UNIVERSITY OF MARYLAND, COLLEGE PARK

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Lattice Field Theory

The smeared R-ratio in isoQCD from first-principles lattice simulations

Francesca Margari, on behalf of the ETMC

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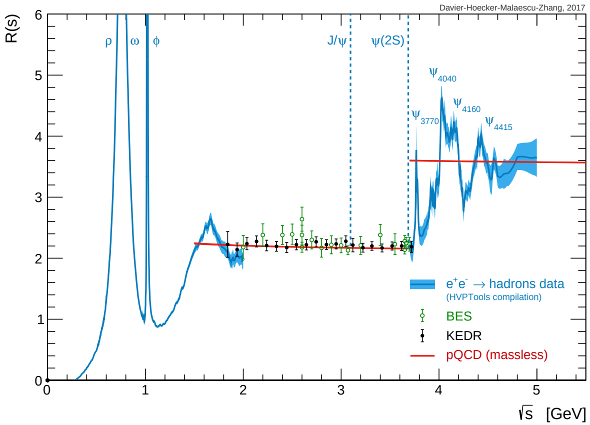
TOR VERGATA
UNIVERSITÀ DEGLI STUDI D ROMA

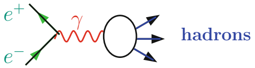
Extended Twisted Mass Collaboration

INFN
Istituto Nazionale di Fisica Nucleare
SEZIONE DI ROMA TOR VERGATA

The R-ratio

The R-ratio plays a fundamental role in particle physics since its introduction in [1, 2].





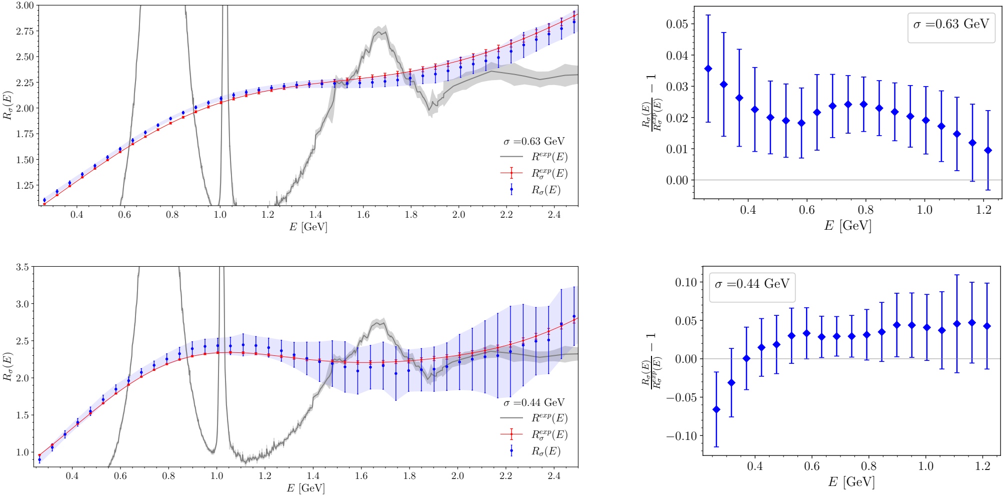
$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} \qquad (1)$$

As any other cross-section, it is an energy-dependent probe of the theory and contains an infinite amount of information.

[1] N. Cabibbo, G. Parisi and M. Testa, *Hadron Production in e+ e− Collisions*, Lett. Nuovo Cim. **4S1**, 35 (1970).

[2] J. D. Bjorken, *Phys. Rev.*, **148**, 1467 (1966)

ETMC, PRL 130 (2023) 241901: the smeared R-ratio in Gaussian energy bins



At $E \simeq m_\rho$:

- $\sigma \simeq 0.6$ GeV : the total relative error $\sim 1\%$, we observe a $\sim 3\sigma$ (or $2.5 - 3$ %) deviation w.r.t. e^+e^- experimental data.
- $\sigma \simeq 0.4$ GeV : the total relative error $\sim 3.5\%$, the larger errors do not allow us to observe significant deviations from experimental data.

ETMC Lattice setup

- The total contribution $R_\sigma(E)$ can be decomposed as:

$$R_\sigma(\text{isoQCD}) = R_\sigma^{\ell\ell} + R_\sigma^{ss} + R_\sigma^{cc} + R_\sigma^{\text{disco}} \tag{2}$$

ETMC Lattice setup

▷ **Isospin decomposition**:

$$R_\sigma(\text{isoQCD}) = \underbrace{\frac{9}{10} R_\sigma^{\ell\ell}}_{R_\sigma(I=1)} + \underbrace{\frac{1}{10} R_\sigma^{\ell\ell} + R_\sigma^{ss} + R_\sigma^{cc} + R_\sigma^{\text{disco}}}_{R_\sigma(I=0)} \tag{2}$$

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- We rely on our effort within **ETMC**, which produced a collection of state-of-the-art $N_f = 2+1+1$ Twisted Mass ensembles at physical pion masses:

ensemble	$(L^3 \times T)/a^4$	a [fm]	L [fm]
B64	$64^3 \times 128$	~ 0.080	5.10
B96	$96^3 \times 192$	~ 0.080	7.65
C80	$80^3 \times 160$	~ 0.068	5.44
D96	$96^3 \times 192$	~ 0.057	5.46
E112	$112^3 \times 224$	~ 0.049	5.47

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- **Low-mode averaging** as noise-reduction techniques. [see Antonio Evangelista's talk, Jul 31]

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▷ **Low-mode averaging** as noise-reduction techniques. [see Antonio Evangelista's talk, Jul 31]

▷ **To match FLAG/Edinburgh scheme (f_π, m_π, m_K, m_{D_s})**, we performed careful analysis of mistuning on m_ℓ, m_s, m_c and m_{cr}. [see Marco Garofalo's talk, Jul 29]

Correlator known at **finite** points $t = n\tau$, $n = 1, \ldots, N$:

$$\underbrace{C(n\tau)}_{\text{correlator}} = \int_{E_0}^{\infty} dE \, e^{-n\tau E} \rho(E), \qquad \rho[\mathcal{S}] = \int_{E_0}^{\infty} dE \, \rho(E) \underbrace{\mathcal{S}(E)}_{\text{target kernel}} \tag{3}$$

In our case, $\rho(E) \to R(\omega)$ and $\mathcal{S}(E) \to G_\sigma(E-\omega)$:

$$R_\sigma(E) = \int_{2m_\pi}^{\infty} d\omega \, R(\omega) \underbrace{G_\sigma(E-\omega)}_{\text{Gaussian kernel}} \; ; \qquad \underbrace{\mathcal{S}(\omega; \mathbf{g})}_{\text{Approximated smearing kernel}} = \sum_{n=1}^{N} g_N(n) \, e^{-n\tau\omega} \tag{4}$$

Theoretical Framework

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$\mathcal{S}(E)$ can be represented *exactly* as

$$\mathcal{S}(E) = \boxed{\lim_{N\to\infty}} \sum_{n=1}^{N} g_{[\mathcal{S}],N}(n)\, e^{-n\tau E} \tag{5}$$

Weierstrass theorem: $x = e^{-\tau E} \Rightarrow e^{-n\tau E} = x^n \Rightarrow$ polynomial approx. of $\mathcal{S}(-\log x/\tau)$ in $x \in (0, 1]$

$\Rightarrow$ exact as $N \to \infty$, for any $\tau > 0$; holds whether τ is fixed in physical units or identified with a.

