A calculation of the time-like pion form factor via an infinite-volume approach

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Outline

What do we study ?

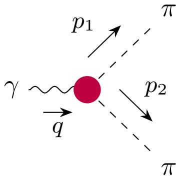
the pion electromagnetic form factor

- The pion form factor F_π is a physical quantity which describes the interaction between the photon and two pions.

Time-like regime, defined by:

$$\langle 0 | j_\mu(q) | \pi\pi(\mathbf{p}_1 + \mathbf{p}_2)\rangle = (p_1 - p_2)_\mu \, F_\pi(q^2)\,,$$

$q \equiv p_1 + p_2$, $q^2 \geq 4m_\pi^2$, $j_\mu =$ **isospin-1** em current.



- Phenomenologically relevant for the HVP contribution to the muon $g-2$, the pion charge radius, properties of ρ meson to name a few.

- *Lattice QCD*: solid, rigorous and systematically improvable non-perturbative setup to calculate hadronic quantities from first principles simulations.

AIM: extract the time-like F_π across the **full** kinematically allowed energy range.

State of the art

finite-volume Lüscher approach

- Limited to **elastic regime**, $(2m_\pi)^2 \leq s \equiv q^2 \leq (4m_\pi)^2$ (isospin limit).

- Phase and modulus from finite-volume energies E_n and matrix elements $c_n(L)$ of j_μ:

  [Lüscher, 1986] [Lüscher, 1991] [Lellouch and Lüscher, 2001] [Meyer, 2011]

$$\delta_1(k_n) = n\pi - \phi(q_n = k_n L/2\pi)\,, \qquad E_n = 2\sqrt{m_\pi^2 + k_n^2}\,,$$

$$\left|F_\pi(E_n^2)\right|^2 = \{k_n \delta_1'(k_n) + q_n \phi'(q_n)\}\,\frac{3\pi E_n^2}{2k_n^5}\,|c_n(L)|^2\,.$$

(see e.g. [Feng et al., 2015] [Bulava et al., 2016] [Erben et al., 2020])

(see e.g. [Ortega-Gama, Dudek, and Edwards, 2024] for inclusion of $K\overline{K}$ channel)

- Watson's theorem:

$$F_\pi(s) = |F_\pi(s)| e^{i\delta_1(s)}\,, \quad \delta_1(s) = \text{P-wave } \pi\pi \text{ scattering phase shift}\,.$$

- Why elastic regime ? Quantization conditions known only for two and three particles.

  (e.g. [Polejaeva and Rusetsky, 2012] [Briceno and Davoudi, 2013] [Hansen and Sharpe, 2014])

Long-term goal

hadronic weak decays into multi-hadron final states

- The pion form factor serves as a **test bed** for application of inverse-like methods in QCD valid in all kinematic regions.

- **Far-reaching goal**: study hadronic weak decays with two-hadron final states $\Longrightarrow$ relevant for flavor physics (e.g. extraction of CKM matrix elements).

- Study long-time behavior of 4pt-correlator to extract matrix elements like

$$\langle B/D|\mathcal{H}_w(0)|\pi\pi\rangle \ ,$$

$\mathcal{H}_w(0) =$ effective weak Hamiltonian.

- Several open channels, multi-particle states are relevant. $\Longrightarrow$ quantization conditions are challenging. Try inverse-method techniques.

Lattice setup

physical pion mass ensembles

- Analysis on ensembles generated by the RBC/UKQCD collaborations. [Blum et al., 2016] [Blum et al., 2025]

- $N_f = 2+1$ chirally symmetric Möbius domain wall fermions, Iwasaki gauge action, **physical pion mass**.

- This work: fixed a but different spatial volumes.

