Tensor Renormalization Group Study of Step-Scaling Function in the CP(1) Model with a θ Term

Lattice 2026

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2d CP(1) model

Toy model of 4d QCD

Common properties : Asymptotic freedom, θ term, etc.

2d CP(1) model with θ term

Action in the continuum

$$S = \int d^2x \left(\frac{1}{g^2} |D_\mu z(x)|^2 + \frac{i\theta}{2\pi} \varepsilon_{\mu\nu} \partial_\mu A_\nu \right)$$

U(1) gauge field

Complex scalar

$$z(x) = \begin{pmatrix} z_1(x) \\ z_2(x) \end{pmatrix} \in \mathbb{C}^2$$

Constraint

$$|z(x)|^2 = 1$$

On the square lattice

$$S = -2\boxed{\beta} \sum_{x,\mu} [z^\dagger(x) z(x+\hat{\mu}) U_\mu(x) + z^\dagger(x+\hat{\mu}) z(x) U_\mu^{-1}(x)] - i \frac{\boxed{\theta}}{2\pi} \sum_x q(x)$$

$$U_\mu(x) = e^{iA_\mu(x)}$$

$$q(x) = \frac{1}{i} \ln U_p(x)$$
$$= \{A_1(x) + A_2(x+\hat{1}) - A_1(x+\hat{2}) - A_2(x)\} \quad \mathrm{mod} 2\pi$$

Nathan Seiberg
Phys. Rev. Lett. 53, 637

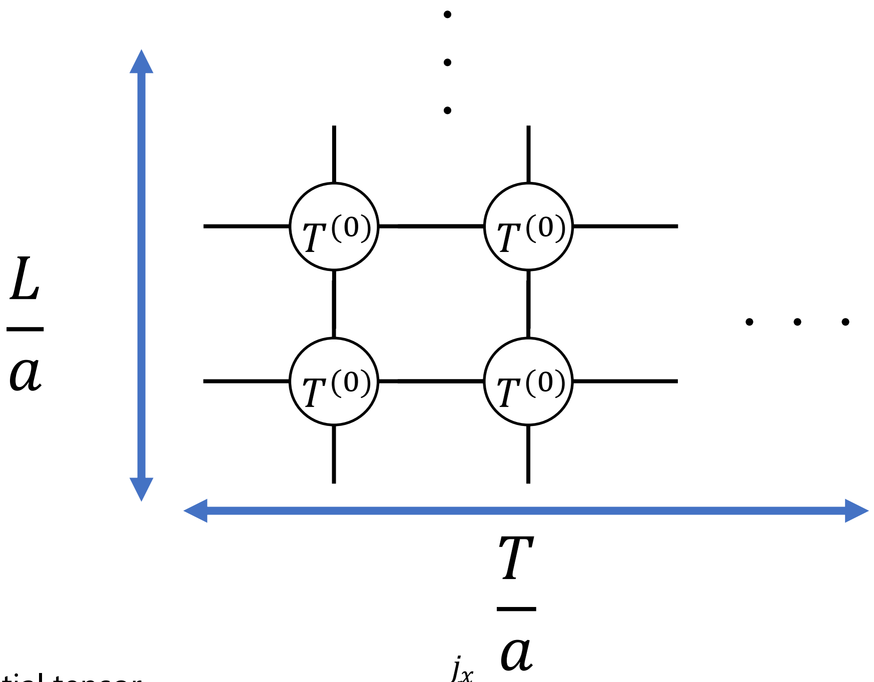
Tensor Renormalization Group (2D Case)

Partition function $$Z = \prod_x \int d\phi(x) e^{-S[\phi]}$$

Tensor network representation

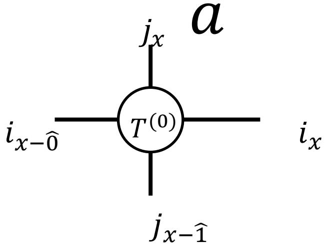
$$Z = \sum_{\{i_x, j_x\}} \prod_x^N T^{(0)}{}_{i_x, j_x, i_{x-\hat{0}}, j_{x-\hat{1}}}$$