Solve the linear system

$\boldsymbol{g} \in \mathbb{R}^N$ are now determined by minimizing the distance between the target and the approximation

$$\frac{\partial A[\boldsymbol{g}]}{\partial g(n)} = 0\,, \qquad \text{where} \quad \boxed{A[\boldsymbol{g}] = \left\| \mathcal{S}(E) - \sum_{n=1}^{N} g(n) e^{-n\tau E} \right\|_{\alpha}^{2}}\,. \tag{6}$$

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$$\boldsymbol{g}_{[\mathcal{S}],N} = \boxed{A_N^{-1} \cdot \boldsymbol{f}_{[\mathcal{S}],N}}\,, \tag{8}$$

Combining with $\rho_N[\mathcal{S}] = \boldsymbol{C}_N^T \cdot \boldsymbol{g}_{[\mathcal{S}],N}$:

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Direct inversion of A_N is **not viable**: A_N is an **ill-conditioned** matrix (more on this later) $\Rightarrow$ **noise is amplified beyond control**.

Regularisations

In the **HLT method** the problem is **naturally** approached in time-space and the regulator is chosen to be the one prescribed by Backus and Gilbert.

$$\boldsymbol{g}_{[\mathcal{S}],N,\lambda} = \underset{\boldsymbol{g}\in\mathbb{R}^N}{\operatorname{argmin}} \left\{ A[\boldsymbol{g}] + \boxed{\lambda\, B[\boldsymbol{g}]} \right\}, \tag{10}$$

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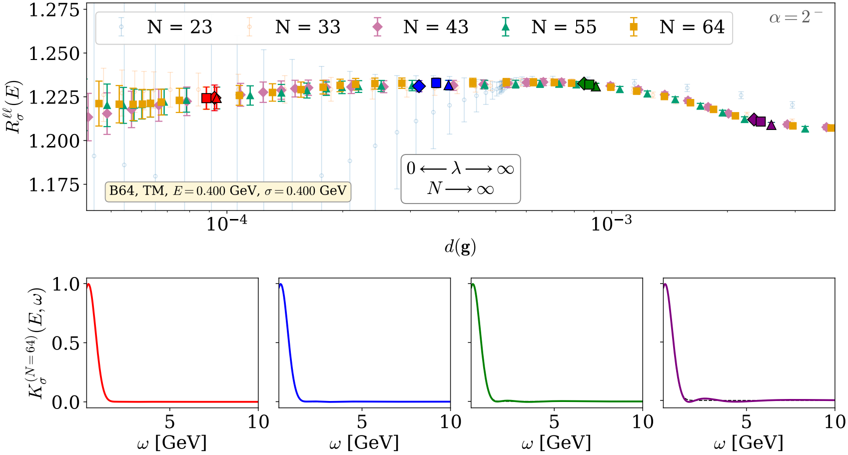
These limits must be taken numerically.

Stability Analysis (SA)



$$W[\lambda, \mathbf{g}] = \frac{A[\mathbf{g}]}{A[0]} + \lambda B[\mathbf{g}]\,, \qquad d[\mathbf{g}] = \sqrt{A[\mathbf{g}]/A[\mathbf{0}]}\,. \tag{13}$$

HLT stability analysis: algorithmic parameters are tuned by identifying a *stability region* where systematic errors are negligible w.r.t. statistical errors.

Why this is hard: a closer look at the ill-conditioning

- A_N is a real, symmetric Cauchy matrix and its condition number grows $\boxed{\textit{exponentially}}$ with N
- As anticipated, inverting A_N directly amplifies statistical noise uncontrollably
  $\Rightarrow$ unusable unregulated solution.

Extraction of spectral densities from lattice correlators: decoupling signal from noise

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Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France

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Dipartimento di Fisica, Università di Roma Tor Vergata and INFN,
Sezione di Roma Tor Vergata, Via della Ricerca Scientifica 1, I-00133 Rome, Italy

We expand the treatment of the problem of the extraction of smeared spectral densities from Euclidean correlators, introduced in Ref. [1], providing an alternative which does not rely on the Backus-Gilbert regularization. This is possible due to the observation that the solution can be decomposed into a sum of terms, in the spirit of the singular value decomposition, where those with the largest contribution to the statistical noise happen to contribute the least to the central value of the smeared spectral density. The analysis of the systematics of the inverse problem is then shifted to finding the optimal truncation of such summation, so that the signal is saturated before the noise explodes. We scrutinise the performance and systematics of this approach either as a standalone procedure, or to complement the stability analysis required to extrapolate the unbiased result in the Backus-Gilbert regulated version of the solution.

arXiv:2605.14652

Eigen-space Analysis (EA)

- We need either (i) a **regulator**, or (ii) a change of basis that separates signal from noise $\Rightarrow$ **diagonalize** A_N.

$$A_N \cdot \boldsymbol{u}_N(k) = \boxed{a_N(k)}\, \boldsymbol{u}_N(k), \qquad a_N(1) > a_N(2) > \cdots > a_N(N) \tag{14}$$

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Idea: it might happen that

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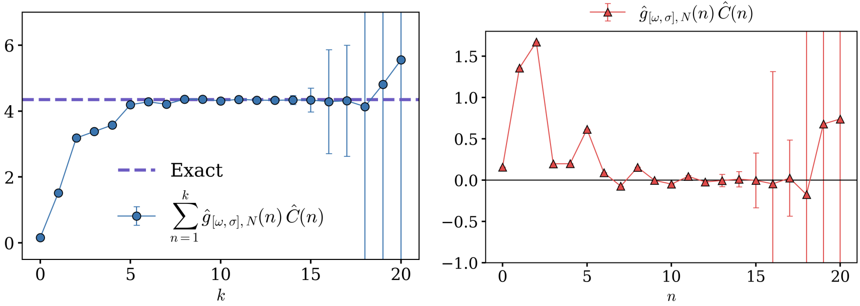
signal and noise partially decouple

$$\rho_N[\mathcal{S}] = \hat{\boldsymbol{C}}_N^T \cdot \hat{\boldsymbol{g}}_{[\mathcal{S}],N} = \boldsymbol{C}_N^T \cdot A_N^{-1} \cdot \boldsymbol{f}_{[\mathcal{S}],N} = \sum_{k=1}^{N} \underbrace{\frac{\hat{C}_N(k)\, \hat{f}_{[\mathcal{S}],N}(k)}{a_N(k)}}_{\text{eigen-space term}}$$