Ensemble	a^{-1}/GeV	$L^3 \times T \times L_s/a^5$	m_π/MeV	m_K/MeV	$m_\pi L$
48I	$1.7312(28)$	$48^3 \times 96 \times 24$	$139.32(30)$	$499.44(88)$	3.9
C	$1.7312(28)$	$64^3 \times 96 \times 24$	$139.32(30)$	$499.44(88)$	5.2

3pt correlator

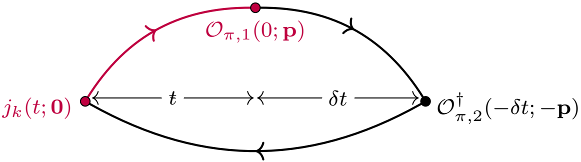
center-of-mass frame and spectral decomposition

Center-of-mass frame ($\mathbf{p}_1 = -\mathbf{p}_2 \equiv \mathbf{p}$), only spatial components survive:

$$\langle 0|j_k(q)|\pi\pi(\mathbf{0})\rangle = 2\mathbf{p}_k F_\pi(4\omega_{\mathbf{p}}^2)\,, \quad k = 1,2,3\,,$$

where $\omega_{\mathbf{p}} = \sqrt{m_\pi^2 + |\mathbf{p}|^2}$ and j_k = isospin-1 local lattice em current.

$C_k(t, \delta t; \mathbf{p}) \equiv$



$$C_k(t,\delta t;\mathbf{p})\frac{e^{E_\pi \delta t}}{Z_{\pi,2}^{1/2}(\mathbf{p})} = G_k(t;\mathbf{p}) + \mathcal{O}\Big(\underbrace{e^{-E_\pi(T-t-2\delta t)}}_{\text{thermal}}, \underbrace{e^{-(E_{3\pi}-E_\pi)\delta t}}_{\text{excited-state}}\Big)\,.$$

[Bulava and Hansen, 2019] [Bruno and Hansen, 2021]

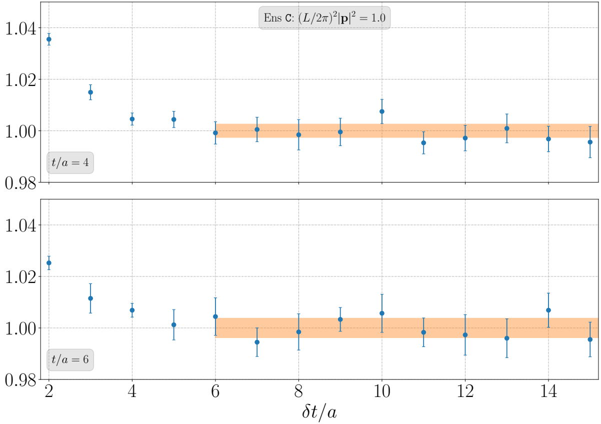
$$\Longrightarrow\ G_k(t;\mathbf{p}) \equiv \langle 0|j_k(t)\mathcal{O}_{\pi,1}(0)|\pi(\mathbf{p})\rangle, \quad (Z_{\pi,i}^{1/2}(\mathbf{p}) \equiv \langle \pi(\mathbf{p})|\mathcal{O}^\dagger_{\pi,i}(0)|0\rangle).$$

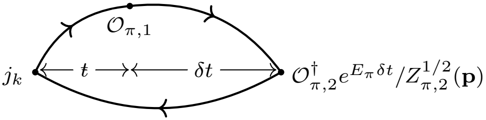
Extraction of $G_k(t;\mathbf{p})$

suppression of contaminations

Extraction of $G_k(t;\mathbf{p})$ by studying the δt-dependence of 3pt corr.

Suppression of excited states and thermal effects well under control.





Spectral density of G

calculating the form factor from LSZ

- Given the spectral decomposition of $G_k(t;\mathbf{p})$ [Bulava and Hansen, 2019]

$$G_k(t;\mathbf{p}) = \sum_n \underbrace{\langle 0|j_k(0)|n(\mathbf{0})\rangle \langle n(\mathbf{0})|\mathcal{O}_{\pi,1}(0)|\pi(\mathbf{p})\rangle}_{M^{(n)}_{k,\mathbf{p}}[\mathcal{O}_{\pi,1}]} e^{-E_n(\mathbf{0})t}$$

- Explicitly construct the spectral density from the convolution

$$\rho_{k,\mathbf{p}}(\mathcal{E}\,|\,\sigma,\mathcal{O}_{\pi,1}) \equiv \sum_n \Delta_\sigma(E_n-\mathcal{E}) M^{(n)}_{k,\mathbf{p}}[\mathcal{O}_{\pi,1}]\,.$$