$$(N = L \times T)$$



Initial tensor

$T^{(0)}{}_{i_x, j_x, i_{x-\hat{0}}, j_{x-\hat{1}}} =$

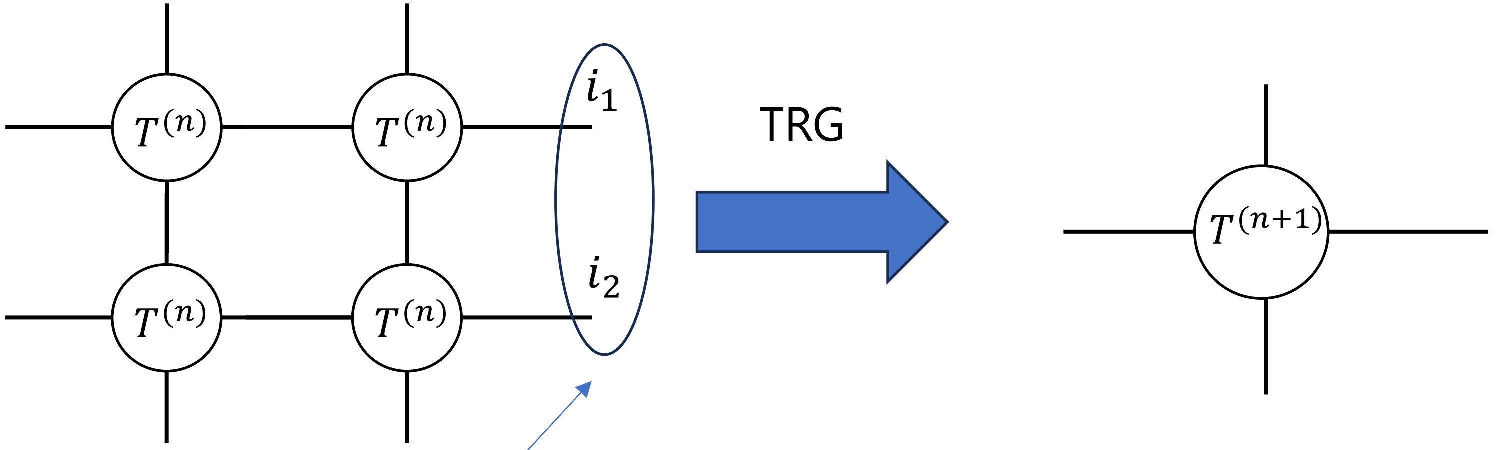


$i_x = 1, \ldots, D$

finite

Using Quadrature or
Truncating the character expansion

Coarse Graining



$i = 1, \cdots, D_c$

$i_1 \otimes i_2 = 1, \dots, D^2$

Singular Value Decomposition

$$A_{ij} = \sum_{k=1}^{D_c} U_{ik} \sigma_k V_{kj}^{\dagger}$$

$i_1 \otimes i_2 \to i$

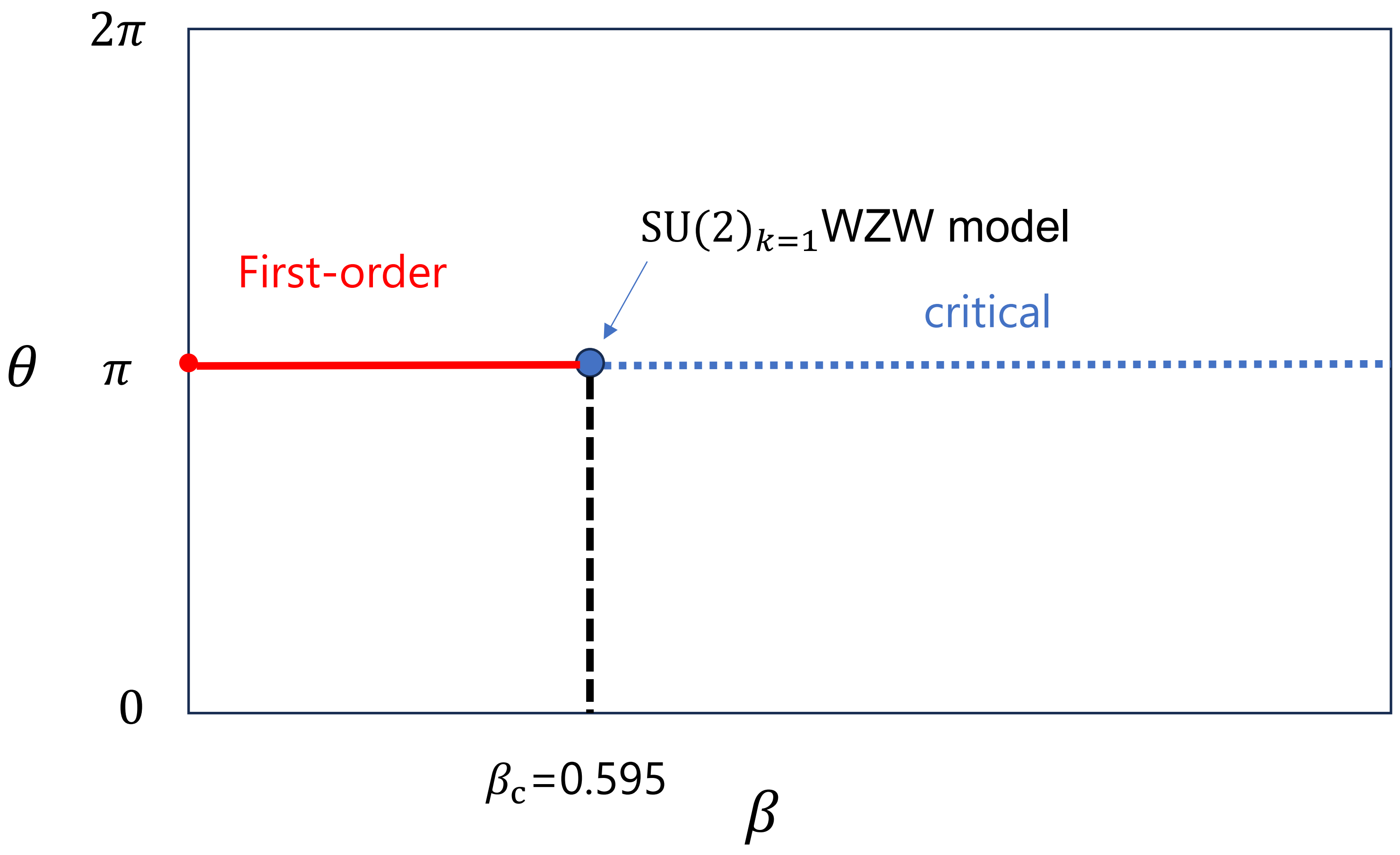
$D^2 \to D_c$

Controlling error and computational cost

This approximation does not rely on sampling → TRG is free from sign problem

Phase structure of CP(1) model on 2d lattice

H. Aizawa, S. Takeda, Y. Yoshimura,
PTEP(2026) 013B07



We will investigate continuum limit as next step

Step Scaling Function

- A non-perturbative tool to determine the running coupling in lattice field theory

- It is interesting to study running coupling at $\theta \neq 0$ non-perturbatively.

In this talk

As a first test, we focus on the case of $\theta = 0$ and examine whether TRG can reproduce the MC results.

We then extend the analysis to the $\theta \neq 0$ case.

Step Scaling Function

Definition of renormalized coupling in CP(1) model

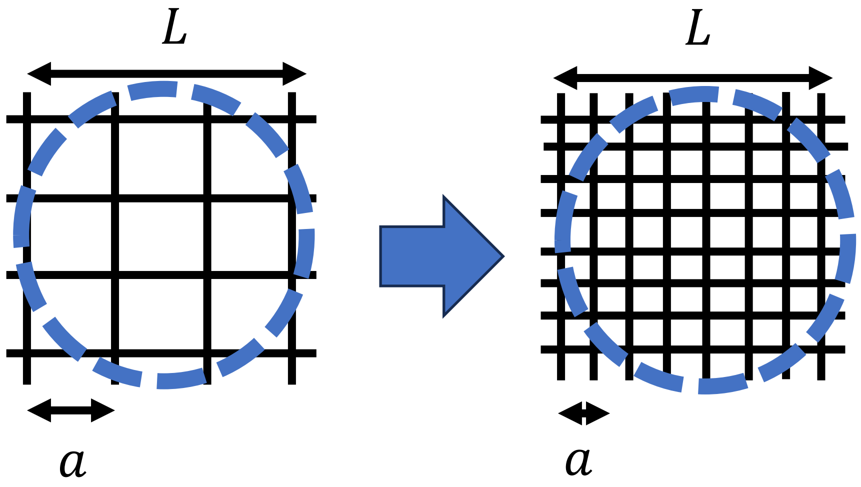
M.Lüscher, P.Weisz, U.Wolff
Nucl. Phys. B359,1,(1991)

$$g^2(L) = mL$$
$$= (E_1 - E_0)a\,\frac{L}{a}$$

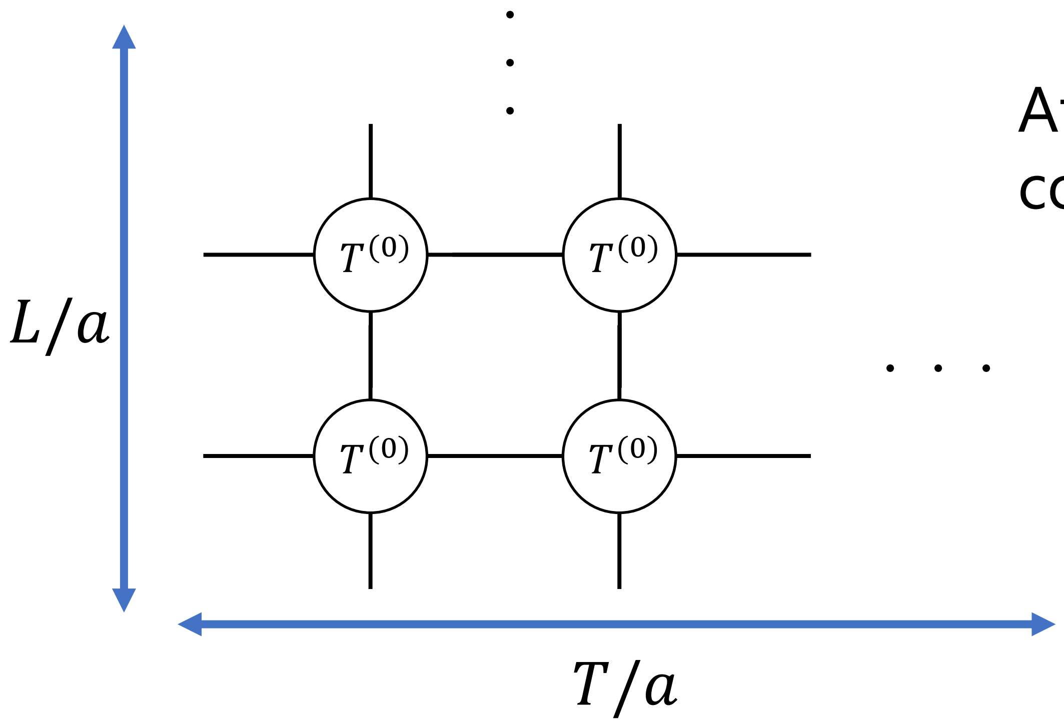
Definition of SSF

$$\Sigma\left(\frac{L'}{L}, g^2(L), \frac{a}{L}\right) = g^2(L')$$

In this study: $L' = 2L$



$\frac{L}{a}$: Lattice size

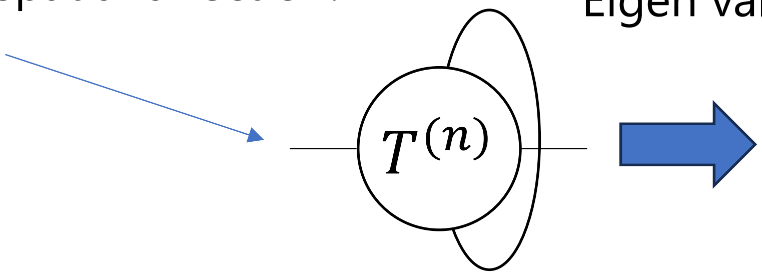


Partition function

After n times coarse-graining

transfer matrix of a system with periodic boundary conditions in the spatial direction.

Eigen value



$$\lambda_i = e^{-E_i T}$$

How to compute SSF from lattice data

$\theta = 0,$ BWTRG, $k = -\frac{1}{2}$

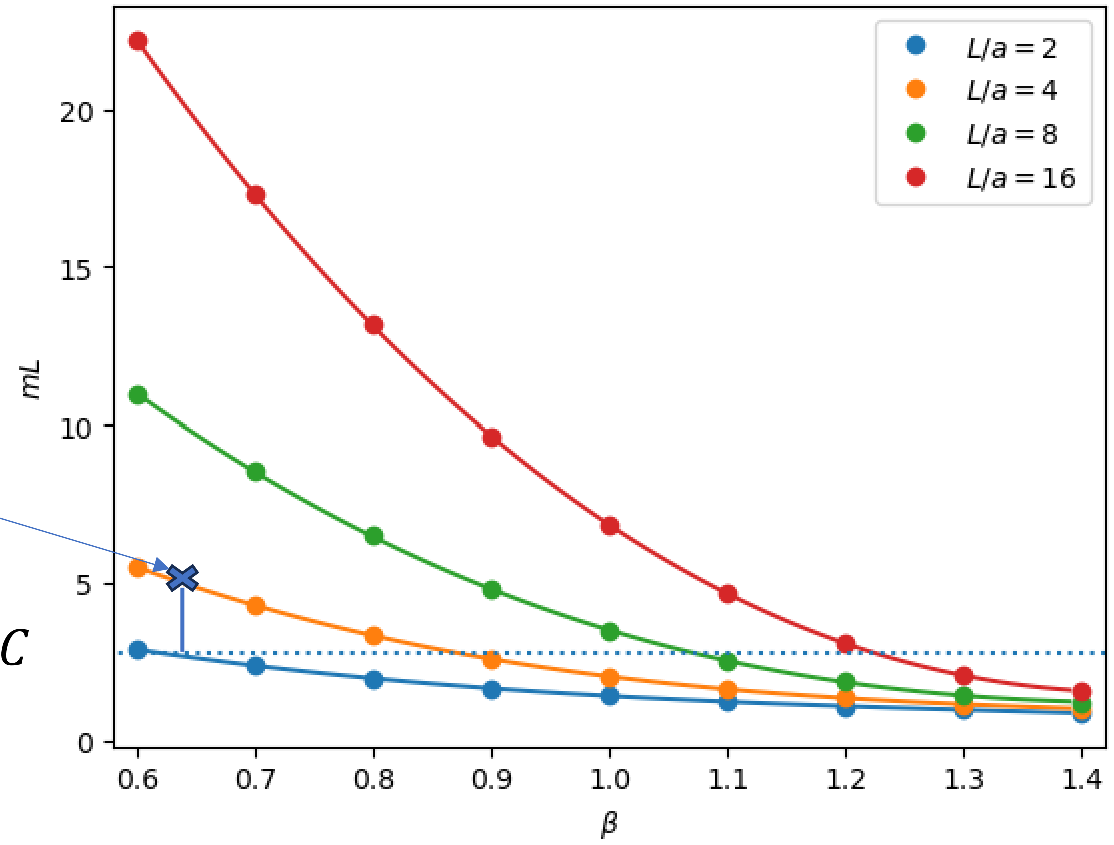
D. Adachi, T. Okubo, and S. Todo,
Phys. Rev. B 105, L060402(2022)