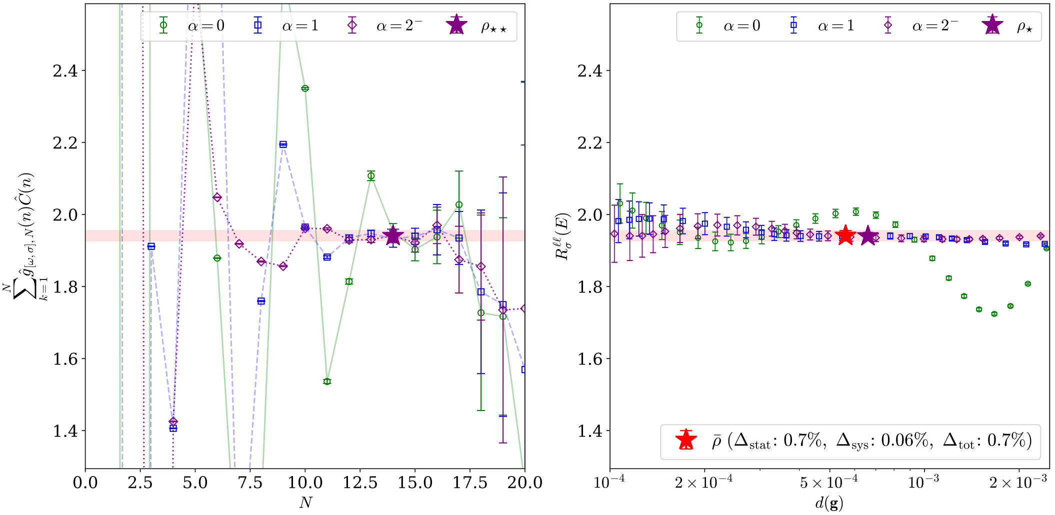
Smeared spectral density reconstructed at $\omega = m_\rho$ with $\sigma = 2m_\pi$, with a **mock vector-vector correlator** with the **statistical noise of B64 TM correlator** used in this calculation.
Left: partial sum. Right: individual terms.

arXiv:2605.14652

Signal saturates long before **noise takes over** $\Rightarrow$ **the sum can be truncated**.

$$\bar{\rho}[\mathcal{S}] = \frac{\rho_\star[\mathcal{S}] + \rho_{\star\star}[\mathcal{S}]}{2}\,, \qquad \Delta_{\rm sys}\bar{\rho}[\mathcal{S}] = |\rho_\star[\mathcal{S}] - \rho_{\star\star}[\mathcal{S}]|\,\mathrm{Erf}\left(\frac{\rho_\star[\mathcal{S}] - \rho_{\star\star}[\mathcal{S}]}{\sqrt{2}\Delta_{\rm stat}\bar{\rho}[\mathcal{S}]}\right) \tag{17}$$

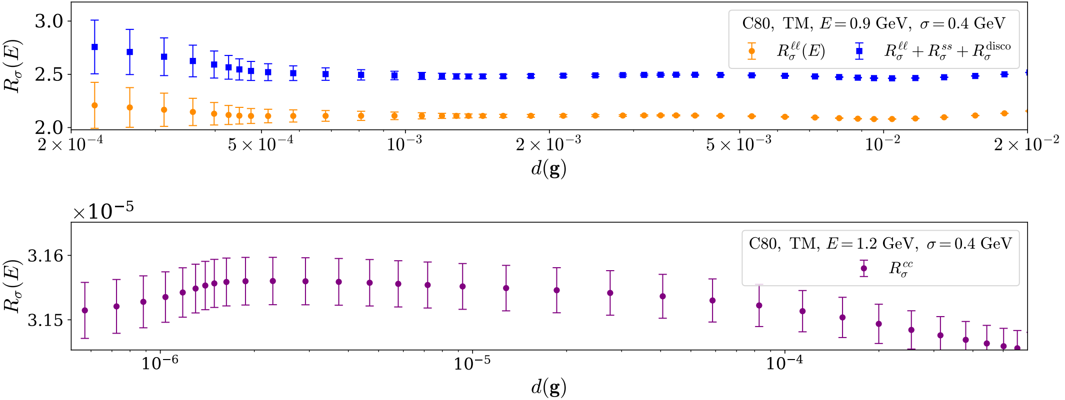
D96, TM, $E = 0.70$ GeV, $\sigma = 0.4$ GeV



Left: Eigen-space Analysis (EA). Right: Stability Analysis (SA).

Isospin decomposition

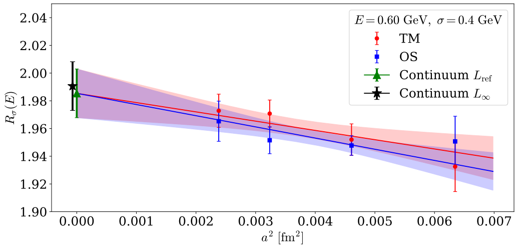
$$R_\sigma(\text{isoQCD}) = \underbrace{\frac{9}{10} R_\sigma^{\ell\ell}}_{R_\sigma(I=1)} + \underbrace{\frac{1}{10} R_\sigma^{\ell\ell} + R_\sigma^{ss} + R_\sigma^{disco} + R_\sigma^{cc}}_{R_\sigma(I=0)} \tag{18}$$

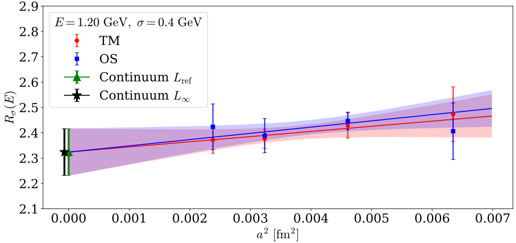


ss, cc interpolated to physical masses

all c contributions neglected (negligible at all energies considered)!

How we proceed so far



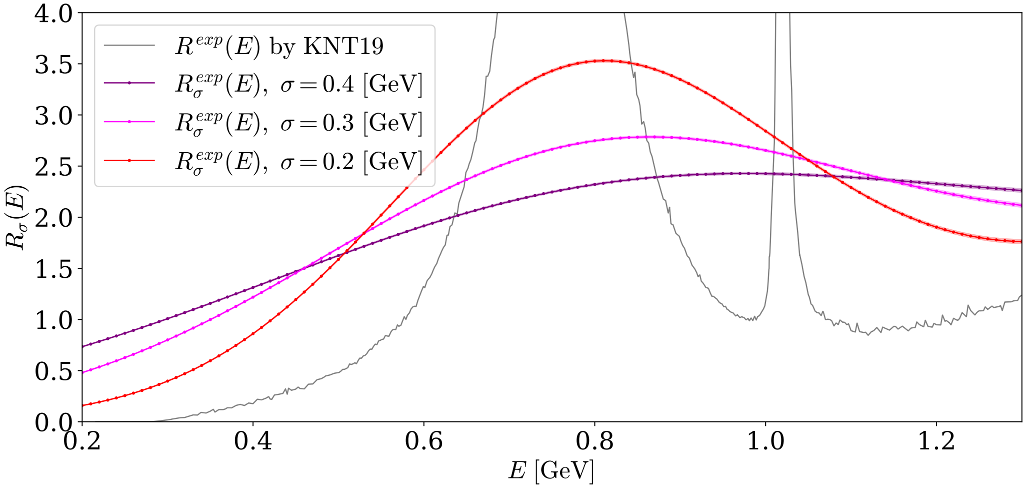


- *Continuum extrapolations*:
  constrained extrapolation, we fit TM and OS data by performing a correlated χ^2-minimization at fixed E, σ at $L_{\mathrm{ref}} = 5.46$ fm.