- The kernel Δ_σ from the LSZ reduction formula $\implies$ $\Delta_\sigma(\mathcal{E}) = \frac{i}{\mathcal{E}+i\sigma}$

$$\boxed{\lim_{\sigma\to 0^+}\lim_{L\to\infty} \frac{2\omega_{\mathbf{p}}}{Z^{1/2}_{\pi,1}(\mathbf{p})}\sigma\rho_{k,\mathbf{p}}(\mathcal{E}\,|\,\sigma,\mathcal{O}_{\pi,1})\Big|_{\mathcal{E}=2\omega_{\mathbf{p}}} = 2\mathbf{p}_k F_\pi(4\omega^2_{\mathbf{p}})\,.}$$

- By looking at $\mathrm{Re}\,\Delta_\sigma$ and $\mathrm{Im}\,\Delta_\sigma \implies$ extract separately $\mathrm{Re}\,F_\pi$ and $\mathrm{Im}\,F_\pi$.
  What about $E_n(\mathbf{0})$ and $M^{(n)}_{k,\mathbf{p}}[\mathcal{O}_{\pi,1}]$?

GEVP

ENERGIES AND MATRIX ELEMENTS OF G_k

- By leveraging GEVP analysis in [Blum et al., 2025], extract $\langle 0|j_3(0)|n(\mathbf{0})\rangle$ and $E_n(\mathbf{0})$:

$$C_{ij}(t) = \left(\begin{array}{c|c} j(t)j(0) & j(t)\mathcal{O}_{\pi\pi}(0) \\ \hline \mathcal{O}_{\pi\pi}(t)j(0) & \mathcal{O}_{\pi\pi}(t)\mathcal{O}_{\pi\pi}(0) \end{array}\right)$$

  * $j(t)$: **local** and smeared currents.
  * $\mathcal{O}_{\pi\pi}(t)$: smeared and polarized two-pion operators with vanishing total momentum but different relative momenta $\{\mathbf{p}, -\mathbf{p}\}$.

- Perform a **constrained fit** on

$$G_k(t;\mathbf{p}) = \sum_{n=1}^{N} M_{k,\mathbf{p}}^{(n)}[\mathcal{O}_{\pi,1}]e^{-E_n(\mathbf{0})t}$$

$\Longrightarrow$ to extract $M_{k,\mathbf{p}}^{(n)}[\mathcal{O}_{\pi,1}]$ for $n = 1, \ldots, N$.

- Is it a problem including only N states ?

Subtracted correlator

bypassing the inverse problem

- Effect for $n > N$ assessed from a **subtracted correlator**: spectral density support lifted from $[\omega_0, \infty)$ to $[\omega_1, \infty)$, with $\omega_0 < \omega_1$.

  [Bruno and Hansen, 2021]

- **Inclusive** approach:
  exploiting integral representation of Δ_σ, inverse problem is obviated.

- $G_k \to G_k^{\mathrm{sub}}$: subtract **at least** the states with $\omega < 2\omega_{\mathbf{p}}$, say r, and calculate

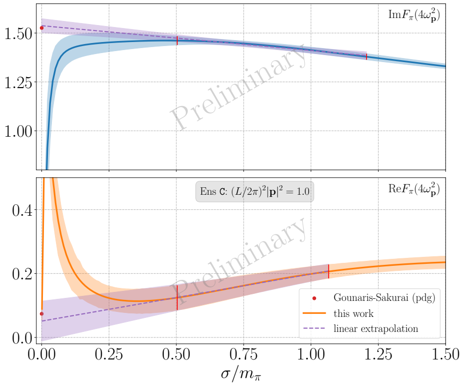
$$\rho_{k,\mathbf{p}}(\mathcal{E} \,|\, \sigma, \mathcal{O}_{\pi,1}) \simeq \sum_{t=t_0}^{t_{\max}} g_\sigma(t; 2\omega_{\mathbf{p}}) G_k^{\mathrm{sub}}(t; \mathbf{p}) + \sum_{n=1}^{r} M_{k,\mathbf{p}}^{(n)}[\mathcal{O}_{\pi,1}] \Delta_\sigma(E_n - 2\omega_{\mathbf{p}}) \,,$$