Renormalization condition

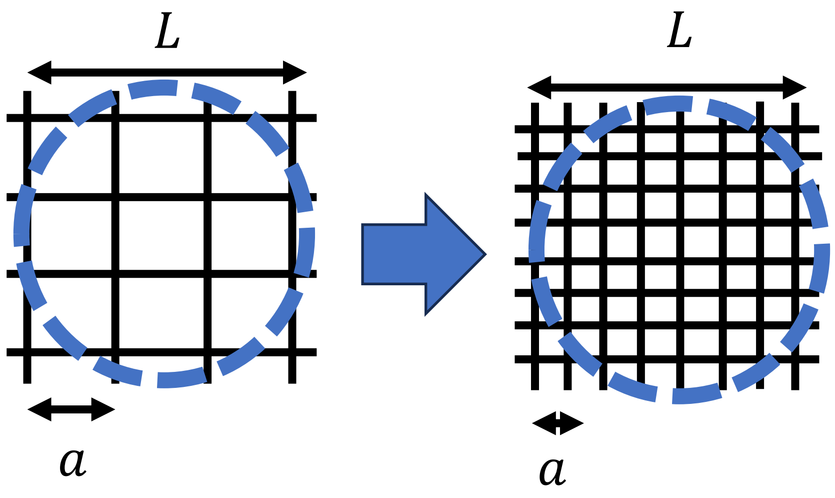
$$g^2(L) = C \text{ (fix)}$$

$$g^2(2L) = \Sigma\left(2, g^2(L), \frac{a}{L} = \frac{1}{2}\right)$$

$$g^2(L) = C$$



How to compute SSF from lattice data



$$\Sigma\left(2, g^2(L), \frac{a}{L}=\frac{1}{2}\right), \Sigma\left(2, g^2(L), \frac{a}{L}=\frac{1}{4}\right), \Sigma\left(2, g^2(L), \frac{a}{L}=\frac{1}{8}\right)$$

