Improving the Wilson action

Jorge Baeza-Ballesteros

Lattice 2026 - 29th July 2026

in collaboration with A. Risch, M. Roost, S. Schaefer and R. Sommer

Deutsches Elektronen
-Synchrotron Zeuthen

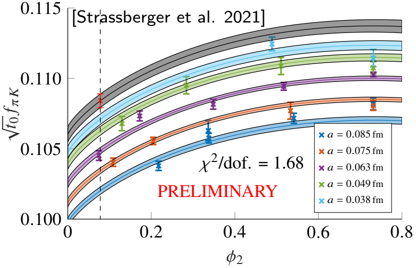
Limitations of the Wilson action

$\mathcal{O}(a)$-improved Wilson action has been very successful in many applications, but experience has shown us **it has some limitations**

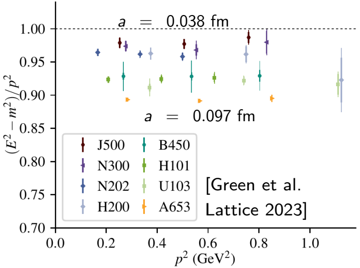
Limitations of the Wilson action

$\mathcal{O}(a)$**-improved Wilson action has been very successful** in many applications, but experience has shown us **it has some limitations**

➤ **Significant discretization effects** found for some observables



Meson decay constants

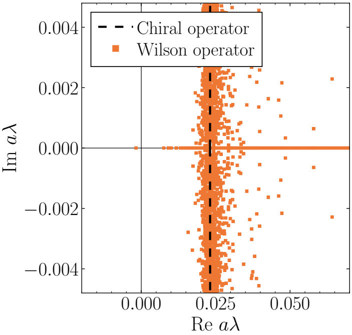


***D*-meson dispersion relation**

Main discretization error affecting two-particle spectroscopy [Hansen and Peterken, 2023]

Limitations of the Wilson action

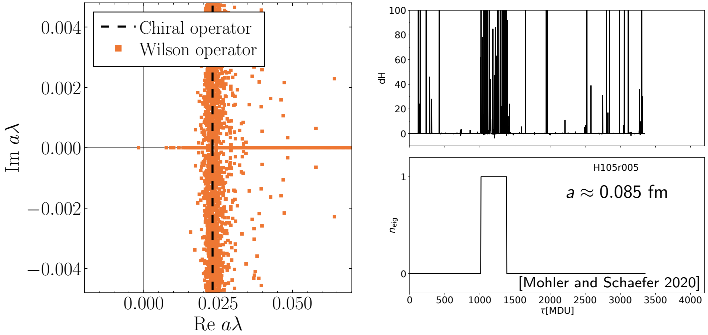
- **Small and negative real eigenvalues** in the spectrum of D



- **Large condition numbers** and slow convergence

Limitations of the Wilson action

- **Small and negative real eigenvalues** in the spectrum of D



- **Large condition numbers** and slow convergence
- Potential barrier ⟶ **Ergodicity problems**
  (long autocorrelation times, reweighting factors...)
- Enhanced by **large values of improvement coeffs.**: $c_{sw} \sim 2$ at $a \approx 0.1$ fm

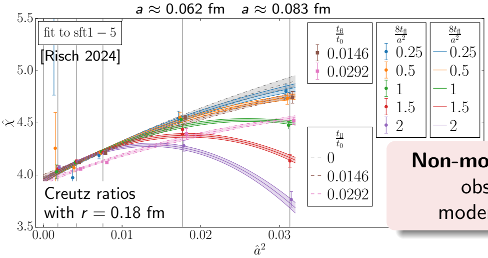
Proposal for a new action

We propose a **new action based on three modifications**

Proposal for a new action

We propose a **new action based on three modifications**

- **"Safe" smearing of links in D**
  (single step of stout-smearing, $\rho = 0.125$)
  - Reduces appearance of small real eigenvalues and magnitude of improvement coefficients
  - Excessive smearing **may induce non-monotonic scaling**