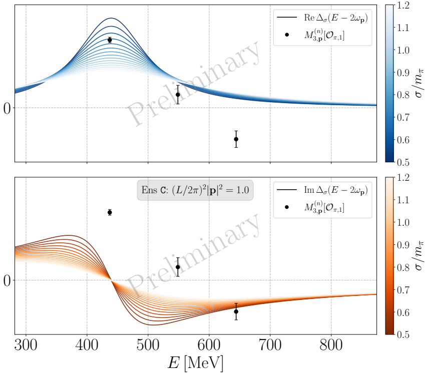
$g_\sigma(t; 2\omega_{\mathbf{p}})$: **analytically** known coefficients.

- Study plateau in $t_{\max}$.

Spectral reconstruction, I

$\sigma \to 0$ extrapolations and $M^{(n)}_{k,\mathbf{p}}[\mathcal{O}_{\pi,1}]$

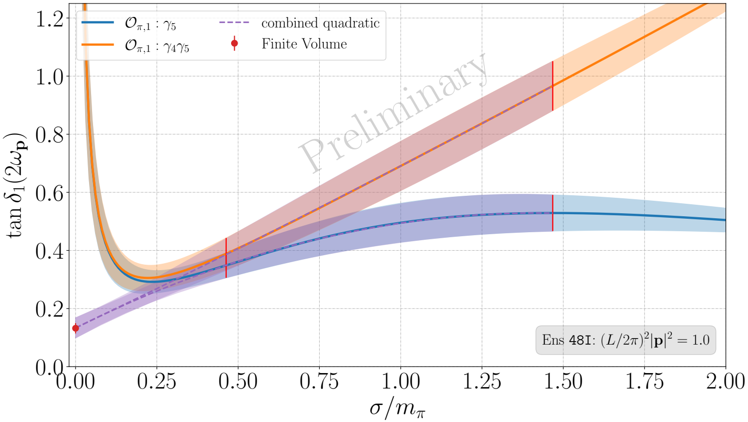




- Window problem: $m_\pi \gg \sigma \gg 1/L$.
  $\implies \sigma \gg 0$: good finite-size, bad LSZ.
  $\implies \sigma \simeq 0$: bad finite-size, good LSZ. [Bresciani, Bruno, and Hansen, 2026], see F. A. Bresciani's talk
- Linear and quadratic extrapolations [Bulava et al., 2022].

Spectral reconstruction, II

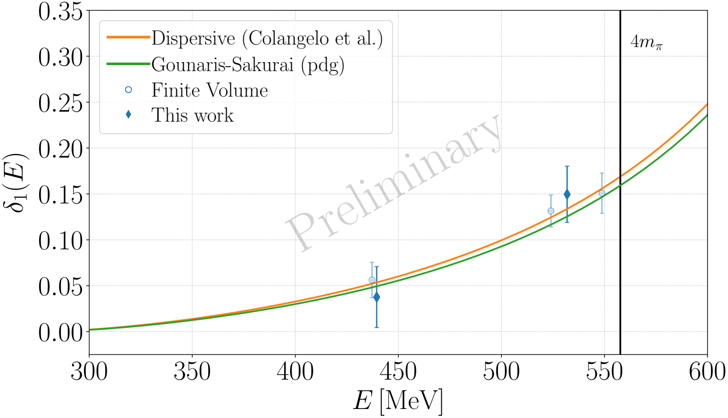
$\sigma \to 0$ extrapolation of $\tan\delta_1(E)$



- $\tan\delta_1(E) \equiv \dfrac{\mathrm{Im}\,F_\pi(E)}{\mathrm{Re}\,F_\pi(E)} = \lim_{\sigma\to 0^+}\lim_{L\to\infty} \dfrac{\mathrm{Re}\,\rho_{3,\mathbf{p}}(2\omega_{\mathbf{p}}\,|\,\sigma,\mathcal{O}_{\pi,1})}{\mathrm{Im}\,\rho_{3,\mathbf{p}}(2\omega_{\mathbf{p}}\,|\,\sigma,\mathcal{O}_{\pi,1})}$.
- Uniqueness of LSZ residue. Strong check with different $\mathcal{O}_{\pi,1}$'s. **Combined** $\sigma \to 0$ extrapolations.