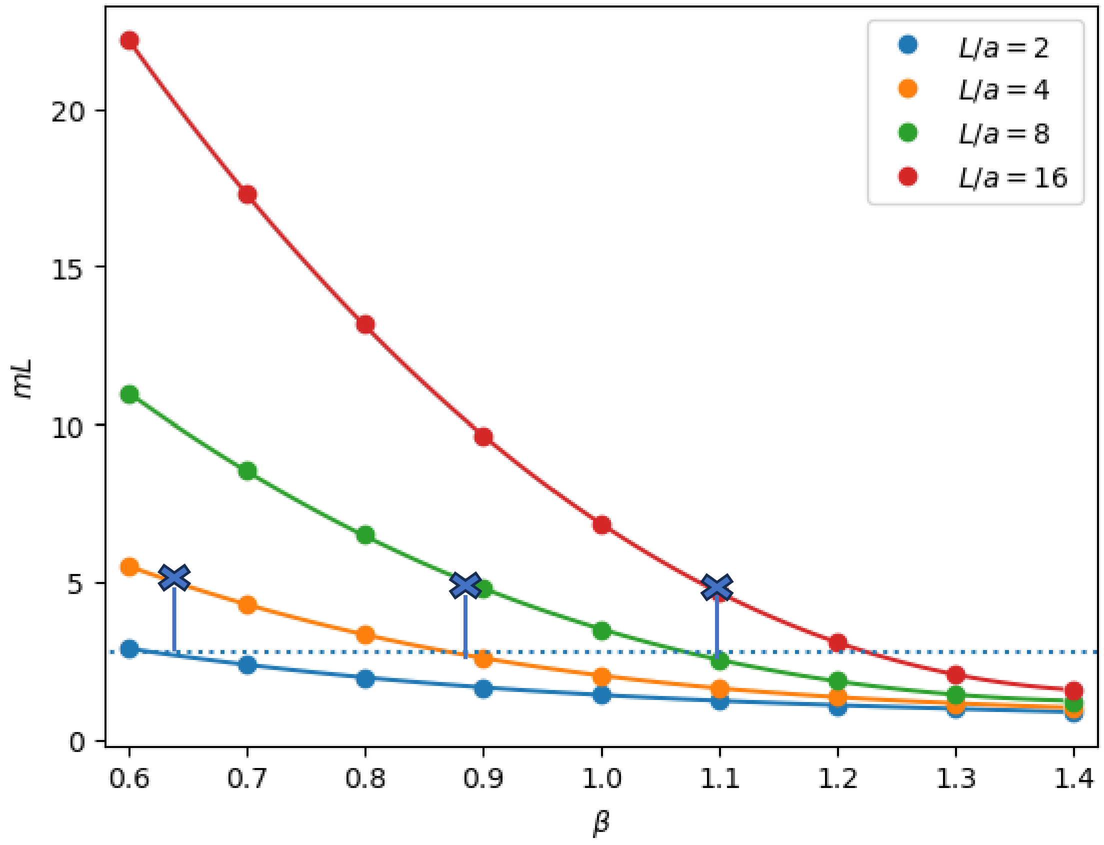
β_1, β_2, β_3, …

a, $\frac{a}{2}$, $\frac{a}{4}$, …

Continuum limit

$\theta = 0$, BWTRG, $k = -\frac{1}{2}$

D. Adachi, T. Okubo, and S. Todo, Phys. Rev. B 105, L060402(2022)



β_1 β_2 β_3

Continuum limit

Renormalization condition

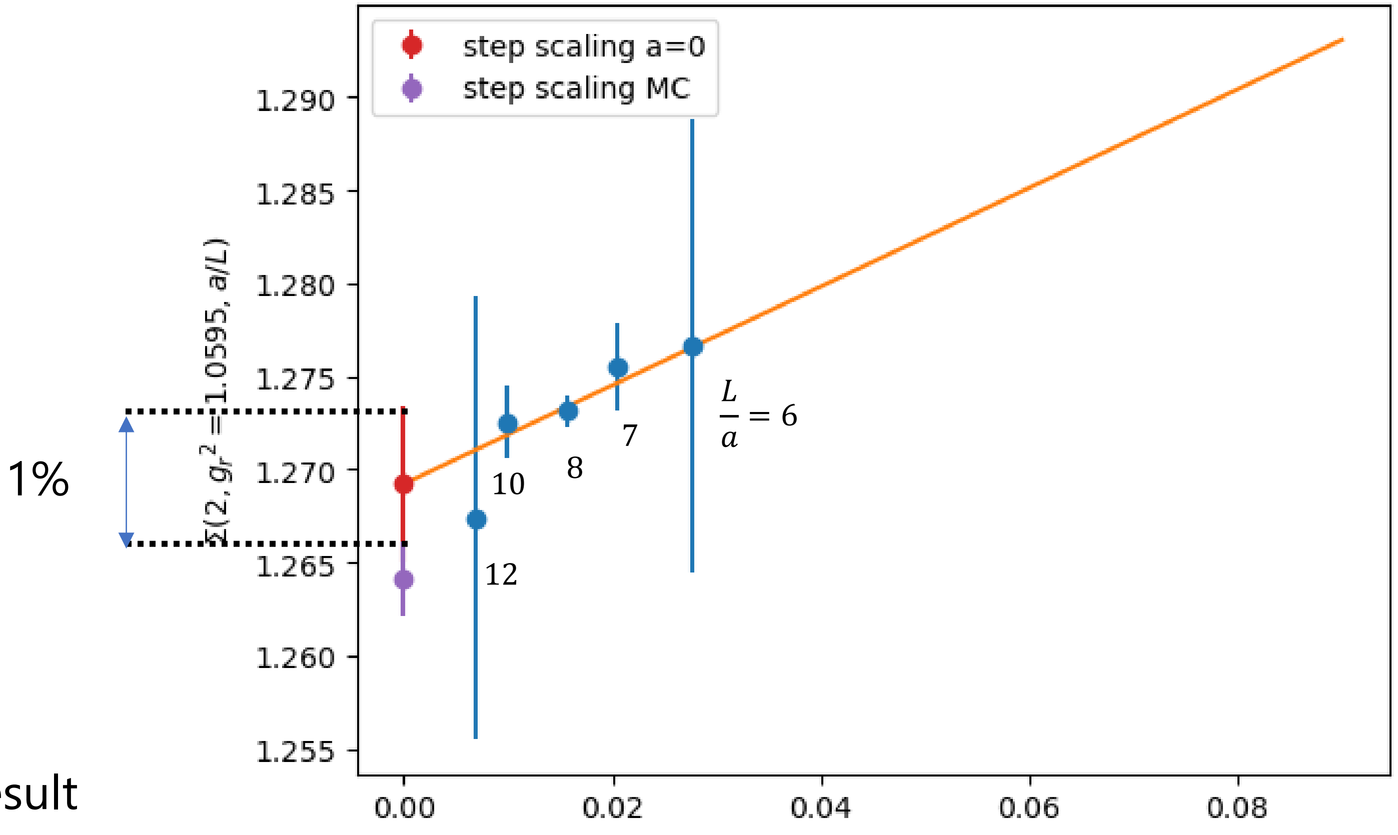
$$g^2(L) = 1.0595$$

Error estimation

- β interpolation
  $\Delta\beta = 0.01$

- D_c extrapolation
  $D_c = 90,100,110,120,128$

Our result is consistent with the MC result within 1% error

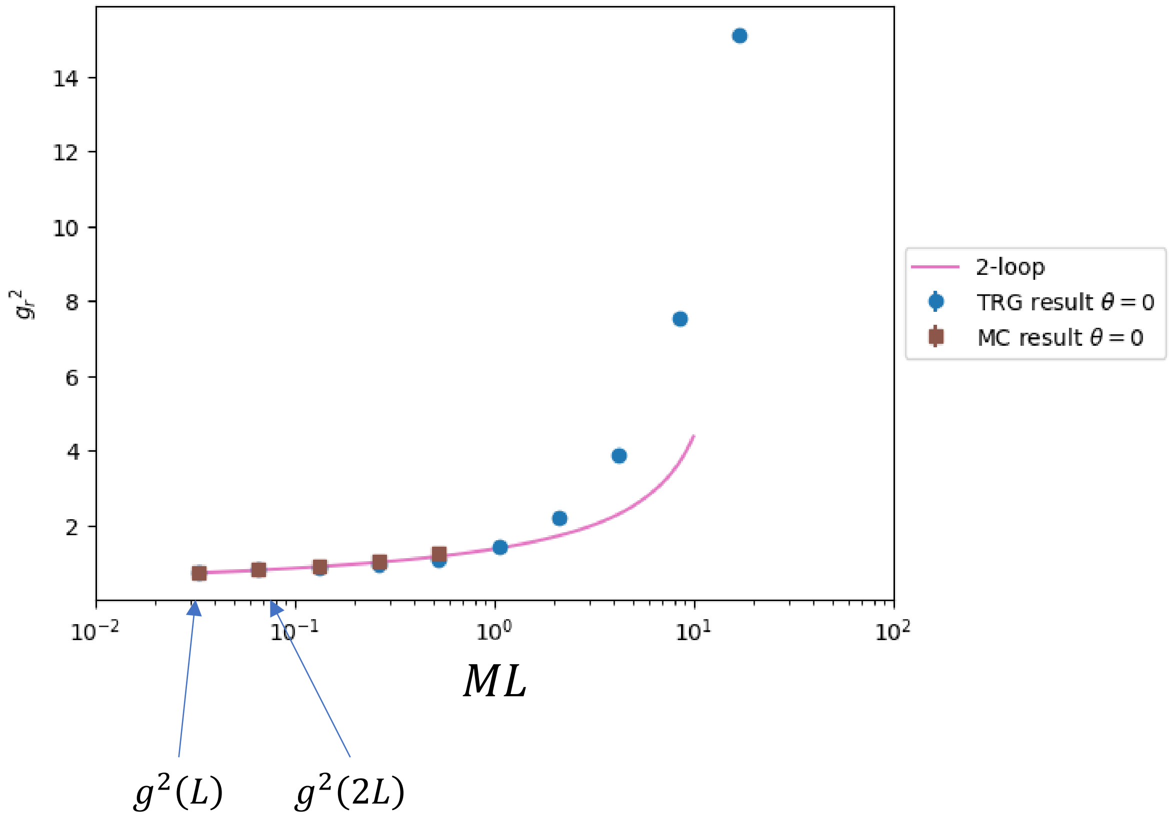


M.Lüscher, P.Weisz, U.Wolff
Nucl. Phys. B359,1,(1991)

Running coupling $(\theta = 0)$

$$g^2(2L) = \Sigma\big(2, g^2(L)\big)$$
$$g^2(4L) = \Sigma\big(2, g^2(2L)\big)$$
$$\vdots$$
$$g^2(2^n L) = \Sigma\big(2, g^2(2^{n-1} L)\big)$$

By repeating this procedure,
we can determine the running coupling
over a wide range of energy scales.



$\theta \neq 0$ case

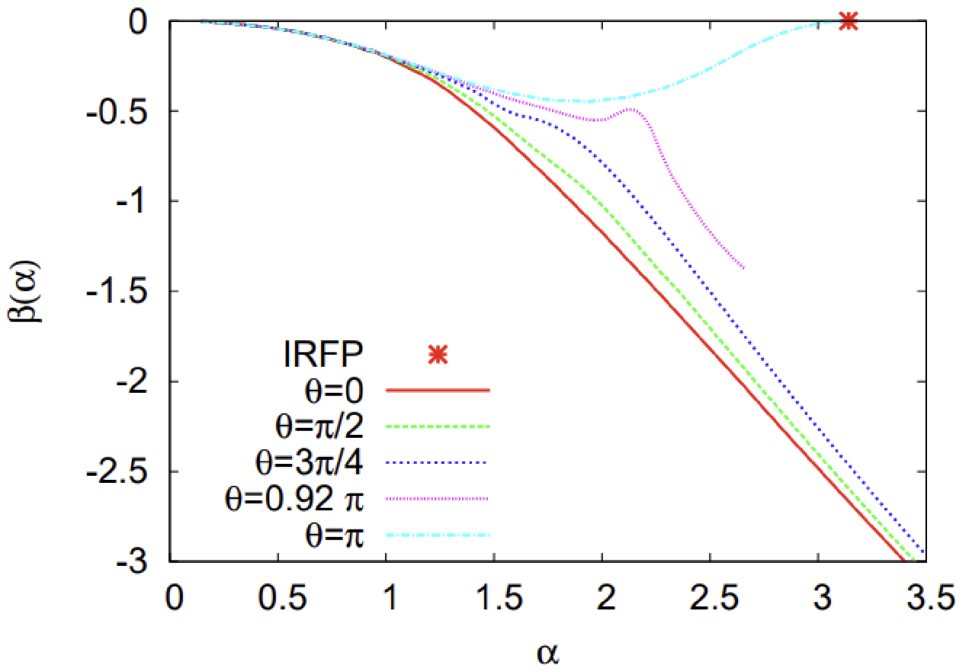
There is IR fixed point at $\theta = \pi$.

J. Balog and A. Hegedus, J. Phys. A37 (2004) 1881;
Nucl. Phys. B725 (2005) 531; Nucl. Phys. B829 (2010) 425.

Meron cluster algorithm demonstrates this IRFP.

P.de Forcrand, M.Pepe and U.J.Wiese,
Phys.Rev.D 86 (2012) 075006

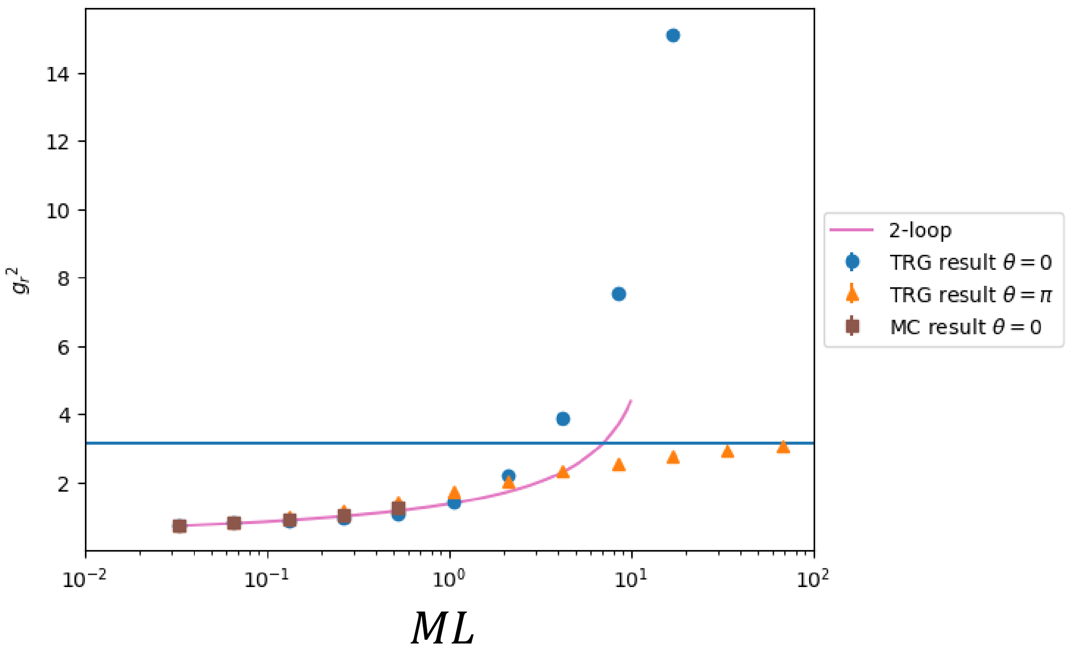
Furthermore, walking behaver is indicated $\theta \sim \pi$.