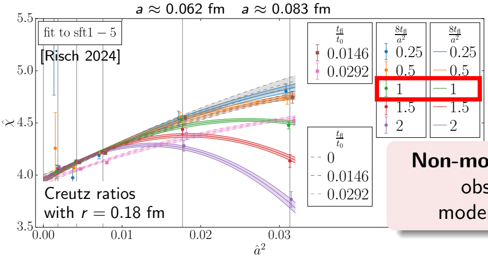


Non-monotonic scaling observed with moderate smearing

Proposal for a new action

We propose a **new action based on three modifications**

- **"Safe" smearing of links in D**
  (single step of stout-smearing, $\rho = 0.125$)
  - Reduces appearance of small real eigenvalues and magnitude of improvement coefficients
  - Excessive smearing **may induce non-monotonic scaling**



Non-monotonic scaling observed with moderate smearing

Proposal for a new action

We propose a **new action based on three modifications**

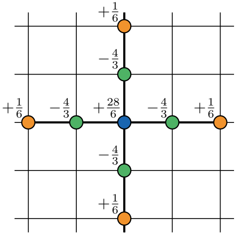
- **"Safe" smearing of links in D**
  (single step of stout-smearing, $\rho = 0.125$)

- **Tree-level improvement at $\mathcal{O}(a^2)$**
  [Sheikholeslami and Wohlert 1985]
  - Restores rotational invariance at $\mathcal{O}(a^2)$ at tree level
  - Reduces logarithmic lattice artifacts

$$\mathcal{O}(0) = \mathcal{O}(a) + \sum_i \hat{c}_i M_i a^2 [\alpha_s(a^{-1})]^{\Gamma_i}$$

with $\Gamma_i \to \Gamma_i + 1$ (no dim-6 operator in SymEFT at tree level) [Husung 2024]

D_{234} operator
[Alford et al. 1997]



Includes dim-6 and -7 operators, and $r = 2/3$

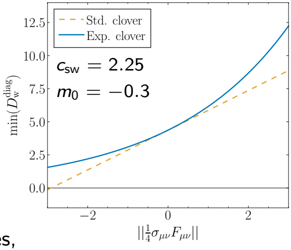
↓

Efficient implementation

Proposal for a new action

We propose a **new action based on three modifications**

- **"Safe" smearing of links in D**
  (single step of stout-smearing, $\rho = 0.125$)
- **Tree-level improvement at $\mathcal{O}(a^2)$**
  [Sheikholeslami and Wohlert 1985]
- **Exponentiated clover term**
  [Francis et al. 2020]
  - Expected to reduce small / negative eigenvalues, especially for large $c_{\rm sw}$



$$D_{234}^{\rm diag} = \frac{28}{6} + m_0 + \frac{ic_{\rm sw}}{6}\sigma_{\mu\nu}F_{\mu\nu} \longrightarrow \left(\frac{28}{6} + m_0\right)\exp\left[\frac{1}{\frac{28}{6} + m_0}\frac{ic_{\rm sw}}{6}\sigma_{\mu\nu}F_{\mu\nu}\right]$$

Proposal for a new action

We propose a **new action based on three modifications**

- **"Safe" smearing of links in D**
  (single step of stout-smearing, $\rho = 0.125$)
- **Tree-level improvement at $\mathcal{O}(a^2)$**
  [Sheikholeslami and Wohlert 1985]
- **Exponentiated clover term**
  [Francis et al. 2020]

This talk: effect of these modifications → **Discretization effects**, **Spectrum of $\gamma_5 D$**

Proposal for a new action

We propose a **new action based on three modifications**

- **"Safe" smearing of links in D**
  (single step of stout-smearing, $\rho = 0.125$)

- **Tree-level improvement at $\mathcal{O}(a^2)$**
  [Sheikholeslami and Wohlert 1985]

- **Exponentiated clover term**
  [Francis et al. 2020]

This talk: effect of these modifications $\rightarrow$ **Discretization effects**, **Spectrum of $\gamma_5 D$**

- **Non-perturbative** c_{sw} to remove $\mathcal{O}(a)$ effects in all cases
- Unless otherwise stated, exp. clover + Lüscher-Weisz