Phase shift

summary of results



- Finite Volume = Lüscher's quantization condition.
- Several $\sigma \to 0$ extrapolations averaged with AIC.
- Blue diamonds: $(L/2\pi)^2|\mathbf{p}|^2 = 1.0$ with two different L's (finite lattice spacing).

Conclusions

...and outlooks

- First tests of inverse problem approach to calculate $F_\pi(s)$
  * using infinite-volume method via LSZ reduction formula;
  * at physical pion mass, with difficult window problem.

- Clear extraction of $\delta_1(E)$ in agreement with Finite-Volume in elastic region and consistency between different operators as $\sigma \to 0$.

[GM et al., in preparation]

- Outlooks:
  * Ongoing computation of G at inelastic energies.
  * Improving precision on $M^{(n)}_{k,\mathbf{p}}[\mathcal{O}_{\pi,1}]$, e.g. GEVP using 4pt functions $\langle \pi(\mathbf{p}) | \mathcal{O}_\pi(t) \mathcal{O}^\dagger_\pi(0) | \pi(\mathbf{p}) \rangle$ and $G_k(t;\mathbf{p})$.

    [Bruno and Hansen, 2021]
  * Complete error budget study (future).

Thanks for your attention!

Backup slides

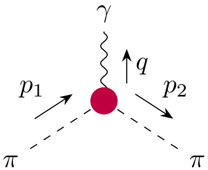
Space-like form factor $F_\pi(0)$

extraction from $C_4(t, \delta t; \mathbf{p})$

Space-like form factor:

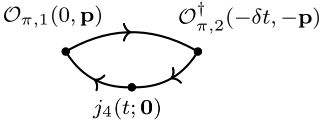
$$\langle \pi(\mathbf{p}_1)|j_\mu(q)|\pi(\mathbf{p}_2)\rangle = (p_1 + p_2)_\mu F_\pi(q^2)\,,$$

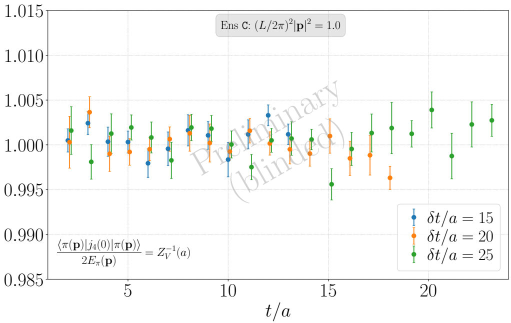
$q = p_1 - p_2$, j_μ = isospin-1 local em current.
It needs renormalization: $j^R_\mu(x) = Z_V(a) j_\mu(x)$.



Extraction from C_4 for $t > T - \delta t$,
t-independent at leading order:

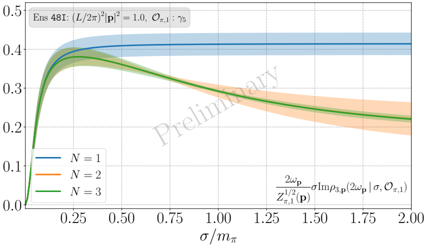
$$\begin{aligned} &C_4(t, \delta t; \mathbf{p}) \\ &\simeq Z^{1/2}_{\pi,1}(\mathbf{p}) Z^{1/2}_{\pi,2}(\mathbf{p}) e^{-E_\pi(\mathbf{p})\delta t} \times \\ &\times \langle \pi(\mathbf{p})|j_4(0)|\pi(\mathbf{p})\rangle\ . \end{aligned}$$