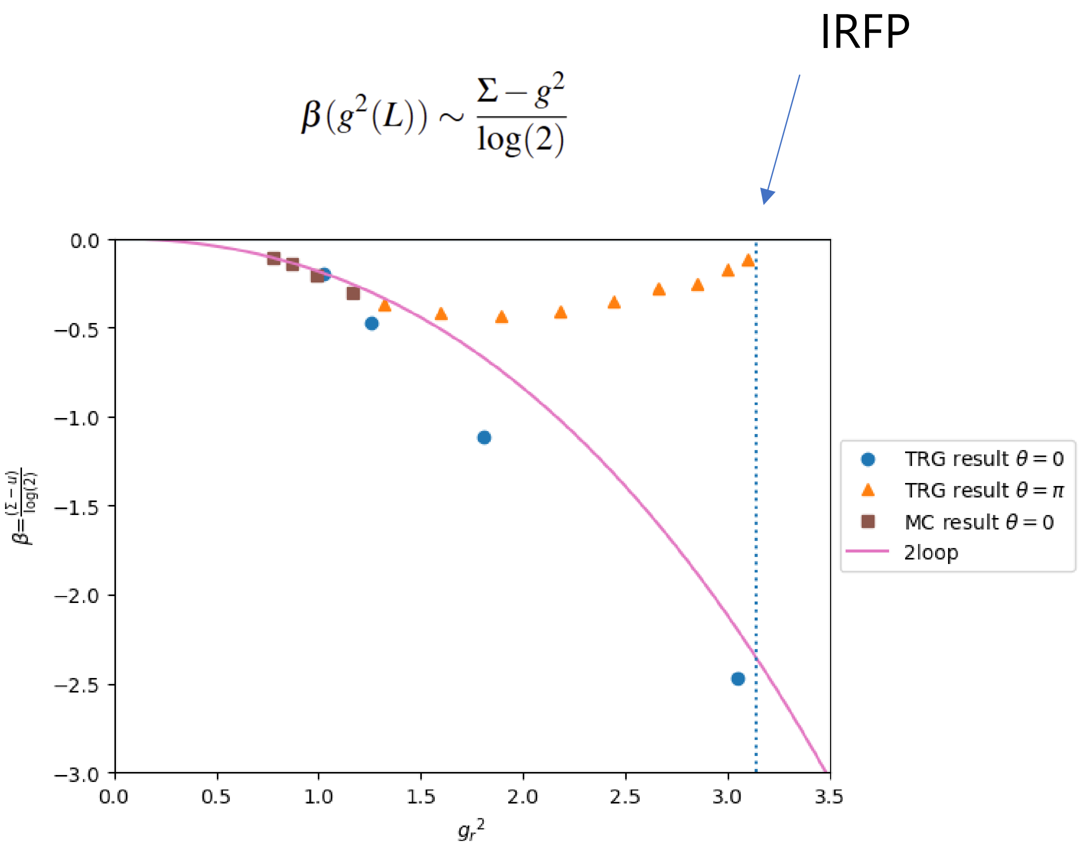


P.de Forcrand, M.Pepe and U.J.Wiese,
Phys.Rev.D 86 (2012) 075006

TRG result at $\theta = \pi$ (preliminary results)

$$\beta(g^2(L)) \sim \frac{\Sigma - g^2}{\log(2)}$$





Summary

- We compute SSF of CP(1) model with θ term by using TRG.
- In $\theta = 0$ case, our result is consistent with MC result within 1% error.
- In $\theta = \pi$ case, our result indicates the presence of IRFP.

Future work

Reducing error for $\theta \neq 0$ case.

Investigating walking behavior of coupling near $\theta = \pi$

Buck up

Difficulty of TRG

Computational cost: $O(D_c^{2d+1})$

d: space-time dimension

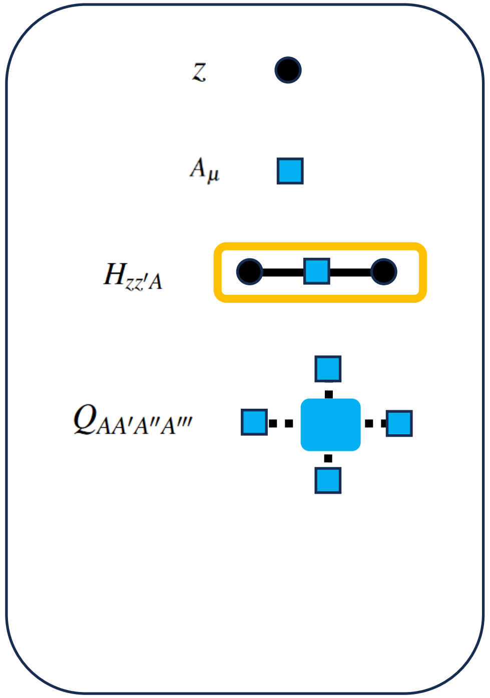
larger internal d.o.f require higher D_c

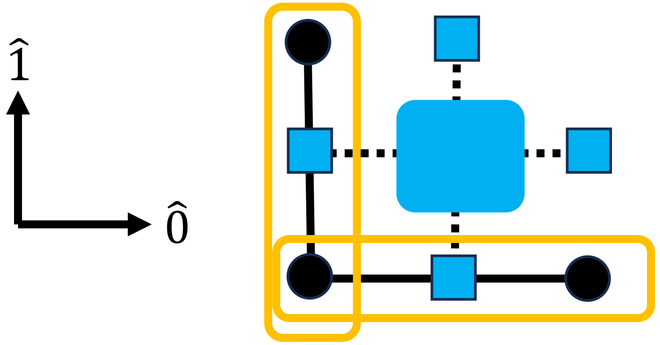
Computing 4d QCD using TRG is still challenging.

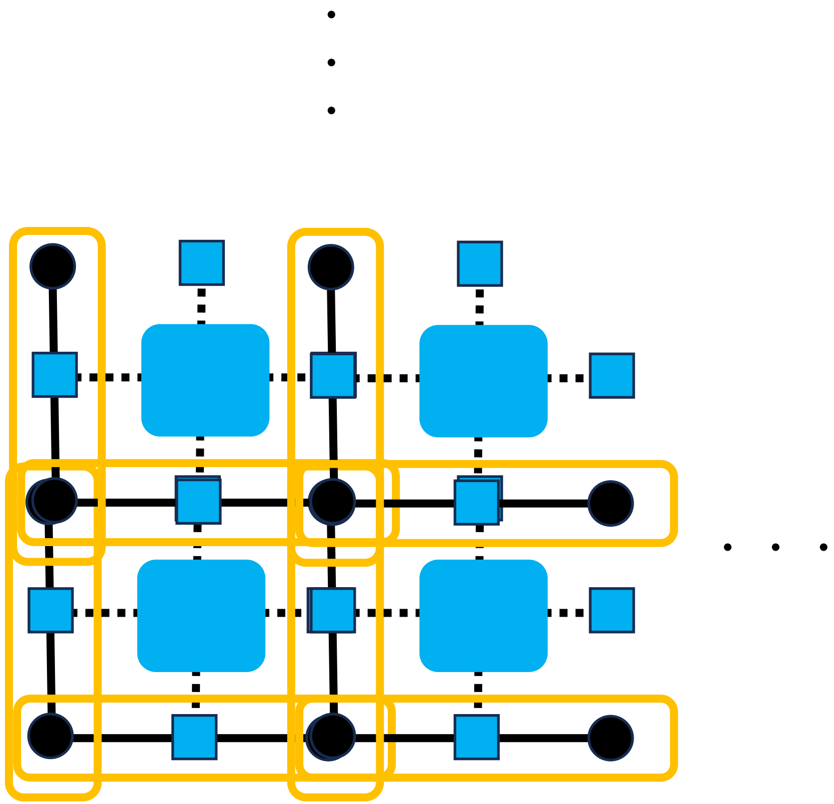
Toy model

2d CP(1) with θ term

Initial tensor





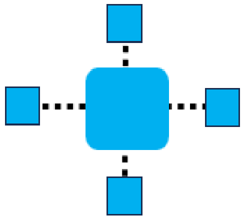


$$H_{zz'A} := \exp\left(2\beta\left[z^\dagger z' e^{iA} + z'^\dagger z e^{-iA}\right]\right),$$

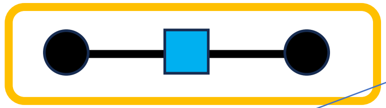
$$Q_{AA'A''A'''} \quad := \quad \exp\left(\frac{\theta}{2\pi}\log\left[e^{i(A+A'-A''-A''')}\right]\right).$$

$$Z(\beta,\theta) = \left(\prod_x \int dz(x)\right)\left(\prod_{x,\mu}\int_{-\pi}^{\pi}\frac{dA_\mu(x)}{2\pi}\right)\prod_{x,\mu} H_{z(x),z(x+\hat{\mu}),A_\mu(x)}\prod_x Q_{A_0(x),A_1(x+\hat{0}),A_0(x+\hat{1}),A_1(x)}.$$

$\theta = 0$ case

$Q_{AA'A''A'''}$  $= 1$

$$\left(\prod_{x,\mu}\int_{-\pi}^{\pi}\frac{dA_\mu(x)}{2\pi}\right)\prod_{x,\mu}H_{z(x),z(x+\hat{\mu}),A_\mu(x)}$$



$$\int dz f(z) \approx \sum_{i=1}^{N_z} W_i^{(z)} f(z_i),$$

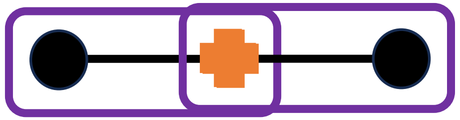
Scalar field

Genz,Keister(1996)

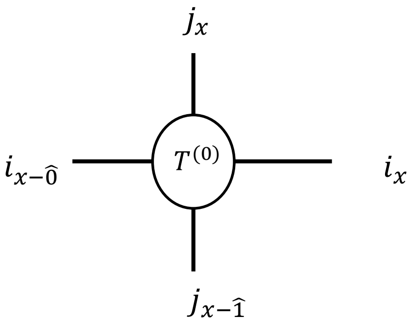
$I_0(4\beta|z^\dagger z'|)$:Modified Bessel function