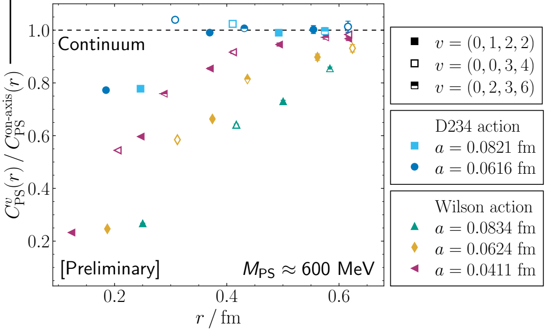
Restoration of rotational invariance

First: **Restoration of rotational symmetry** at short distances

Ratio of two-point function along off- and on-axis directions

$C^v_{\rm PS} = \langle O_{\rm PS}(x + r\hat{v}) O^\dagger_{\rm PS}(x) \rangle \quad O_{\rm PS} = \bar{u}\gamma_5 d$

No $\mathcal{O}(a^2)$ counterterm needed at tree level



LW gauge action + D_{234} + smearing

Plaquette gauge action + $D_{\rm W}$ + std. clover

- **Smaller discretization effects** at short distances
- **Rotational invariance restored after few lattice spacings**

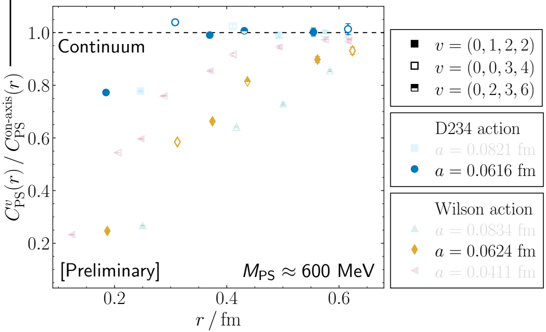
Restoration of rotational invariance

First: **Restoration of rotational symmetry** at short distances

Ratio of two-point function along off- and on-axis directions

$C^v_{\mathrm{PS}} = \langle O_{\mathrm{PS}}(x + r\hat{v}) O^\dagger_{\mathrm{PS}}(x) \rangle \quad O_{\mathrm{PS}} = \bar{u}\gamma_5 d$

No $\mathcal{O}(a^2)$ counterterm needed at tree level



LW gauge action + D_{234} + smearing

Plaquette gauge action + D_{W} + std. clover

- **Smaller discretization effects** at short distances
- **Rotational invariance restored after few lattice spacings**

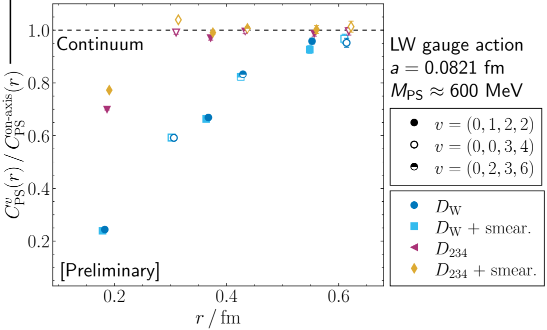
Restoration of rotational invariance

First: **Restoration of rotational symmetry** at short distances

Ratio of two-point function along off- and on-axis directions

$C^v_{\mathrm{PS}} = \langle O_{\mathrm{PS}}(x + r\hat{v}) O^\dagger_{\mathrm{PS}}(x) \rangle \quad O_{\mathrm{PS}} = \bar{u}\gamma_5 d$

No $\mathcal{O}(a^2)$ counterterm needed at tree level



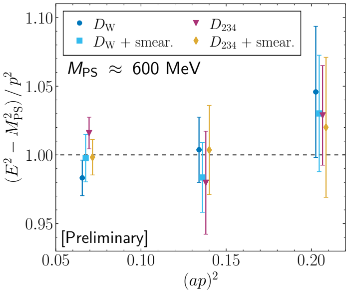
LW gauge action
$a = 0.0821$ fm
$M_{\mathrm{PS}} \approx 600$ MeV