- *Finite-volume effects*:
  corrected using the **Gounaris–Sakurai** model after data-driven validation.

F. A. Bresciani, M. Bruno, M. T. Hansen, *Finite-volume effects on smeared spectral densities*, arXiv:2606.14349

$R_\sigma^{\rm exp}(E)$ based on KNT19 compilation



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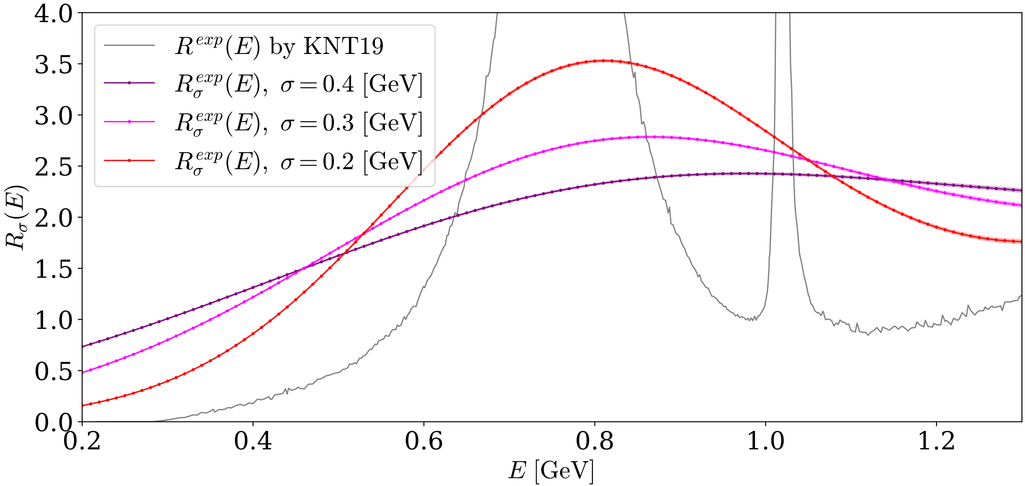
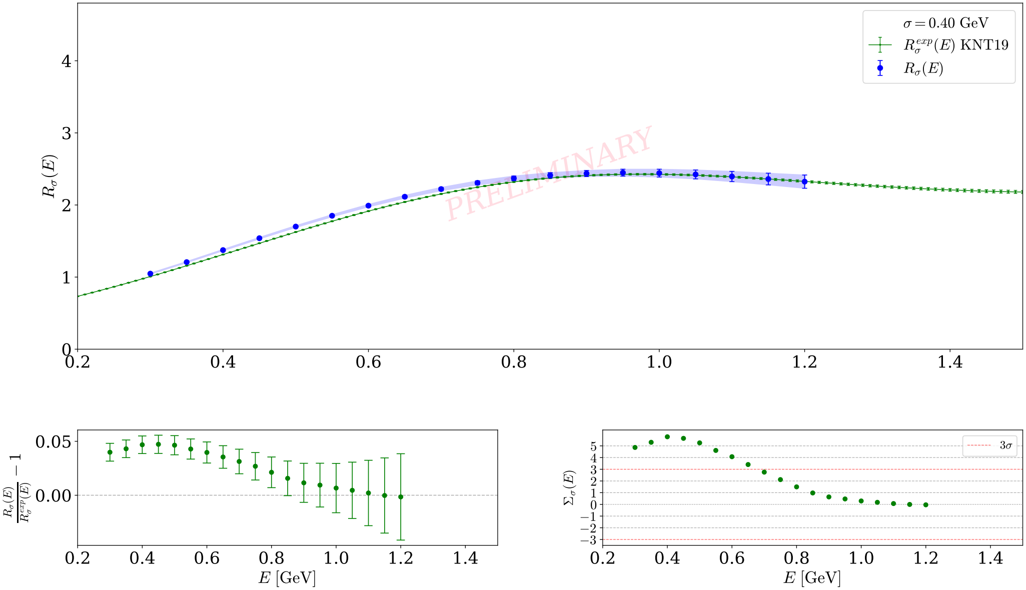
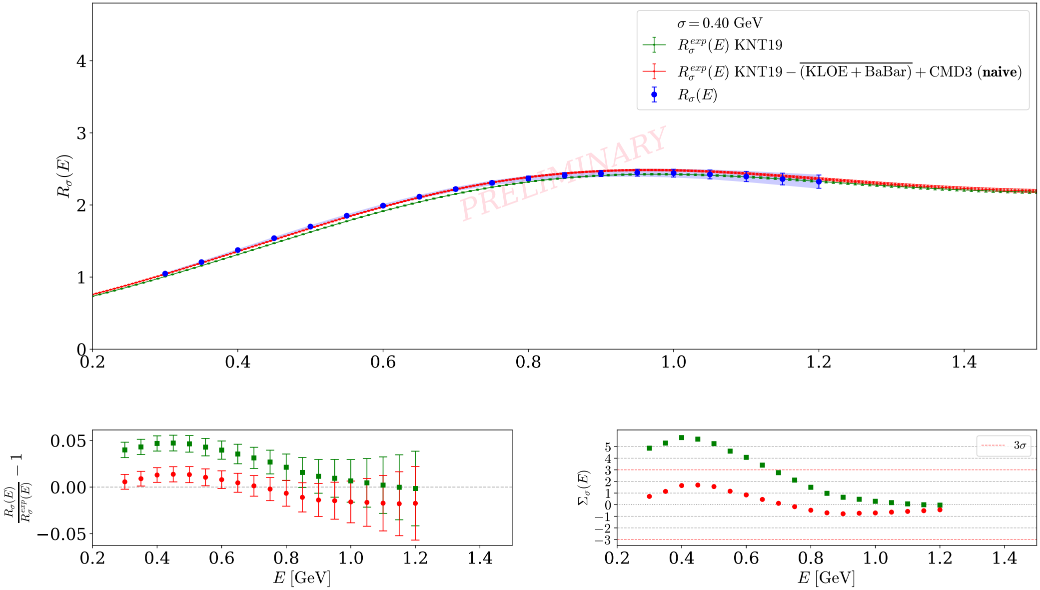


We have also **naively** replaced the BaBar and KLOE data with CMD-3.