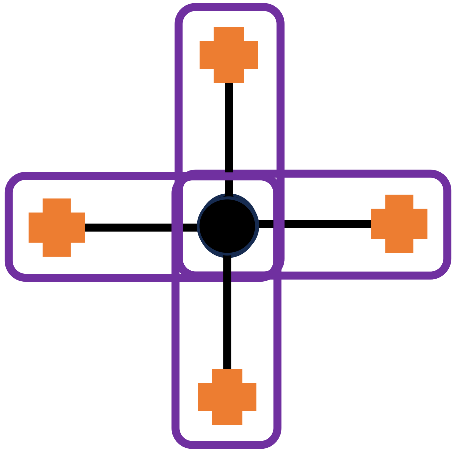
Matrix

SVD

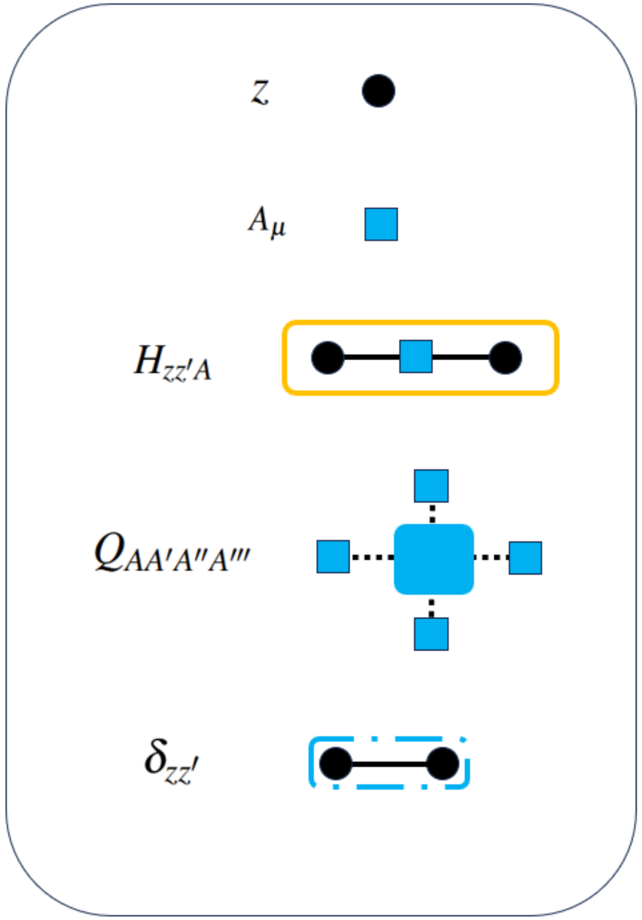


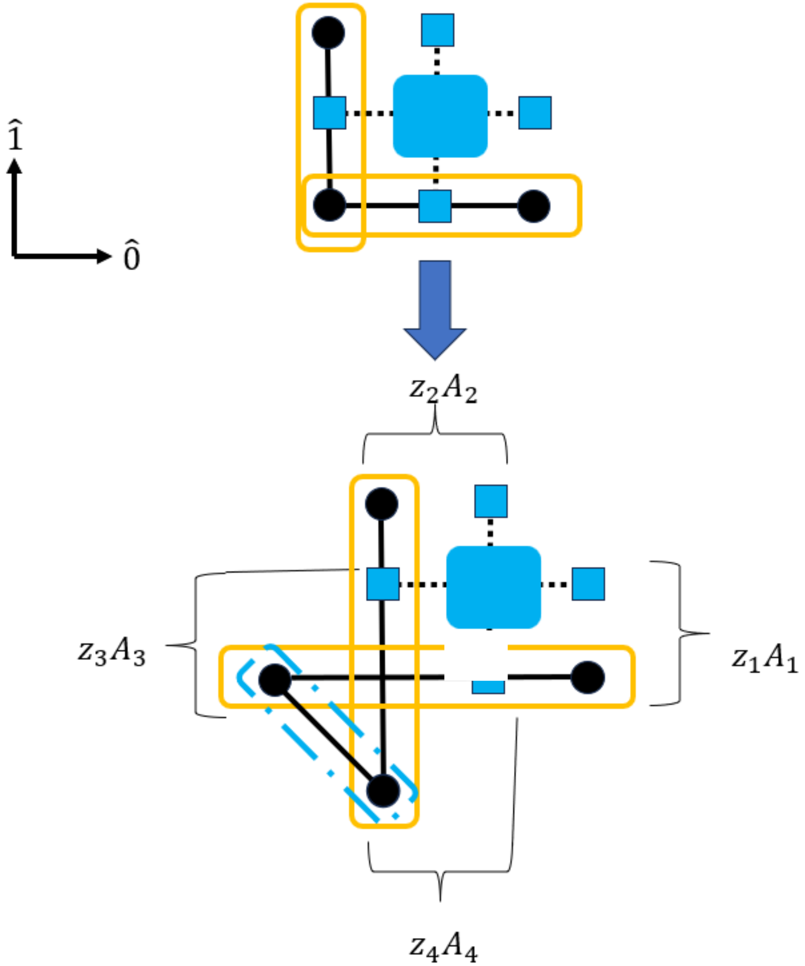
Initial tensor

$T^{(0)}{}_{i_x,j_x,i_{x-\hat{0}},j_{x-\hat{1}}} =$ 

$= \sum_{i=1}^{N_z} W_i^{(z)}$ 

$\theta \neq 0$





Using quadrature

$$\int dz f(z) \approx \sum_{i=1}^{N_z} W_i^{(z)} f(z_i),$$

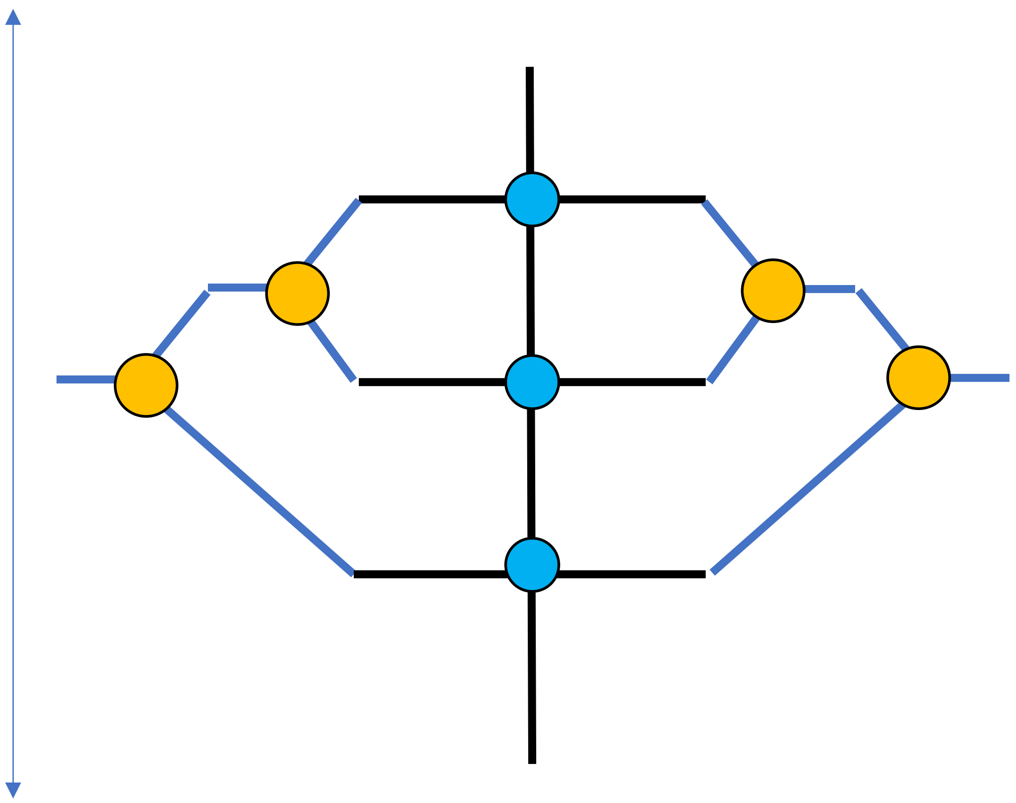
Scalar field   Genz,Keister(1996)

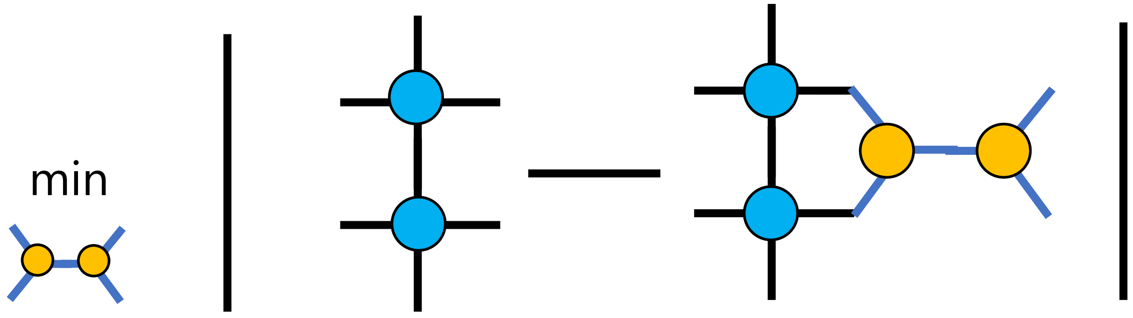
$$\int dU f(A) \approx \sum_{a=1}^{N_A} W_a^{(A)} f(A_a)$$