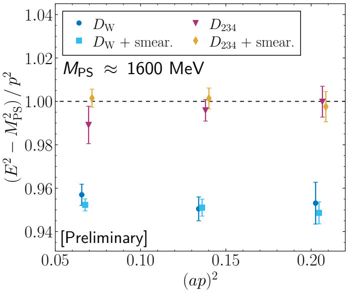
All with exp. clover

- **Smaller discretization effects** at short distances
- **Rotational invariance restored after few lattice spacings**
- **$\mathcal{O}(a^2)$ improvement plays the crucial role**

Meson dispersion relation

Second: **Dispersion relations of pseudoscalar mesons** ($a = 0.0821$ fm)



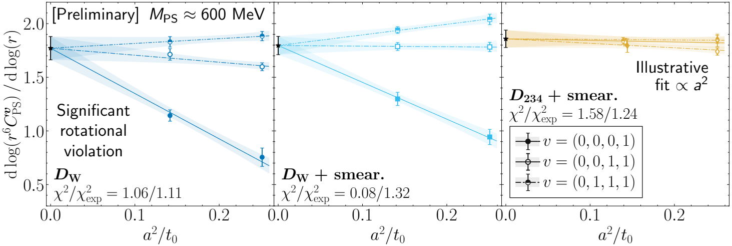


- Correct dispersion relations for *light* mesons
- ***D_{234}* operator recovers continuum result** for *heavier* mesons, even at $M_{\mathrm{PS}} \approx 1600$ MeV

Continuum limit of log-derivative

Third: **Continuum limit of log-derivative** at intermediate distances

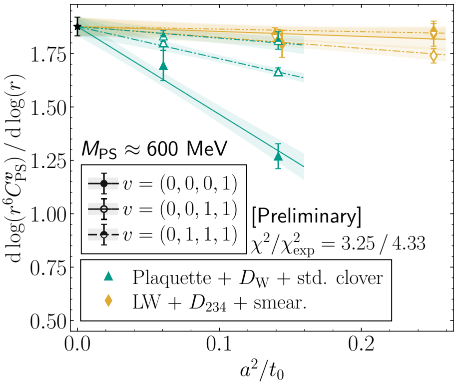
$$L^v_{\mathrm{PS}}(r) = \left. \frac{\mathrm{d}\,\log[r^6 C^v_{\mathrm{PS}}(r)]}{\mathrm{d}\,\log r} \right|_{r=r^*} \qquad r^* \approx 0.5\,\mathrm{fm}$$



- Combine multiple directions for a constrained extrapolation
- Again, **significant improvement from D_{234} operator**
- Moderate error reduction from smearing

Continuum limit of log-derivative

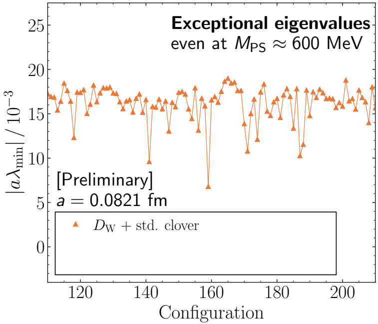
Third: **Continuum limit of log-derivative** at intermediate distances

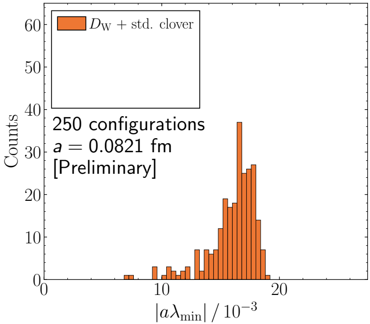


Results for **old and new actions can be combined** for constrained continuum extrapolations

Spectrum of $\gamma_5 D$ and chiral symmetry

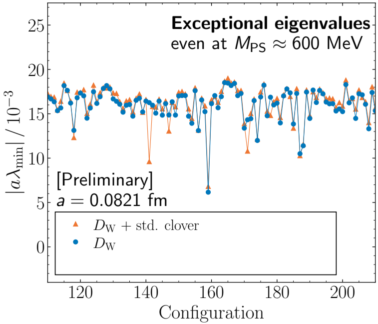
We **investigate lowest eigenvalue of $\gamma_5 D$** at $M_{\mathrm{PS}} \approx 600$ MeV