$\sigma = 0.40$ GeV
$R_\sigma^{exp}(E)$ KNT19
$R_\sigma(E)$
PRELIMINARY
$R_\sigma(E)$
$\frac{R_\sigma(E)}{R_\sigma^{exp}(E)} - 1$
$\Sigma_\sigma(E)$
3σ
E [GeV]
E [GeV]



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$R_\sigma^{exp}(E)$ KNT19 $- \overline{(\mathrm{KLOE}+\mathrm{BaBar})} + \mathrm{CMD3}$ (**naive**)
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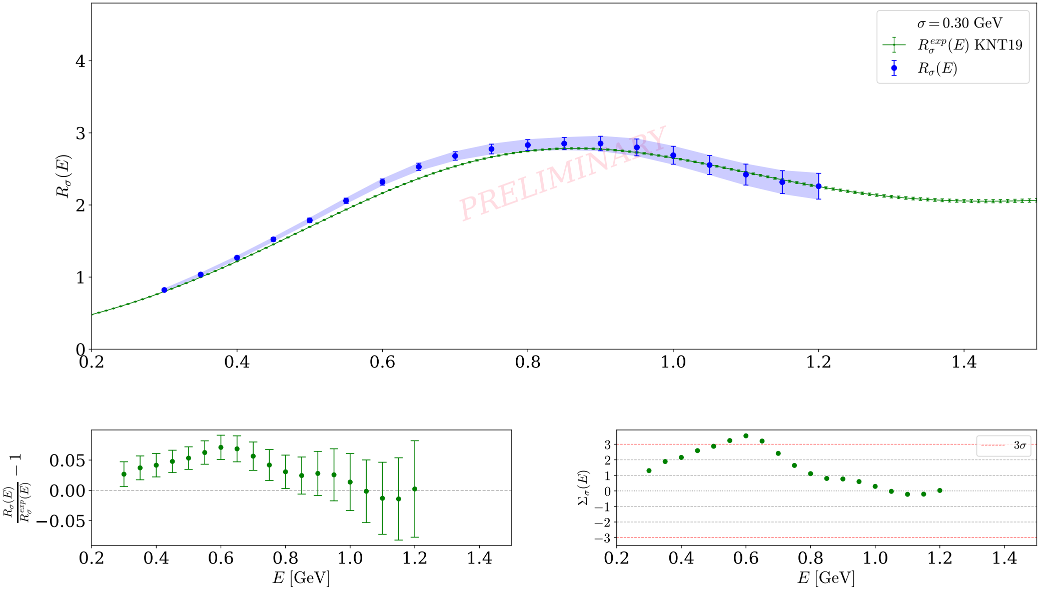
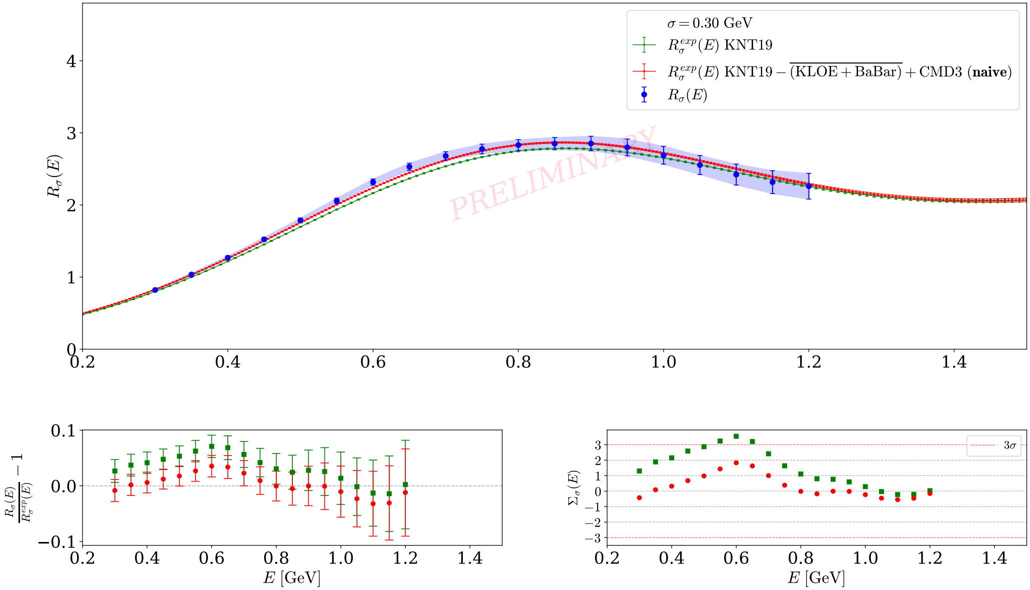


$\sigma = 0.30$ GeV
$R_\sigma^{exp}(E)$ KNT19
$R_\sigma(E)$
$R_\sigma(E)$
PRELIMINARY
$\frac{R_\sigma(E)}{R_\sigma^{exp}(E)} - 1$
$\Sigma_\sigma(E)$
3σ
E [GeV]
E [GeV]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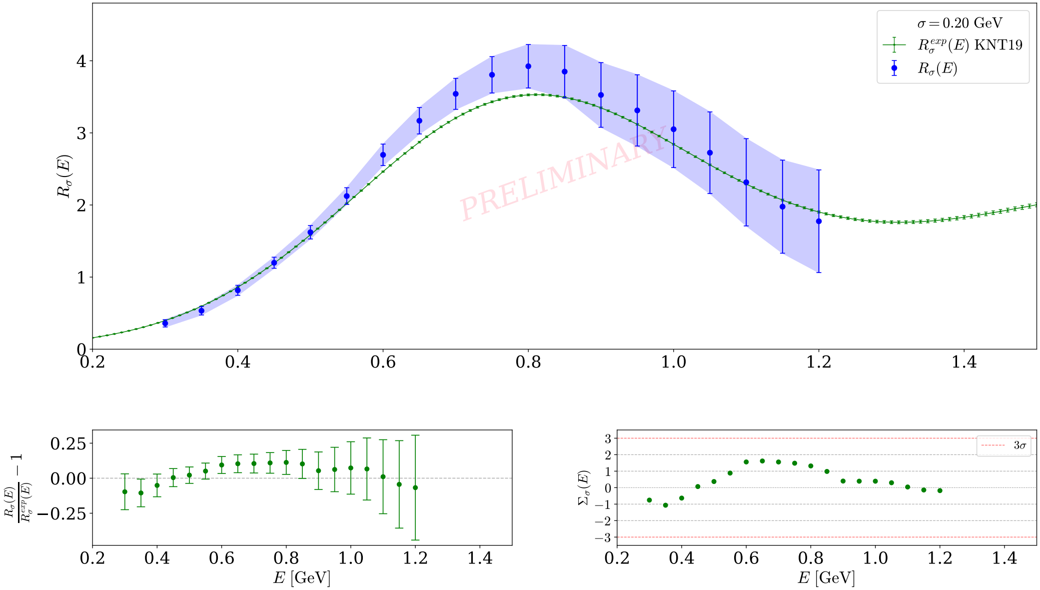
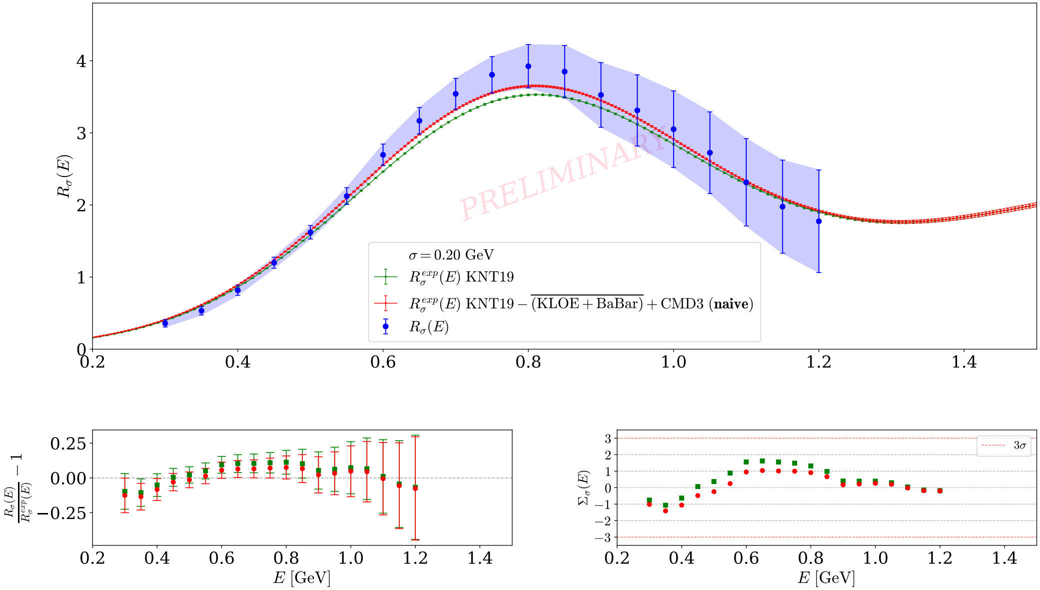