Gauge field   Ryo Sakai et al.(2018)

i, a become tensor index

$$\frac{L}{a} = 3$$





Z.Y. Xie, et al. Phys. Rev. B 86 (2012)045139

Initial tensor

Previous study

Using character expansion

$$e^{i\frac{\theta}{2\pi}q_p} = \sum_{k\in\mathbb{Z}} e^{ik(A_1+A_2-A_3-A_4)} C_k(\theta)$$

truncate

$$C_k(\theta) \propto \frac{1}{k}$$

Converge slowly

New tensor

Using quadrature

$$\int dz f(z) \approx \sum_{i=1}^{N_z} W_i^{(z)} f(z_i), \qquad \int dU f(A) \approx \sum_{a=1}^{N_A} W_a^{(A)} f(A_a)$$

i, a become tensor index

Scalar field Genz,Keister(1996)

Gauge field Ryo Sakai et al.(2018)

Comparison of initial tensor

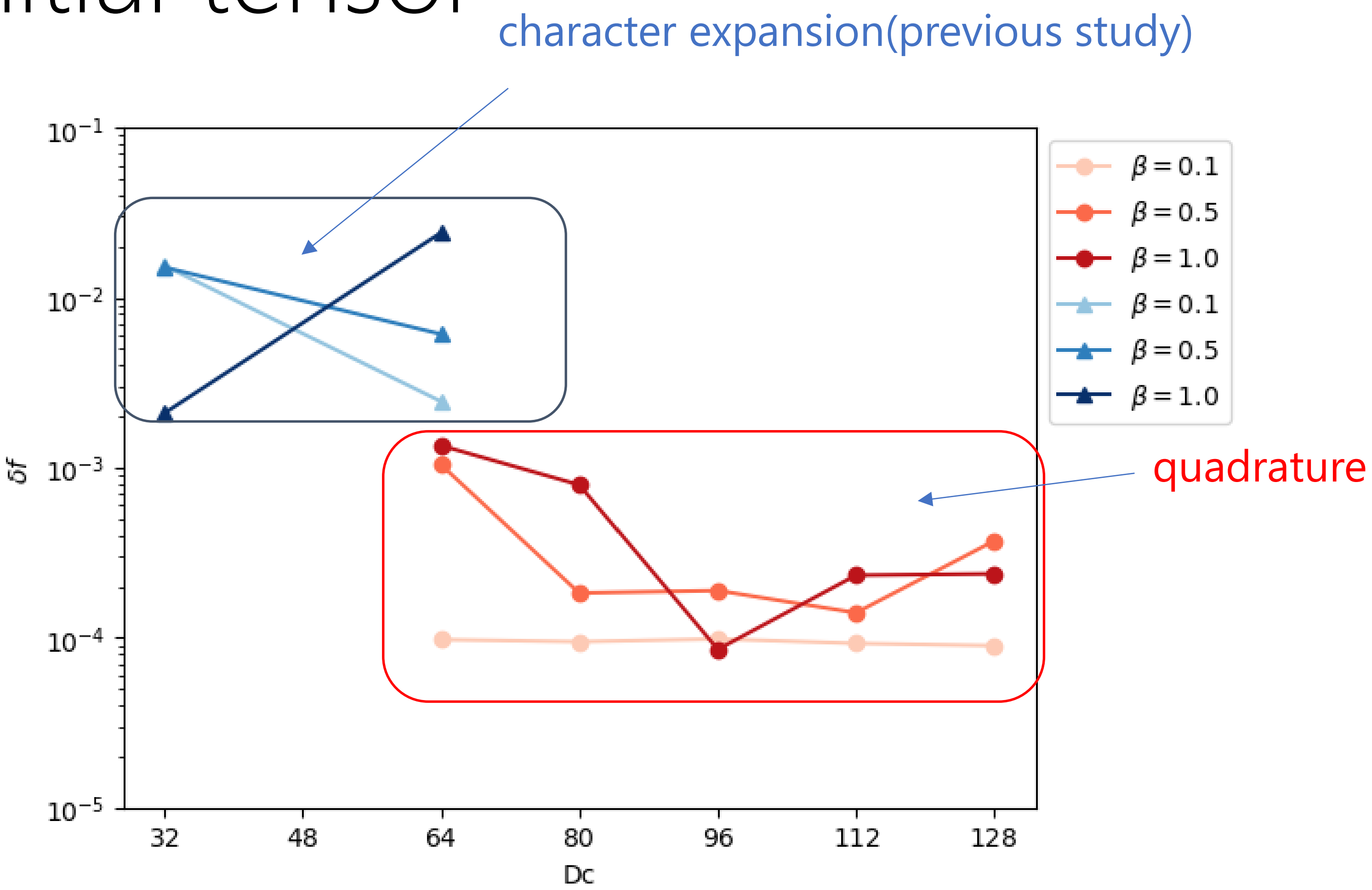
we investigate the error comparing the exact value on 2×2 lattice.

