow$ **Algebraic inverse problem** $\Longleftarrow$

$$\boxed{\rho(\omega) = \int ds\, \hat{\mathrm{u}}_s^*(\omega)\hat{\rho}_s}$$

$$\boxed{\rho(\omega) = \int ds\, \hat{\mathrm{u}}_s^*(\omega)\frac{\overline{\mathrm{C}}_s}{\lambda_s}}$$

$\Longrightarrow$ from $\overline{\mathrm{C}}_s = \int dt\, C(t)\overline{\mathrm{u}}_s(t) = \lambda_s\hat{\rho}_s$ we can compute $\rho(\omega)$

$$\boxed{\rho(\omega) = \int ds\, \hat{\mathrm{u}}_s^*(\omega) \frac{\overline{\mathrm{C}}_s}{\lambda_s}}$$

$\Longrightarrow$ from $\overline{\mathrm{C}}_s = \int dt\, C(t) \overline{\mathrm{u}}_s(t) = \lambda_s \hat{\rho}_s$ we can compute $\rho(\omega)$, $\rho_\kappa$

- Smearing is straightforward

$$\rho_\kappa = \int_0^\infty d\omega\, \kappa(\omega) \rho(\omega) = \int ds\, \hat{\kappa}_s^* \frac{\overline{\mathrm{C}}_s}{\lambda_s}\,, \quad \hat{\kappa}_s^* = \int_0^\infty d\omega\, \kappa(\omega) \hat{\mathrm{u}}_s^*(\omega)$$

$$\boxed{\rho(\omega) = \int ds\, \hat{\mathrm{u}}^*_s(\omega)\frac{\overline{\mathrm{C}}_s}{\lambda_s}}$$

$\Longrightarrow$ from $\overline{\mathrm{C}}_s = \int dt\, C(t)\overline{\mathrm{u}}_s(t) = \lambda_s \hat{\rho}_s$ we can compute $\rho(\omega)$, $\rho_\kappa$

- ⋆ Smearing is straightforward

$$\rho_\kappa = \int_0^\infty d\omega\, \kappa(\omega)\rho(\omega) = \int ds\, \hat{\kappa}^*_s \frac{\overline{\mathrm{C}}_s}{\lambda_s}\,, \quad \hat{\kappa}^*_s = \int_0^\infty d\omega\, \kappa(\omega)\hat{\mathrm{u}}^*_s(\omega)$$

- ⋆ UV subtractions *must* be treated separately

$$\rho_m(\omega) = \frac{\rho(\omega)}{\omega^m}\,, \quad C_m(t) = \int_0^\infty d\omega\, e^{-\omega t}\rho_m(\omega)$$

$$C(t) = (-\partial_t)^m C_m(t) \to C_m(t) = \frac{1}{\Gamma(m)}\int_t^\infty dt'\, C(t')(t-t')^{m-1}$$

Idea: $C(t) \mapsto C_m(t) \mapsto \hat{\rho}_{m,s} = \frac{\overline{\mathrm{C}}_{m,s}}{\lambda_s} \mapsto \rho_m(\omega)$ $(\mapsto \rho_{\kappa,m})$

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- Mellin solution from $C(t)$ on $t \in (0, \infty)$ [Bruno, Giusti and MS 2025]
- $C(t)$ affected by large discretization errors at $t \approx 0$:

$$C(t) = \int_0^\infty d\omega \, e^{-\omega t} \rho(\omega) \,, \quad t \in [t_0, \infty)$$

- Introduce the basis functions
  [Mehler 1881; Fock 1943; Kontorovich and Lebedev 1939; Lebedev 1972]

$$\overline{\mathrm{u}}_s(t) \equiv \overline{u}_s(t, t_0) = \sqrt{\frac{s \tanh(\pi s)}{t_0}} P_{-\frac{1}{2}+is}\left(\frac{t}{t_0}\right)$$

$$\hat{\mathrm{u}}_s(\omega) \equiv \hat{u}_s(\omega, t_0) = \frac{\sqrt{2s \sinh(\pi s)}}{\pi} \frac{K_{is}(\omega t_0)}{\sqrt{\omega}}$$