$\sigma = 0.20$ GeV
$R_\sigma^{exp}(E)$ KNT19
$R_\sigma(E)$
PRELIMINARY
$R_\sigma(E)$
$\frac{R_\sigma(E)}{R_\sigma^{exp}(E)} - 1$
$\Sigma_\sigma(E)$
3σ
E [GeV]
E [GeV]



PRELIMINARY
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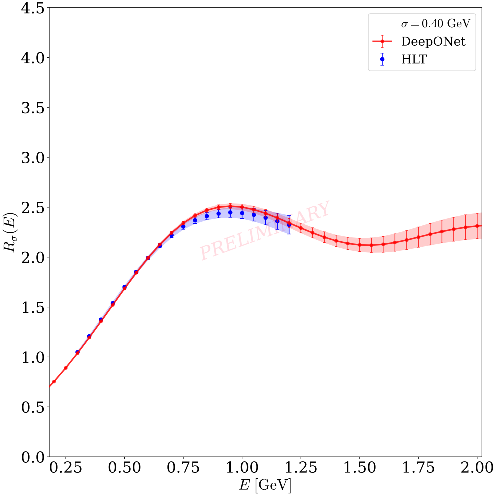
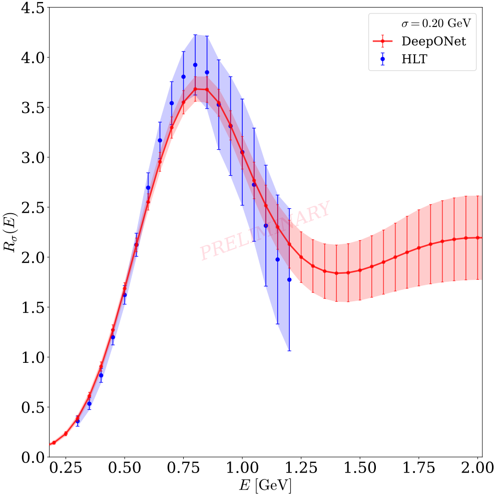
Outlook





Operator Learning for spectral reconstruction in lattice QCD, based on arXiv:2607.03311

see Alessandro De Santis's talk, Jul 29

Conclusions

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- We plan to account for **QED+IB effects** from first principles.

Thank you!

Backup slides

In recent years, the importance of the R-ratio has been mainly associated with muon $g-2$ experiments ...

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Dispersive approach

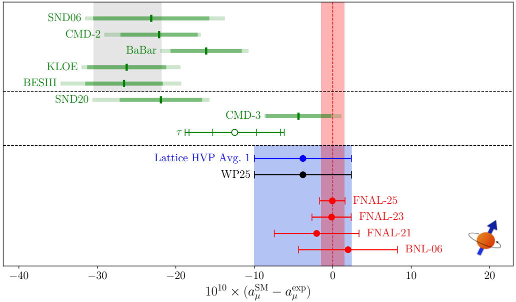
$$a_\mu^{\mathrm{HVP-LO}} = \int_{2m_\pi}^{\infty} \mathrm{d}E\; R(E) \underbrace{\tilde{K}(E)}_{\text{analytic function}} \qquad (19)$$

- The idea is to replace $R(E) \to R^{\mathrm{exp}}(E)$ and use previous formula.

Lattice QCD

$$a_\mu^{\mathrm{HVP-LO}} = \int_{0}^{\infty} \mathrm{d}t\; C(t) \underbrace{K(t)}_{\text{analytic function}} \qquad (20)$$

- **ETM Collaboration** effort in computing light-quark contribution to $a_\mu^{\mathrm{HVP-LO}}$ at subpercent accuracy. [see Marco Garofalo's talk, Jul 29]



HLT method

$$R_\sigma(E) = \int_{2m_\pi}^{\infty} d\omega\, R(\omega) \underbrace{G_\sigma(E-\omega)}_{\text{Gaussian kernels}} \quad ; \qquad \underbrace{K(\omega;\mathbf{g})}_{\text{Approximated smearing kernels}} = \sum_{\tau=1}^{T/a} g_\tau\, e^{-a\omega\tau} \,.$$

which allows us to evaluate $R_\sigma(E)$ from $C(t)$ using

$$\sum_{\tau=1}^{\tau_{\max}} g_\tau(E,\sigma)\, C(\tau a) = \int_{2m_\pi}^{\infty} d\omega \left(\frac{1}{12\pi^2} \sum_{\tau=1}^{\tau_{\max}} g_\tau(E,\sigma) e^{-\omega a \tau} \right) R(\omega)\, \omega^2 \simeq R_\sigma(E)$$

Accuracy of the approximated kernel

$$A[\mathbf{g}] = \int_{2m_\pi}^{\infty} \mathrm{d}\omega \, \left\{ K(\omega;\mathbf{g}) - \frac{12\pi^2 G_\sigma(E-\omega)}{\omega^2} \right\}^2 \cdot e^{\alpha\omega} \,, \qquad \alpha = 2^-$$

The coefficients $\mathbf{g}$ are obtained by minimizing

$$W[\lambda,\mathbf{g}] = (1-\lambda) \underbrace{\frac{A[\mathbf{g}]}{A[0]}}_{d^2[\mathbf{g}]} + \lambda B[\mathbf{g}], \qquad B[\mathbf{g}] = \sum_{\tau_1,\tau_2=1}^{T/a} g_{\tau_1} g_{\tau_2} \, \mathrm{Cov}_{\tau_1\tau_2} \,.$$

Stability Analysis

Systematic error splits into two contributions:

$$|\rho[S] - \rho_{N,\lambda}[S]| \;\leq\; \underbrace{|\rho[S] - \rho_N[S]|}_{\text{Weierstrass: } \to 0 \text{ as } N\to\infty} + \underbrace{|\rho_N[S] - \rho_{N,\lambda}[S]|}_{\text{regulator bias}}$$