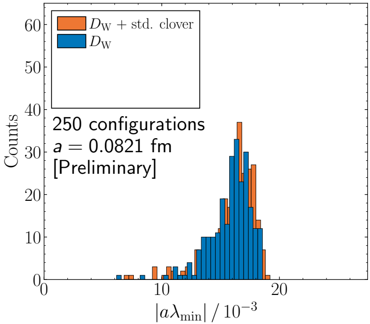




Spectrum of $\gamma_5 D$ and chiral symmetry

We **investigate lowest eigenvalue of $\gamma_5 D$** at $M_{\mathrm{PS}} \approx 600$ MeV

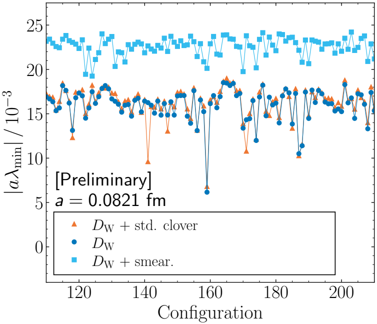


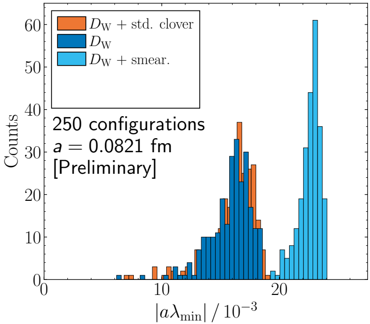


➤ Exp. clover cures some exceptional eigenvalues, but not all

Spectrum of $\gamma_5 D$ and chiral symmetry

We **investigate lowest eigenvalue of $\gamma_5 D$** at $M_{\rm PS} \approx 600$ MeV

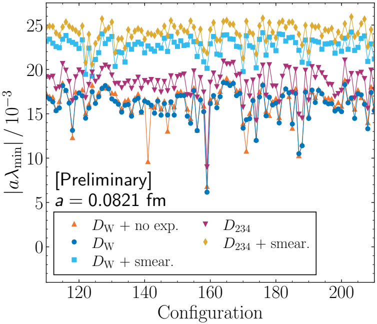


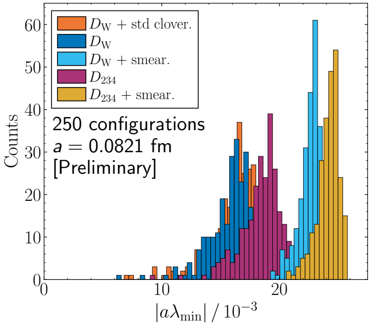


- Exp. clover cures some exceptional eigenvalues, but not all
- **Significant improvement from smearing**

Spectrum of $\gamma_5 D$ and chiral symmetry

We **investigate lowest eigenvalue of $\gamma_5 D$** at $M_{\mathrm{PS}} \approx 600$ MeV



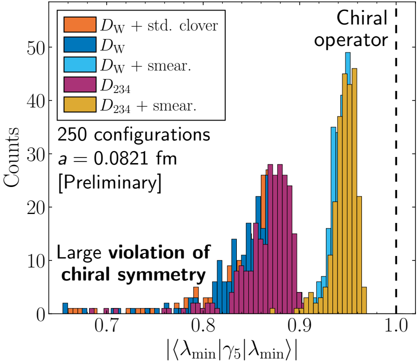


- Exp. clover cures some exceptional eigenvalues, but not all
- **Significant improvement from smearing**
- Some **minor improvement from D_{234} operator**, possibly due to $r = 2/3$

Reduction of chiral symmetry violation

Lowest eigenvalue leads to largest **chiral-symmetry violation**

- For a **chiral operator**, $\langle \lambda_{\text{min}} | \gamma_5 | \lambda_{\text{min}} \rangle = \pm 1$



- **Smearing reduces violation of chiral symmetry**
- Significant reduction of eigenvalues that violate chiral symmetry the most