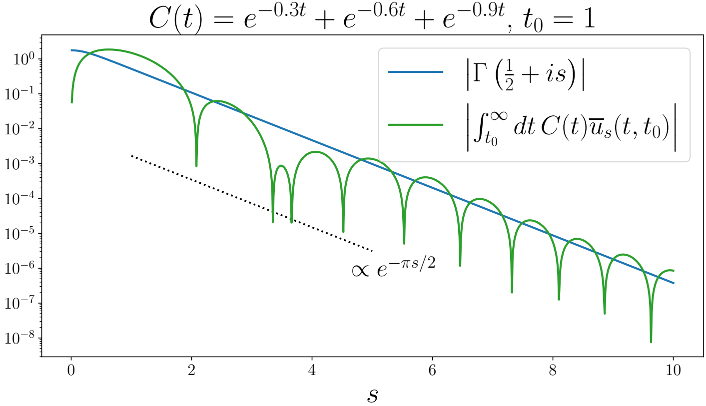
- They satisfy

$$\int_{t_0}^\infty dt \, \overline{u}_s(t, t_0) e^{-\omega t} = \left| \Gamma\left(\tfrac{1}{2} + is\right) \right| \hat{u}_s(\omega, t_0)$$

$$\int_{t_0}^\infty dt \, \frac{1}{t + t'} \overline{u}_s(t, t_0) = \left| \Gamma\left(\tfrac{1}{2} + is\right) \right|^2 \overline{u}_s(t', t_0)$$

Delicate cancellations are at play to recover the exact solution

$$\rho(\omega) = \int_0^\infty ds\, \hat{u}_s(\omega, t_0) \frac{\int_{t_0}^\infty dt\, C(t) \overline{u}_s(t, t_0)}{\left| \Gamma\left(\frac{1}{2} + is\right) \right|}$$



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$$\mathcal{C}(t_0 + na) = \int_0^\infty d\omega \, e^{-\omega(t_0+na)} \varrho(\omega) \,, \quad n \in \mathbb{N}$$

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$$\mathcal{C}(t_0+na) = \int_0^\infty d\omega\, e^{-\omega(t_0+na)} \varrho(\omega)\,, \quad n \in \mathbb{N}$$

⋆ Introduce the basis functions
[Magnus 1950; Rosenblum 1958; Hill 1960]

$$\overline{\mathrm{u}}_s(t) \equiv \overline{v}_s(n,t_0,a) \propto \sqrt{\frac{s\tanh(\pi s)}{t_0}}\, {}_3F_2\left(-n, \tfrac{1}{2}+is, \tfrac{1}{2}-is; 1, 2t_0/a; 1\right)$$

$$\hat{\mathrm{u}}_s(\omega) \equiv \hat{v}_s(\omega,t_0,a) \propto \frac{\sqrt{2s\sinh(\pi s)}}{\pi}\, {}_2F_1\left(\tfrac{1}{2}+is, \tfrac{1}{2}-is; 2t_0/a; -\tfrac{e^{-a\omega}}{1-e^{-a\omega}}\right)$$

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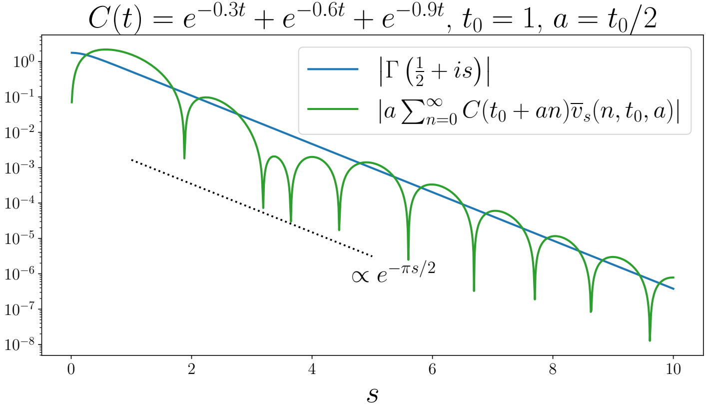
- They satisfy