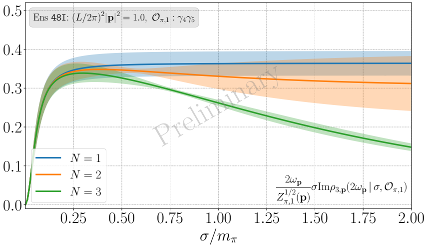


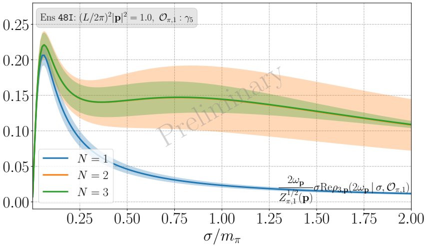


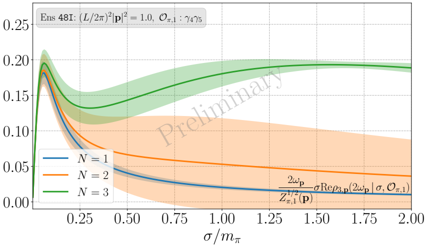
Spectral reconstruction, III

stability with respect to N









More on subtracted correlator

bypassing the inverse problem

- Consider $\rho(\omega)$ with support $[\omega_0, \infty)$ $(\omega_0 > 0)$ s.t.

$$G(t) = \int \mathrm{d}\omega \, \rho(\omega) e^{-\omega t} \, .$$

Assume the following exists

$$\rho_\Delta(\overline{\omega}) = \int \mathrm{d}\omega \, \Delta_\sigma(\omega - \overline{\omega}) \rho(\omega) \, , \quad \Delta_\sigma(\omega - \overline{\omega}) = \frac{1}{\omega - \overline{\omega} + i\sigma} \, .$$

- Introduce integral representation of the kernel

[Bruno and Hansen, 2021] [Bruno et al., 2026]

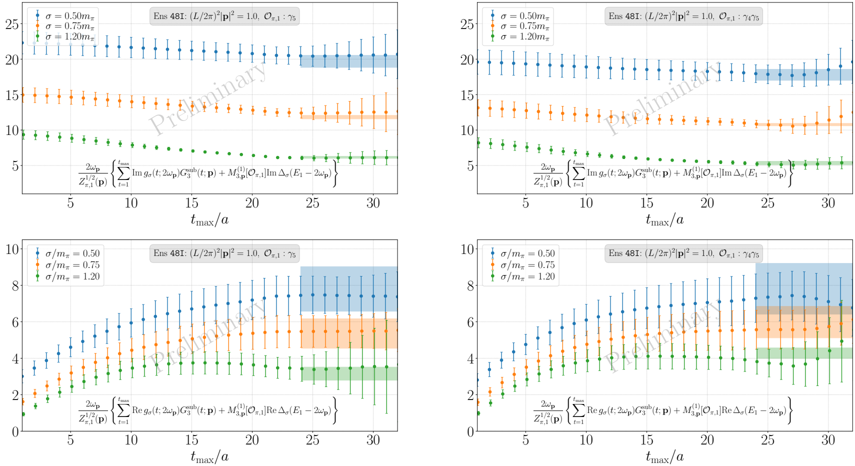
$$\rho_\Delta(\overline{\omega}) = \lim_{T\to\infty} \int \mathrm{d}\omega \int_0^T \mathrm{d}t \, e^{-(\omega - \overline{\omega} + i\sigma)t} \rho(\omega) = \lim_{T\to\infty} \int_0^T \mathrm{d}t \, e^{(\overline{\omega} - i\sigma)t} G(t) \, .$$

- Limit exists only if $\overline{\omega} < \omega_0$. Direct analytic continuation is possible if poles are located on the left of the particle-production branch cut
$\Rightarrow$ subtract poles in $G(t)$ and compute

$$\Longrightarrow \quad \rho_\Delta(\overline{\omega}) = \int \mathrm{d}t \, g_\sigma(t; \overline{\omega}) G^{\mathrm{sub}}(t) \, .$$

Spectral reconstruction, IV

stability in $t_{\max}$ using analytic coefficients approach.



Reconstruction by analytic coefficients subtracting only $E_1 < 2\omega_\mathbf{p}$.

Comparison with colored bands $\to \frac{2\omega_\mathbf{p}}{Z^{1/2}_{\pi,1}(\mathbf{p})} \sum_{n=1}^{N=3} M^{(n)}_{k,\mathbf{p}}[\mathcal{O}_{\pi,1}]\Delta_\sigma(E_n - 2\omega_\mathbf{p})$.