Summary and outlook

We propose a **new Wilson-like action** with "safe" smearing, $\mathcal{O}(a^2)$ improvement at tree level and exp. clover

- We study the impact of all three features in the **quenched approximation**
- $\mathcal{O}(a^2)$ improvement at tree level **significantly reduces violations of rotational invariance and other discretization effects**
- Smearing has a **major impact on reducing exceptional eigenvalues and chiral-symmetry violation**

Summary and outlook

We propose a **new Wilson-like action** with “safe” smearing, $\mathcal{O}(a^2)$ improvement at tree level and exp. clover

- We study the impact of all three features in the **quenched approximation**
- $\mathcal{O}(a^2)$ improvement at tree level **significantly reduces violations of rotational invariance and other discretization effects**
- Smearing has a **major impact on reducing exceptional eigenvalues and chiral-symmetry violation**

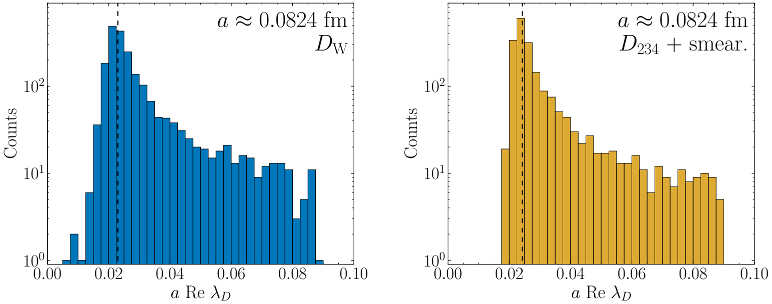
Paper in preparation: improvement coefficients, eigenvalues of D...

Next steps: Dynamical simulations and improvement

Thank you for your attention!

Eigenvalues of D

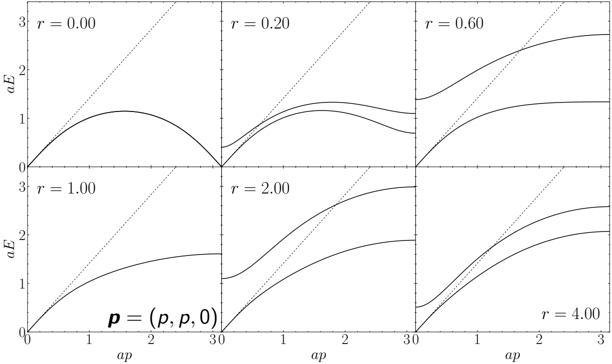
We have studied the **eigenvalues of the non-Hermitian Dirac operator** ($M_{\mathrm{PS}} \approx 600$ MeV)



Significant reduction of small eigenvalues

The role of r in the Wilson action

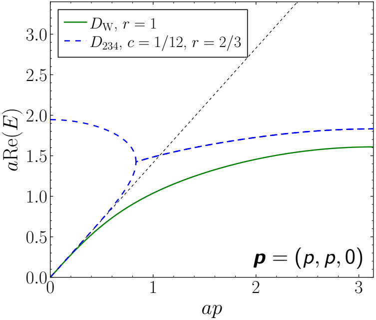
The choice $r = 1$ **eliminates ghost branches in the free dispersion relation** and allows for the **projection trick**



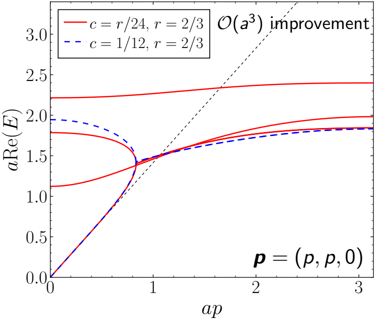
$$D_{\mathrm{W}}^{\mathrm{free}}\psi(x) = -P_{\mu}^{-}\psi(x + a\hat{\mu}) + \psi(x) - P_{\mu}^{+}\psi(x - a\hat{\mu}) + m_0\psi(x)$$