$$a\sum_{n=0}^{\infty} \overline{v}_s(n,t_0,a) e^{-\omega(t_0+an)} = \left|\Gamma\left(\tfrac{1}{2}+is\right)\right| \hat{v}_s(\omega,t_0,a)$$

$$a\sum_{n=0}^{\infty} \frac{1}{a(n+n')+2t_0} \overline{v}_s(n,t_0,a) = \left|\Gamma\left(\tfrac{1}{2}+is\right)\right|^2 \overline{v}_s(n',t_0,a)$$

Delicate cancellations are at play to recover the exact solution

$$\varrho(\omega) = \int_0^\infty ds\, \hat{v}_s(\omega, t_0, a) \frac{a \sum_{n=0}^\infty \mathcal{C}(t_0 + na)\overline{v}_s(n, t_0, a)}{\left|\Gamma\left(\frac{1}{2} + is\right)\right|}$$



Basis terms have $O(a)$-effects in the continuum limit...

$$\overline{v}_s(n, t_0, a) = \left\{ 1 + \frac{a}{4t_0} \left[1 + 2t\partial_t\right] \right\} \overline{u}_s(t, t_0) + O(a^2)$$

$$\hat{v}_s(\omega, t_0, a) = \left\{ 1 - \frac{a}{4t_0} \left[1 + 2\omega\partial_\omega\right] \right\} \hat{u}_s(\omega, t_0) + O(a^2)$$

Basis terms have $O(a)$-effects in the continuum limit...

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... but they combine to yield $O(a^2)$ effects, if $\mathcal{C}$ is $O(a)$-improved

$$\varrho(\omega) = \int_0^\infty ds\, \hat{v}_s(\omega, t_0, a)\frac{a\sum_{n=0}^{\infty}\mathcal{C}(t_0 + na)\overline{v}_s(n, t_0, a)}{\left|\Gamma\left(\frac{1}{2} + is\right)\right|}$$

$$\xrightarrow{a\to 0} \int_0^\infty ds\, \hat{u}_s(\omega, t_0)\frac{\int_{t_0}^{\infty} dt\, C(t)\overline{u}_s(t, t_0)}{\left|\Gamma\left(\frac{1}{2} + is\right)\right|} \qquad \mapsto \rho(\omega)$$

$$+ \frac{a}{2t_0}\int_0^\infty d\omega' \cancel{\left\{\int_0^\infty ds\, \hat{u}_s(\omega, t_0)\hat{u}_s(\omega', t_0) - \delta(\omega - \omega')\right\}}\left[\omega'\partial_{\omega'}\rho(\omega')\right] \qquad \mapsto 0$$

$$+ O(a^2)$$

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$$C(t) = \int_0^\infty d\omega\, e^{-\omega t} \rho(\omega)$$

$$\overline{\mathrm{C}}_s \equiv \int dt\, C(t) \overline{\mathrm{u}}_s(t)\,, \quad \hat{\rho}_s \equiv \int_0^\infty d\omega\, \rho(\omega) \hat{\mathrm{u}}_s(\omega)\,, \quad \int dt\, \overline{\mathrm{u}}_s(t) e^{-\omega t} = \lambda_s \hat{\mathrm{u}}_s(\omega)$$

$$\boxed{\overline{\mathrm{C}}_s \;=\; \lambda_s \hat{\rho}_s\,, \quad \rho(\omega) \;=\; \int ds\, \hat{\mathrm{u}}_s^*(\omega) \frac{\overline{\mathrm{C}}_s}{\lambda_s}}$$