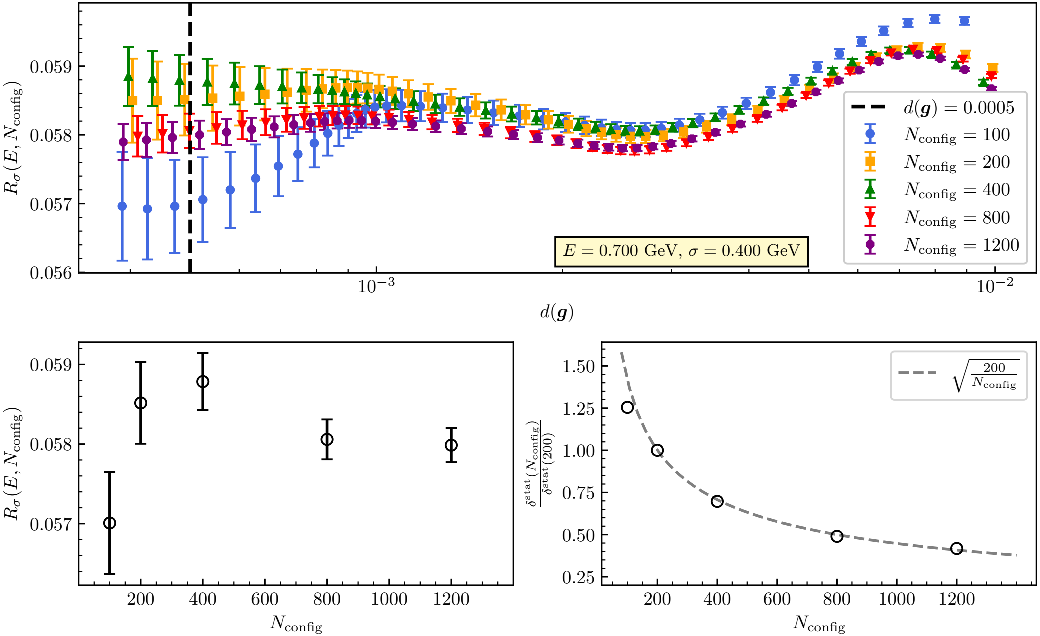
$$|\rho_{N,\lambda}[S] - \rho_N[S]| \leq \lambda \left\| \sqrt{\mathrm{Cov}_N} \cdot \mathbf{g}_{[S],N,\lambda} \right\| \left\| \sqrt{\mathrm{Cov}_N} \cdot A_N^{-1} \cdot \mathbf{C}_N \right\| = \mathcal{O}\left(\lambda \sqrt{B[\mathbf{g}_{[S],N,\lambda}]} \right)$$

- **SA stopping criterion**: start at λ with $A[\mathbf{g}] \gg B[\mathbf{g}]$; stop at $\lambda_\star$ when

$$A[\mathbf{g}_{[S],N,\lambda_\star}] \ll B[\mathbf{g}_{[S],N,\lambda_\star}]\,, \qquad \left|\rho_{N,\lambda_{\star+1}}[S] - \rho_\star[S]\right|^2 \ll B[\mathbf{g}_{[S],N,\lambda_\star}]$$

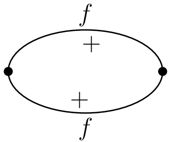
- $N \to \infty$ limit taken analogously: onset reached when residual N-dependence is negligible w.r.t. the statistical error.

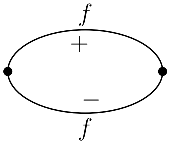
What happens if the statistics is increased?



ETMC Lattice setup

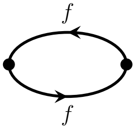
$$J_\mu^{f,\mathrm{OS}} \propto \bar{\psi}_f^+ \gamma_\mu \psi_f^+ \qquad\qquad J_\mu^{f,\mathrm{TM}} \propto \bar{\psi}_f^+ \gamma_\mu \psi_f^-$$



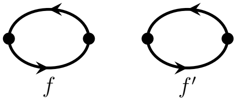


$$R_\sigma^{ff,C,\mathrm{OS}}(a) = A + B^{\mathrm{OS}}\, a^2 \qquad\qquad R_\sigma^{ff,C,\mathrm{TM}}(a) = A + B^{\mathrm{TM}}\, a^2$$

$$V_{\mathrm{conn}}^f(t) \equiv -\frac{1}{3} \sum_{i=1,2,3} a^3 \sum_{\mathbf{x}} \left\langle J_i^f(\mathbf{x},t)\; J_i^f(0) \right\rangle^{(C)} = q_f^2 \times$$



$$V_{\mathrm{disco}}^{ff'}(t) \equiv -\frac{1}{3} \sum_{i=1,2,3} a^3 \sum_{\mathbf{x}} \left\langle J_i^f(\mathbf{x},t)\; J_i^{f'}(0) \right\rangle^{(D)} = -q_f q_{f'} \times$$



Precision and resolution to be improved with more statistics and by increasing the statistical precision of our lattice correlators.

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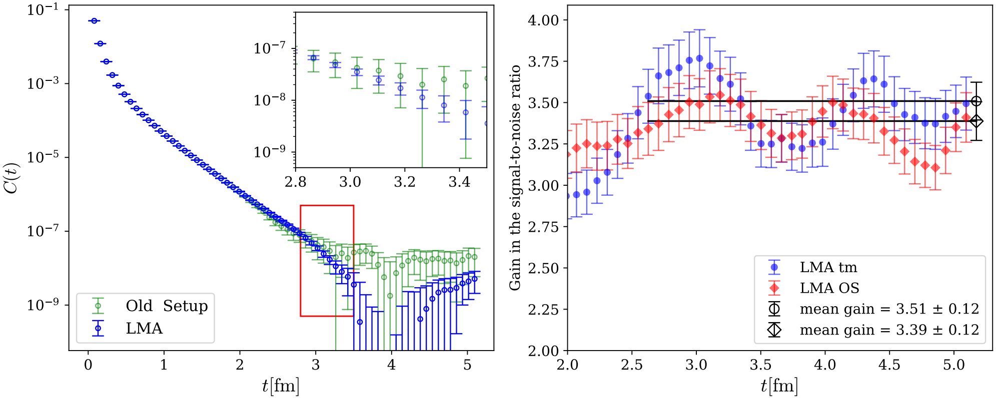
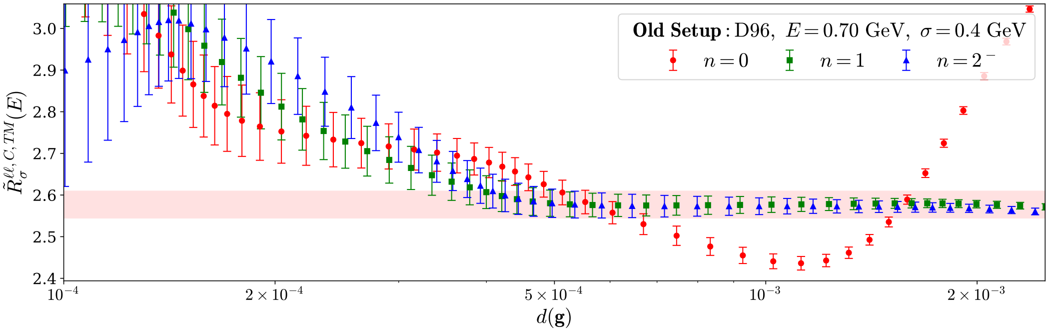


We employ noise-reduction techniques:

by computing a relatively small number of low modes of the Dirac operator exactly, which we refer to as low-mode averaging (LMA).