The D_{234} Dirac operator

Reduce the number of ghosts in the **free-quark dispersion relation**



Small discretization effects for $aE \lesssim 1$



Minimum number of ghosts, with $aE_\mathrm{ghost} \gg 1$

The D_{234} Dirac operator

Rewrite the operator using the **projection trick**

$$
\begin{aligned}
D_{234}\psi(x) = &\tfrac{1}{6} U_\mu(x) U_\mu(x+a\hat{\mu}) \boldsymbol{P}_{\boldsymbol{\mu}}^{-} \psi(x+2a\hat{\mu}) \\
&- \tfrac{4}{3} U_\mu(x) \boldsymbol{P}_{\boldsymbol{\mu}}^{-} \psi(x+a\hat{\mu}) \\
&+ \tfrac{1}{6} U_\mu^\dagger(x-a\hat{\mu}) U_\mu^\dagger(x-2a\hat{\mu}) \boldsymbol{P}_{\boldsymbol{\mu}}^{+} \psi(x-2a\hat{\mu}) \\
&- \tfrac{4}{3} U_\mu^\dagger(x-a\hat{\mu}) \boldsymbol{P}_{\boldsymbol{\mu}}^{+} \psi(x-a\hat{\mu}) \\
&+ \tfrac{7}{6}\psi(x) + m_0\psi(x) + \tilde{D}_{\text{clover}}^{\text{exp}}\psi(x)\,.
\end{aligned}
$$

Only two spin components from neighbors needed

- Simple algorithmic implementation

$$\mathrm{Cost}(D_{234}) \approx 2 \times \mathrm{Cost}(D_{\mathrm{W}})$$

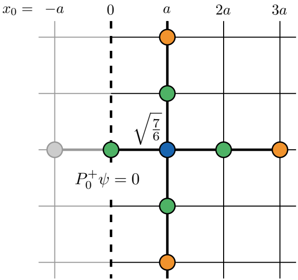
Expected **smaller factor for deflated solver**

- Convenient structure for SF boundaries (**ongoing work**)

D_{234} operator in the SF

Complication: cannot define two-hop derivative in the first time slice next to the boundaries

- Naive definition leads to $\mathcal{O}(a)$ boundary errors



PT at tree level and ideas from [Lüscher 2006]

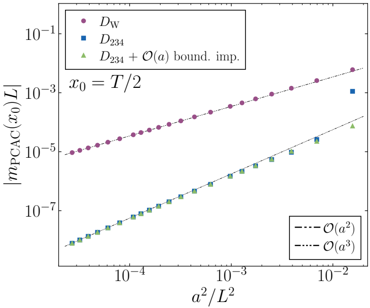
- Need to include **boundary counterterm**:

$$\textbf{Counterterm} \quad \left(\tilde{c}_t^{234} - \frac{7}{6}\right)\bar{\psi}(a)P_0^+ D_0^- \psi(a) \quad \longrightarrow \quad \tilde{c}_t^{234} = \sqrt{\frac{7}{6}}$$

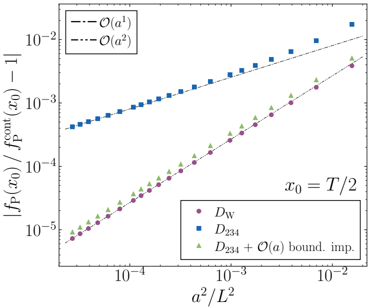
$$\textbf{Boundary field} \quad \zeta = \sqrt{\frac{7}{6}}U_0^\dagger(0)P_0^-\psi(a) - \frac{1}{6}U_0^\dagger(0)U_0^\dagger(a)P_0^-\psi(2a)$$

SF in perturbation theory

We have studied the scaling of observables at **TL in perturbation theory**
($T = L$, $am_0 = 0$, background field from [Lüscher et al. 1993])