- ⋆ Integral equations become algebraic equations
- ⋆ Delicate cancellations in $\frac{\overline{\mathrm{C}}_s}{\lambda_s}$ must be at play for exact solutions
- ⋆ Integral transforms provide target continuum formulas to extract spectral densities from Euclidean correlators $C(t)$
- ⋆ Discrete solutions from $O(a)$-improved correlators are $O(a)$-improved
- ⋆ Generalizable to smeared and/or regulated spectral functions

# Outlook

## Applications to realistic scenarios [Toniolo Tue 14:20]

- ⋆ Effects of finite temporal/spatial extents studied in the continuum
- ⋆ Applications require $t \leq t_{\text{max}}$
  - ⇒ Reliable reconstructions of $\overline{\mathsf{C}}_s$ only up to $|s| = s_{\text{max}} \propto t_{\text{max}}$
  - ⇒ This spoils the delicate cancellations in $\frac{\overline{\mathsf{C}}_s}{\lambda_s}$ at large $|s|$
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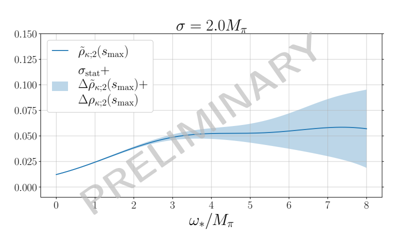
$N_f = 2$ multi-level (two-level) data from
[Dalla Brida, Giusti, Harris and Pepe 2020]
at $a = 0.065$ fm, $96 \times 48^3$, $M_\pi = 270$ MeV

$$\mathcal{C}(an) = -\frac{a^3}{3} \sum_{\vec{n},k} \langle V_k^{ud}(an, a\vec{n}) V_k^{du}(0) \rangle$$

$$\kappa(\omega) = \exp\left\{-\frac{(\omega-\omega_*)^2}{2\sigma^2}\right\} \Big/ \sqrt{2\pi\sigma^2}$$



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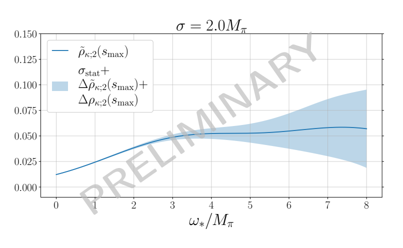



Thank you for your attention!

# Backup slides

# Properties of $C(t)$ and $\rho(\omega)$

$$
\begin{aligned}
C(t) &= \int d^3\vec{x}\, \langle O_1(t,\vec{x}) O_2(0,\vec{0}) \rangle_c \\
&= \int d^3\vec{x} \int \frac{d^4 p}{(2\pi)^4} e^{ipx} \int_0^\infty ds\, \frac{\rho(\sqrt{s})}{p^2+s} \\
[s=\omega^2] \rightarrow &= \int_0^\infty d\omega\, e^{-\omega|t|} \rho(\omega)
\end{aligned}
$$

UV

$$\lim_{t \to 0^+} C(t) = O(t^{-p})$$

$$\lim_{\omega \to +\infty} \rho(\omega) = O(\omega^{p-1})$$

IR

$$\lim_{t \to +\infty} C(t) = O(e^{-\omega_{\text{thr}} t})$$

$$\rho(\omega)\Big|_{\omega \in [0,\omega_{\text{thr}})} = 0$$

$$\int_0^\infty dt\, u_s(t) e^{-\omega t} = \Gamma\left(\tfrac{1}{2} + is\right) u_s^*(\omega)\,, \quad u_s(x) = \frac{e^{is\log(x)}}{\sqrt{2\pi x}}$$

Basis in time: if also $C \in L^2(0,\infty)$

$$\int_0^\infty dt\, \mathcal{A}(t+t') u_s(t) = \left|\Gamma\left(\tfrac{1}{2} + is\right)\right|^2 u_s^*(t')$$

$$\mathcal{A}(t+t') = \int_0^\infty d\omega\, e^{-\omega(t+t')} = \frac{1}{t+t'}$$