$$\delta f = \left| \frac{f_{tensor} - f_{exact}}{f_{exact}} \right|$$

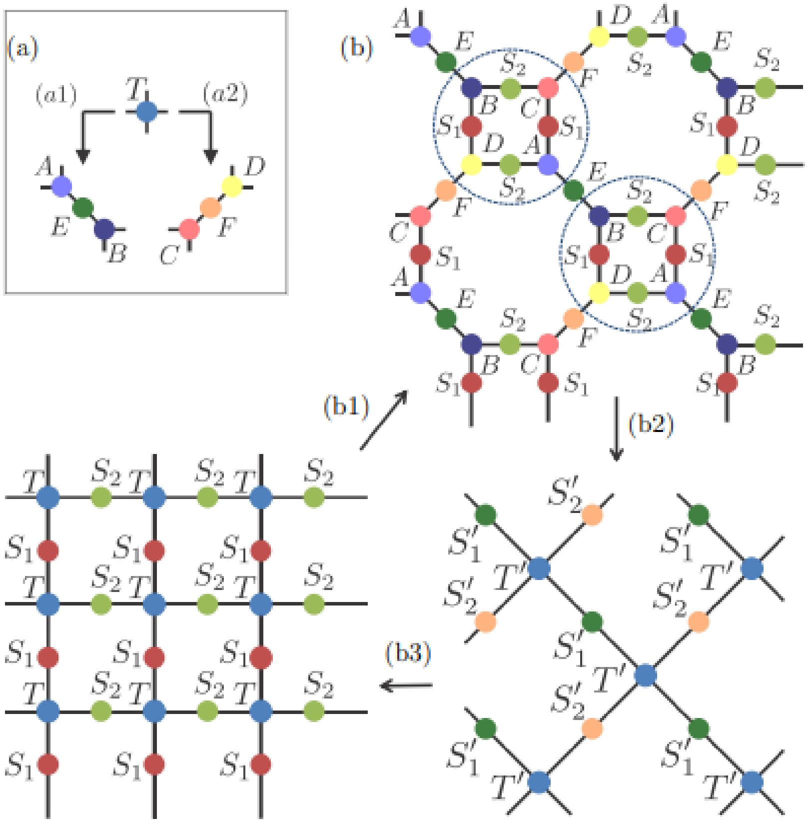
Parameters

$N_z = 224,\ N_A = 120$

$\theta = \pi$



New initial tensor is better



D. Adachi, T. Okubo, and S. Todo,
Phys. Rev. B 105, L060402(2022)

$$A_{(x_0,y_0),i} = U_{1\,(x_0,y_0),i}\sigma_{1\,ii}^{(1-k)/2},$$

$$E_{i,j} = \delta_{ij}\sigma_{1\,ii}^{k},$$

$$B_{i,(x_1,y_1)} = \sigma_{1\,ii}^{(1-k)/2}V_{1\,i,(x_1,y_1)},$$

$$C_{(x_0,y_1),i} = U_{2\,(x_0,y_1),i}\sigma_{2\,ii}^{(1-k)/2},$$

$$F_{i,j} = \delta_{ij}\sigma_{2\,ii}^{k},$$

$$D_{i,(x_1,y_0)} = \sigma_{2\,ii}^{(1-k)/2}V_{2\,i,(x_1,y_0)}.$$

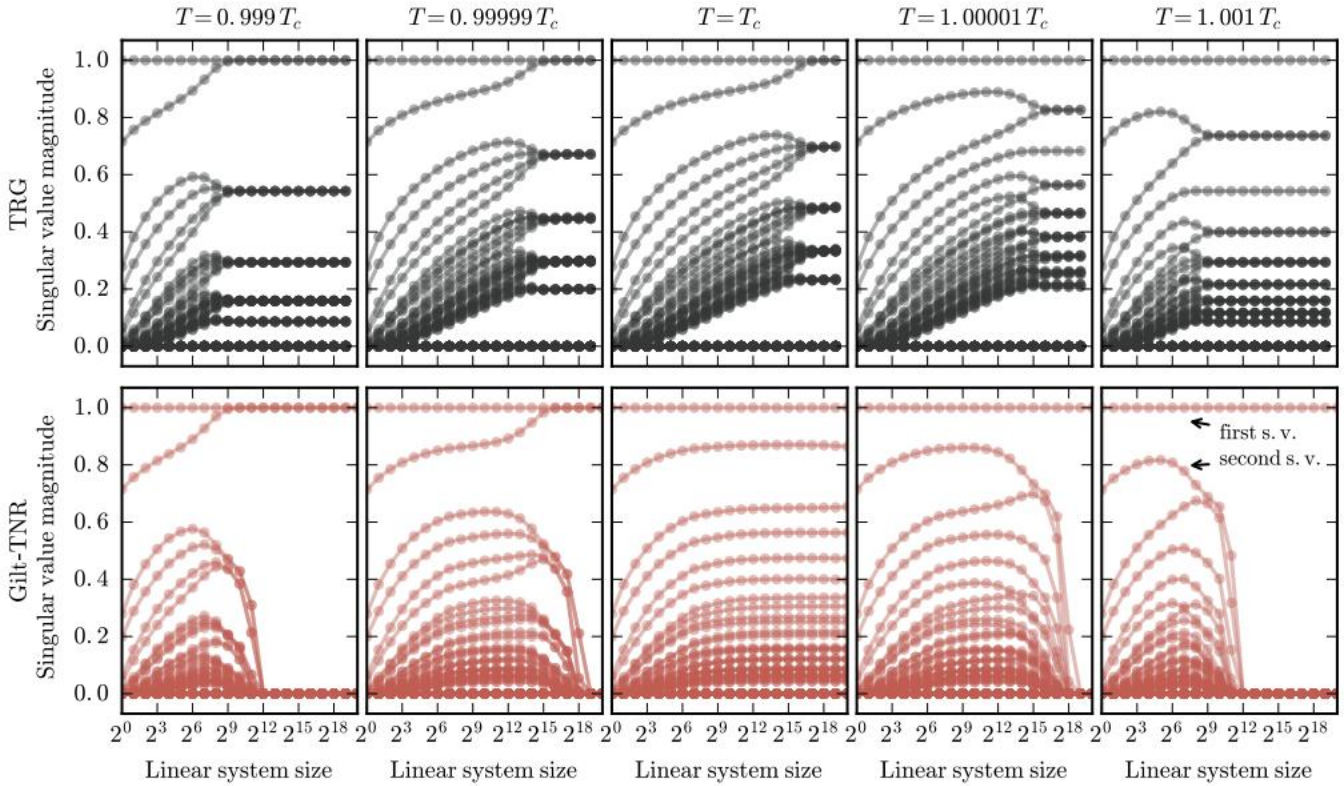
$$T_{x_0,x_1,y_0,y_1} \approx \sum_{i}^{\chi} U_{1\,(x_0,y_0),i}\sigma_{1\,ii}V_{1\,i,(x_1,y_1)},$$

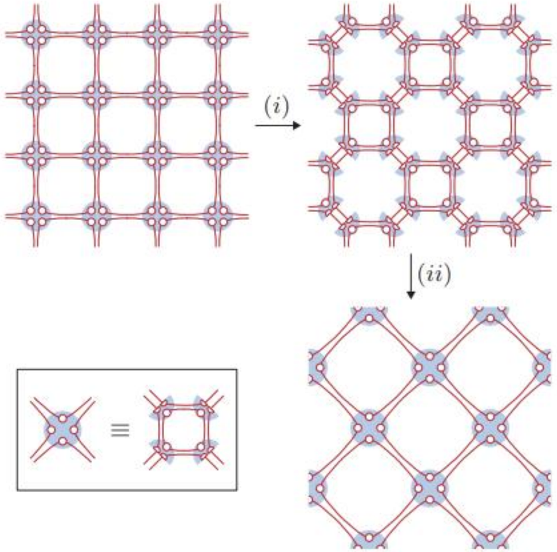
$$T_{x_0,x_1,y_0,y_1} \approx \sum_{i}^{\chi} U_{2\,(x_0,y_1),i}\sigma_{2\,ii}V_{2\,i,(x_1,y_0)},$$

$$T'_{x_0,x_1,y_0,y_1} = \sum_{i_0,i_1,i_2,i_3}\left[B_{x_0,(i_0,i_2)}C_{(i_0,i_3),y_0}D_{y_1,(i_1,i_2)}\right.$$
$$\left.\times A_{(i_1,i_3),x_1}S_{2\,i_0,i_0}S_{2\,i_1,i_1}S_{1\,i_2,i_2}S_{1\,i_3,i_3}\right],$$

$$S'_1 = E,$$

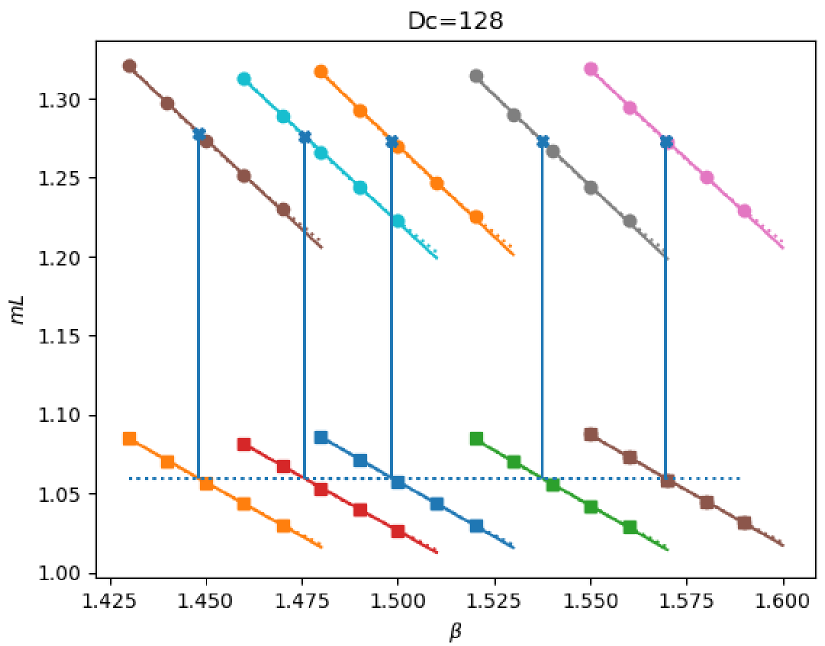
$$S'_2 = F.$$

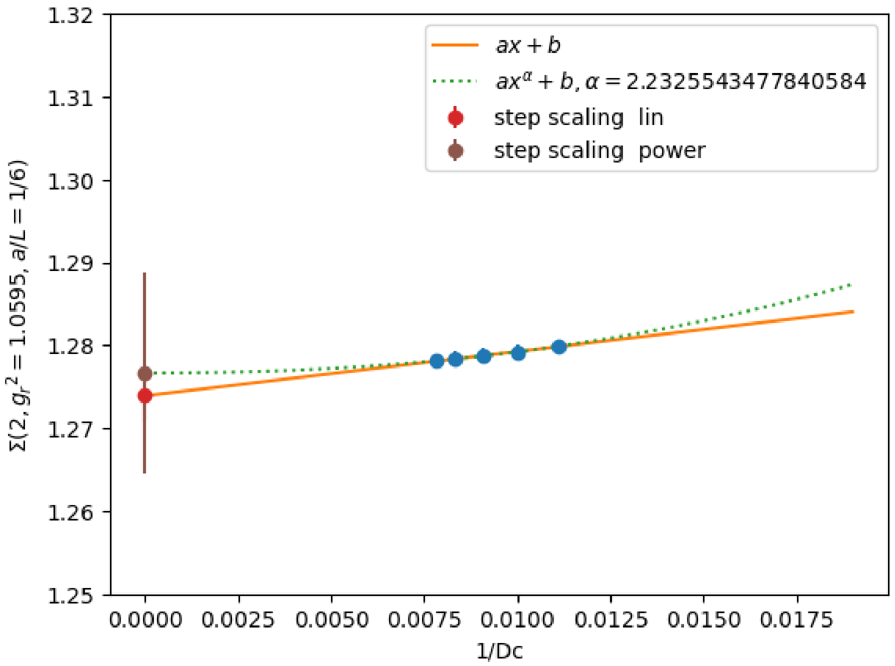




M. Hauru, C. Delcamp, and S. Mizera
Phys. Rev. B **97**, 045111

Error Estimation



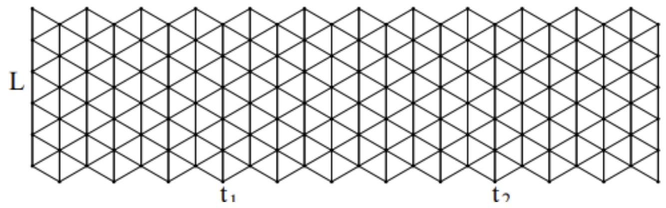


W. Bietenholz, A. Pochinsky, and U.-J. Wiese, Phys. Rev. Lett. 75 (1995) 4524.

$$S[\vec{e}] = \sum_{\langle xy \rangle} s(\vec{e}_x, \vec{e}_y)\,,$$

$$s(\vec{e}_x, \vec{e}_y) = \frac{1}{g^2}(1 - \vec{e}_x \cdot \vec{e}_y)$$

$$s(\vec{e}_x, \vec{e}_y) = \begin{cases} \frac{1}{g}(1 - \vec{e}_x \cdot \vec{e}_y), & \text{if } \vec{e}_x \cdot \vec{e}_y > -1/2 \\ +\infty, & \text{otherwise}\,, \end{cases}$$



$$S = -2\beta \sum_{x,\mu} \left[z^\dagger(x) z(x+\hat{\mu}) U_\mu(x) + z^\dagger(x+\hat{\mu}) z(x) U_\mu^{-1}(x) \right] - i\frac{\theta}{2\pi} \sum_x q(x)$$