PCAC relation is protected from boundary effects



Boundary improvement is **required at TL**

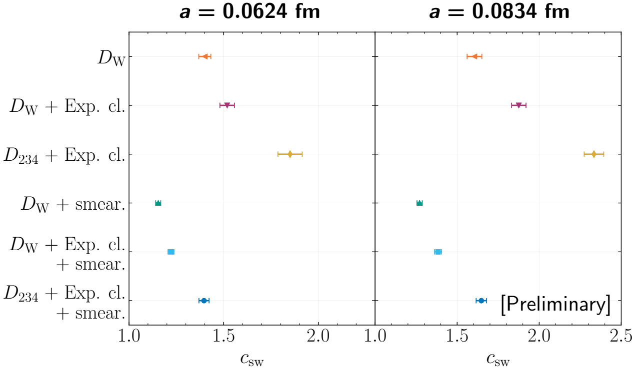
$\mathcal{O}(a^2)$ impr. in the **bulk**

$\mathcal{O}(a)$ impr. at the **boundary**

Non-perturbative tuning of c_{sw}

Study the impact of modifications in the **non-perturbative** value of $\boldsymbol{c_{sw}}$

- Understood how to **define $\boldsymbol{D_{234}}$ operator in the Schrödinger functional**

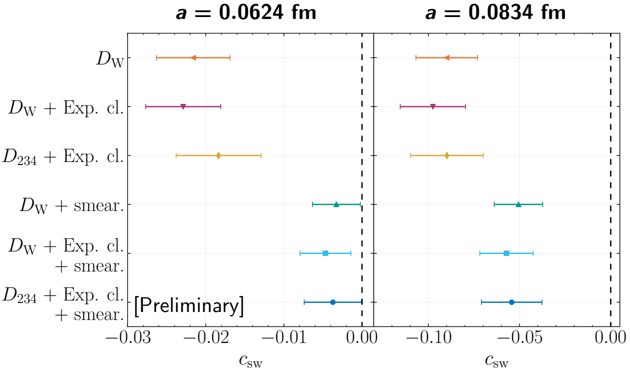


Effect of exp. clover and D_{234} operators **may change in full QCD**

- In full QCD, c_{sw} is smaller than with exp. clover [Francis et al. 2020]

Non-perturbative tuning of c_A

Study the impact of modifications in the **non-perturbative** value of $\boldsymbol{c_A}$



Smearing significantly **reduces the non-perturbative value of $\boldsymbol{c_A}$**

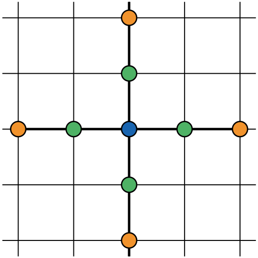
Deflated solver for the D_{234} operator

We have implemented a **deflated solver for the D_{234} operator**

D_{234} only contains straight links

$\downarrow$

Structure of little system remains the same



SAP could be implemented, with a measured **cost $\sim$ 1.6$\times$ that of D_W**

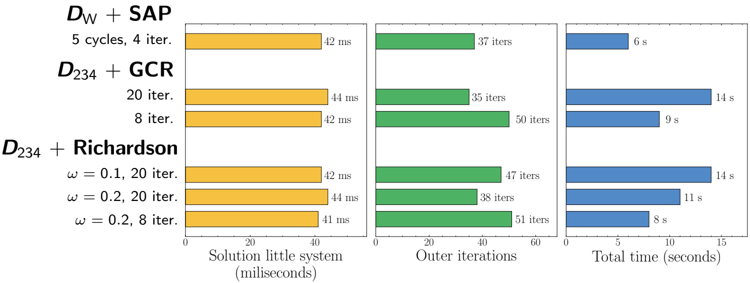
- We have tried **alternatives** suggested by A. Frommer
  - Fixed number of **GCR iterations**
  - **Richardson iteration** (no scalar products)

$$D\psi = \eta \quad \longrightarrow \quad \psi_{k+1} = \psi_k + \omega(\eta - D\psi_k)$$

Tests of the deflated solver

We have **tested the deflated solver on the CLS N451r000 ensemble**

- $T \times L^3 = 128 \times 48^3$, $a \approx 0.076$ fm, $M_\pi \approx 286$ MeV
- Tuned quark mass for D_{234} operators with $c_{sw} = 1.3$
- Results for 48 nodes of JUWELS, no SSE/AVX acceleration for D_W



Solving for the D_{234} operator **only $\sim$ 1.5 $\times$ more expensive**

- Need further exploration (optimal parameters...)