Basis in energy space: if also $\rho \in L^2(0,\infty)$

$$\int_0^\infty d\omega\, \mathcal{H}(\omega,\omega') u_s(\omega) = \left|\Gamma\left(\tfrac{1}{2} + is\right)\right|^2 u_s(\omega')$$

$$\mathcal{H}(\omega+\omega') = \int_0^\infty dt\, e^{-t(\omega+\omega')} = \frac{1}{\omega+\omega'}$$

# Integral transforms for $t \in (0, \infty)$
## The Mellin transform

1. Simplest case:

$$C(t) = \int_0^\infty d\omega\, e^{-\omega t} \rho(\omega)\,, \quad t \in (0, \infty)$$

2. The *Mellin transform* diagonalizes Laplace:

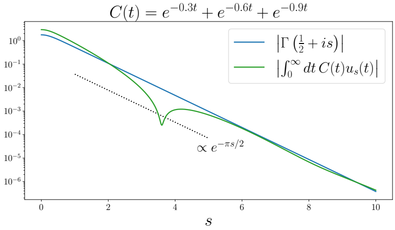
$$\int_0^\infty dt\, u_s(t) e^{-\omega t} = \Gamma\left(\tfrac{1}{2} + is\right) u_s^*(\omega)\,, \quad u_s(x) = \frac{e^{is\log(x)}}{\sqrt{2\pi x}}\,, \quad s \in \mathbb{R}$$

$$\iff \int_0^\infty dt\, \frac{1}{t + t'} u_s(t) = \left|\Gamma\left(\tfrac{1}{2} + is\right)\right|^2 u_s(t')$$

3. Construct the algebraic equation and invert

$$\int_0^\infty dt\, u_s(t) C(t) = \Gamma\left(\tfrac{1}{2} + is\right) \int_0^\infty d\omega\, \rho(\omega) u_s^*(\omega)$$

$$\rho(\omega) = \int_{-\infty}^\infty ds\, u_s(\omega) \frac{\int_0^\infty dt\, C(t) u_s(t)}{\Gamma\left(\tfrac{1}{2} + is\right)}$$



# The Mehler-Fock transform

$$\overline{u}_s(t,t_0) = \sqrt{\frac{s\tanh(\pi s)}{t_0}}\, P_{-\frac{1}{2}+is}(t/t_0)$$

$$\int_{t_0}^{\infty} dt\, \mathcal{A}(t+t')\overline{u}_s(t,t_0) = \left|\Gamma\left(\frac{1}{2}+is\right)\right|^2 \overline{u}_s(t',t_0)$$

⋆ Time basis if $u < 1/4$ and $v > 1/2$ with

$$\lim_{t\to t_0^+} C(t) = O((t-t_0)^{-u}), \quad \lim_{t\to+\infty} C(t) = O(t^{-v})$$

$$\hat{u}_s(\omega, t_0) = \frac{\sqrt{2s\sinh(\pi s)}}{\pi} \frac{K_{is}(\omega t_0)}{\sqrt{\omega}}$$

$$\int_0^\infty d\omega\, \mathcal{H}_{t_0}(\omega, \omega')\hat{u}_s(\omega, t_0) = \left|\Gamma\left(\tfrac{1}{2} + is\right)\right|^2 \hat{u}_s(\omega', t_0)$$

$$\mathcal{H}_{t_0}(\omega + \omega') = \int_{t_0}^\infty dt\, e^{-t(\omega+\omega')} = \frac{e^{-t_0(\omega+\omega')}}{\omega + \omega'}$$

- ⋆ Energy basis if $u < 1/2$ and $v > 1$ with

$$\lim_{\omega\to 0^+} \rho(\omega) = O(\omega^{-u})\,, \qquad \lim_{\omega\to+\infty} \rho(\omega) = O(\omega^{-v})$$

- ⋆ Note:

$$\int_{t_0}^\infty dt\, e^{-\omega t}\overline{u}_s(t, t_0) = \left|\Gamma\left(\tfrac{1}{2} + is\right)\right| \hat{u}_s(\omega, t_0)$$