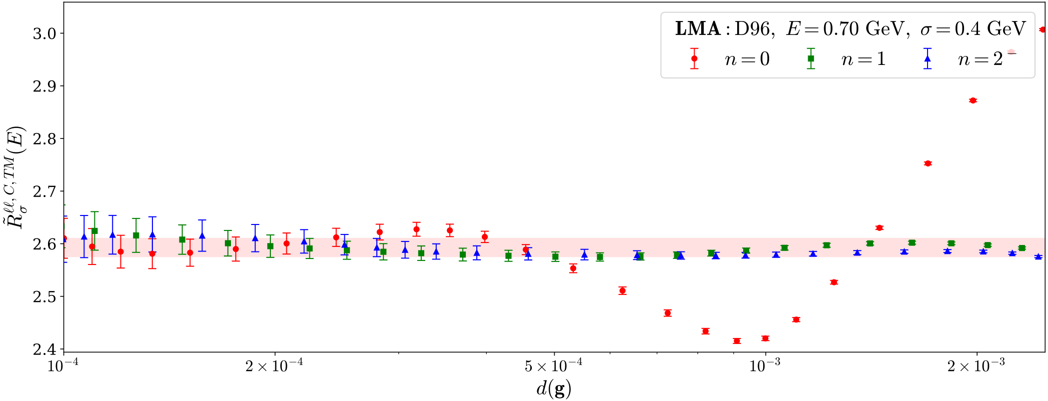
Previously, it was ...



Blinding data: $R_\sigma^{\ell\ell,C,TM}(E) \to \tilde{R}_\sigma^{\ell\ell,C,TM}(E)$.

HLT stability analysis: algorithmic parameters are tuned by identifying a *stability region* where systematic fluctuations lie within statistical errors and results from different norms are compatible.

Now, it is ...



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Quark mass mistuning corrections

$$\mathcal{O}(m_f, m_{\rm cr}|m_f^{\rm val}, m_{\rm cr}^{\rm val}) = \mathcal{O}^{\rm sim}(m_f^{\rm val}, m_{\rm cr}^{\rm val}) + \sum_f (m_f - m_f^{\rm sim})\,\partial_f^{\rm sea}\mathcal{O} + (m_{\rm cr} - m_{\rm cr}^{\rm sim})\,\partial_{\rm cr}^{\rm sea}\mathcal{O}$$

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- ▷ $m_{\rm cr}$ — **both sea and valence**: maximal twist requires $\kappa = \kappa_{\rm crit}$, fixed by imposing $m_{\rm PCAC} = 0$; a residual mistuning $m_{\rm PCAC} \neq 0$ induces corrections on both sea and valence;

Calculation of $\ell - s$ quark-disconnected contribution

- We indicate with $B(f)$ the vector loop for flavour f.

- $B(\ell) - B(s)$ computable via Axial WI [R. Di Palma et al. – Phys.Rev.D 111 (2025)]

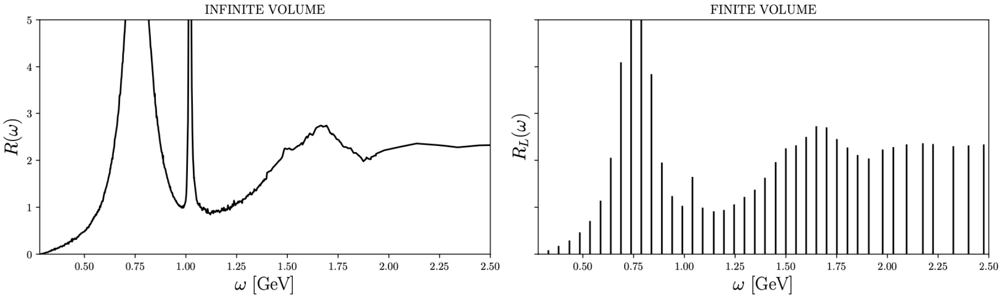
$$\sum_{f=u,d,s} q_f B_f^{\mu}(z) = Z_V (m_\ell + m_s)\, a^4 \sum_x \langle P_{\ell s}(x) A_{\ell s}^{\mu}(z) \rangle \,, \tag{21}$$

- frequency splitting improvement with 4 levels ($N = 3$) [L. Giusti et al. – Eur.Phys.J.C 79 (2019) 7, 586]

$$\begin{aligned} B^{\mu}(m_\ell) - B^{\mu}(m_s) = \big[B^{\mu}(m_\ell) - B^{\mu}(m_1)\big] \\ + \big[B^{\mu}(m_1) - B^{\mu}(m_2)\big] \\ + \cdots + \big[B^{\mu}(m_N) - B^{\mu}(m_s)\big] \,. \end{aligned} \tag{22}$$

- The axial WI trick has been key to get a precise determination of the quark-disconnected contribution.

Spectral density in a finite volume



$$R_\sigma(E) = \int_{2m_\pi}^{+\infty} \mathrm{d}\omega \, R(\omega) \underbrace{G_\sigma(E-\omega)}_{\text{Gaussian kernels}} \quad ; \qquad G_\sigma(E-\omega) = \frac{e^{-\frac{(E-\omega)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} \,. \tag{23}$$

Finite-volume effects: O(3) non-linear σ-model

In JHEP 07 (2022) 034, the authors addressed the question of **finite-volume effects in a model smeared spectral density**.

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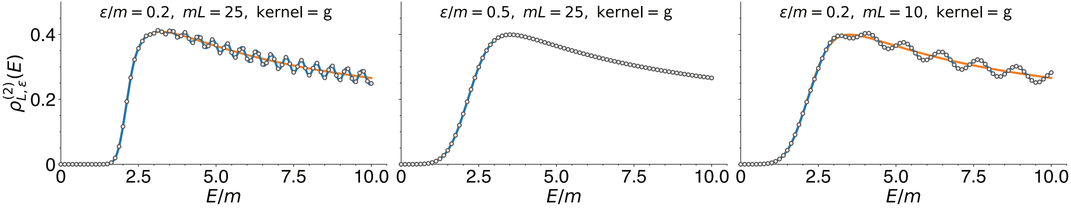

J. Bulava, M. T. Hansen, M. W. Hansen, A. Patella, and N. Tantalo, *Inclusive rates from smeared spectral densities in the two-dimensional O(3) non-linear σ-model*, JHEP **07**, 034 (2022).

Finite-volume effects: coming back to $R_\sigma(E)$ ...

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