# Discrete bases

Basis in discrete time space: if $\{C(t_0+an)\}_n \in \ell^2(\mathbb{N})$

$$a\sum_{n=0}^{\infty}\mathcal{A}(a(n+n')+2t_0)\overline{v}_s(n,t_0,a)=\left|\Gamma\left(\tfrac{1}{2}+is\right)\right|^2\overline{v}_s(n',t_0,a)$$

$$\overline{v}_s(n,t_0,a)=\sqrt{\frac{s\tanh(\pi s)}{t_0}}\frac{|\Gamma(2t_0/a-1/2+is)|}{\sqrt{a/(2t_0)}\Gamma(2t_0/a)}\,{}_3F_2(-n,\tfrac{1}{2}+is,\tfrac{1}{2}-is;1,\tfrac{2t_0}{a};1)$$

Basis in energy space:

$$\int_0^{\infty}d\omega\,\mathcal{H}_{t_0,a}(\omega,\omega')\hat{v}_s(\omega,t_0,a)=\left|\Gamma\left(\tfrac{1}{2}+is\right)\right|^2\hat{v}_s(\omega',t_0,a)$$

$$\mathcal{H}_{t_0,a}(\omega+\omega')=a\sum_{n=0}^{\infty}e^{-(t_0+an)(\omega+\omega')}$$

$$\hat{v}_s(\omega,t_0,a)=\sqrt{\frac{s\tanh(\pi s)}{t_0}}\frac{|\Gamma(2t_0/a-1/2+is)|}{\sqrt{a/(2t_0)}\Gamma(2t_0/a)}$$

$$\times\sqrt{\frac{\pi}{2t_0}}\frac{ae^{-\omega t_0}}{1-e^{-a\omega}}\,{}_2F_1\left(\tfrac{1}{2}+is,\tfrac{1}{2}-is;1,\tfrac{2t_0}{a};-\frac{e^{-\omega a}}{1-e^{-a\omega}}\right)$$

Note: $a\sum_{n=0}^{\infty}e^{-\omega(t_0+an)}\overline{v}_s(n,t_0,a)=\left|\Gamma\left(\tfrac{1}{2}+is\right)\right|\hat{v}_s(\omega,t_0,a)$

$$C(t) = \int_0^\infty d\omega\, e^{-\omega t} \omega^m \rho_m(\omega) \quad \Longrightarrow \quad \rho_m(\omega) = \int ds\, \hat{u}_s^*(\omega) \hat{\rho}_{m,s}$$

UV subtractions *must* be treated separately: only $\hat{\rho}_{m,s}$ may exist

1. **Reconstruct unsubtracted $C$**

$$C(t) = (-1)^m \partial_t^m C_m(t)$$

$$C_m(t) = \int_0^\infty d\omega\, e^{-\omega t} \rho_m(\omega) = \frac{1}{\Gamma(m)} \int_t^\infty dt'\, C(t')(t - t')^{m-1}$$

i. Reconstruct $C_m(t)$ from $C(t)$

ii. Reconstruct $\rho_m$ from $\hat{\rho}_{m,s} = \frac{\overline{C}_{m,s}}{\lambda_s}$

2. **Reconstruct unsubtracted $\rho$**

$$C_{(m)}(t) = C(t) t^m \equiv \int_0^\infty d\omega\, e^{-\omega t} \rho_{(m)}(\omega) = \int_0^\infty d\omega\, e^{-\omega t} (-\partial_\omega)^m \rho_m(\omega)$$

$$\rho_m(\omega) = \int_0^\omega d\omega'\, \frac{(\omega - \omega')^{m-1}}{\Gamma(m)\omega^m} \rho_{(m)}(\omega')$$

i. Reconstruct $\rho_{(m)}$ from $C_{(m)}$

ii. Reconstruct $\rho_m$ from $\rho_{(m)}$

$\Rightarrow$ *Unsubtracted* $(m = 0)$ spectral reconstructions are all we need

# Regulated spectral densities

## Relative errors of discretized strategy 1 in the continuum limit