

Learning Proton Structure from Lattice QCD

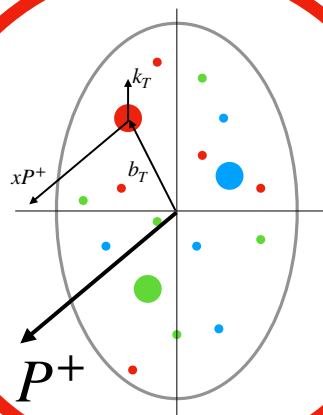
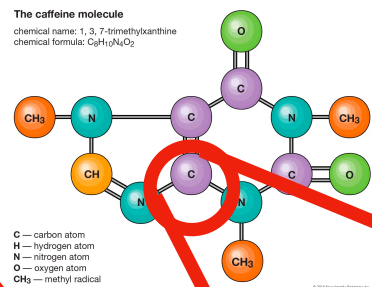
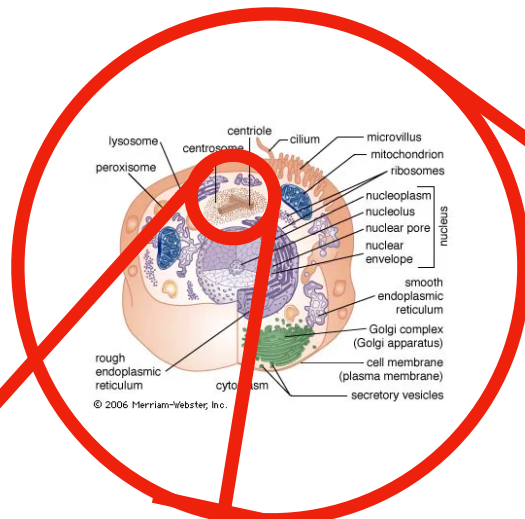
Joe Karpie (A219)



Jefferson Lab



Digging Deeper and Deeper



- Typical humans in Highline Park (NYC)

Overview

- Spatial structure of Hadrons comes from GPDs measured by certain Exclusive Virtual Processes
- Ioffe time Distributions define Generalized Parton Distributions and Double Distributions
- They cannot be directly calculated on the lattice, but there are many methods to get by that
- Modeling lattice QCD data with polynomials for Mellin moments and with Gaussian Processes Regression for Double Distributions
- Results for the H and E GPD on from one ensemble (single lattice spacing, pion mass, and volume)

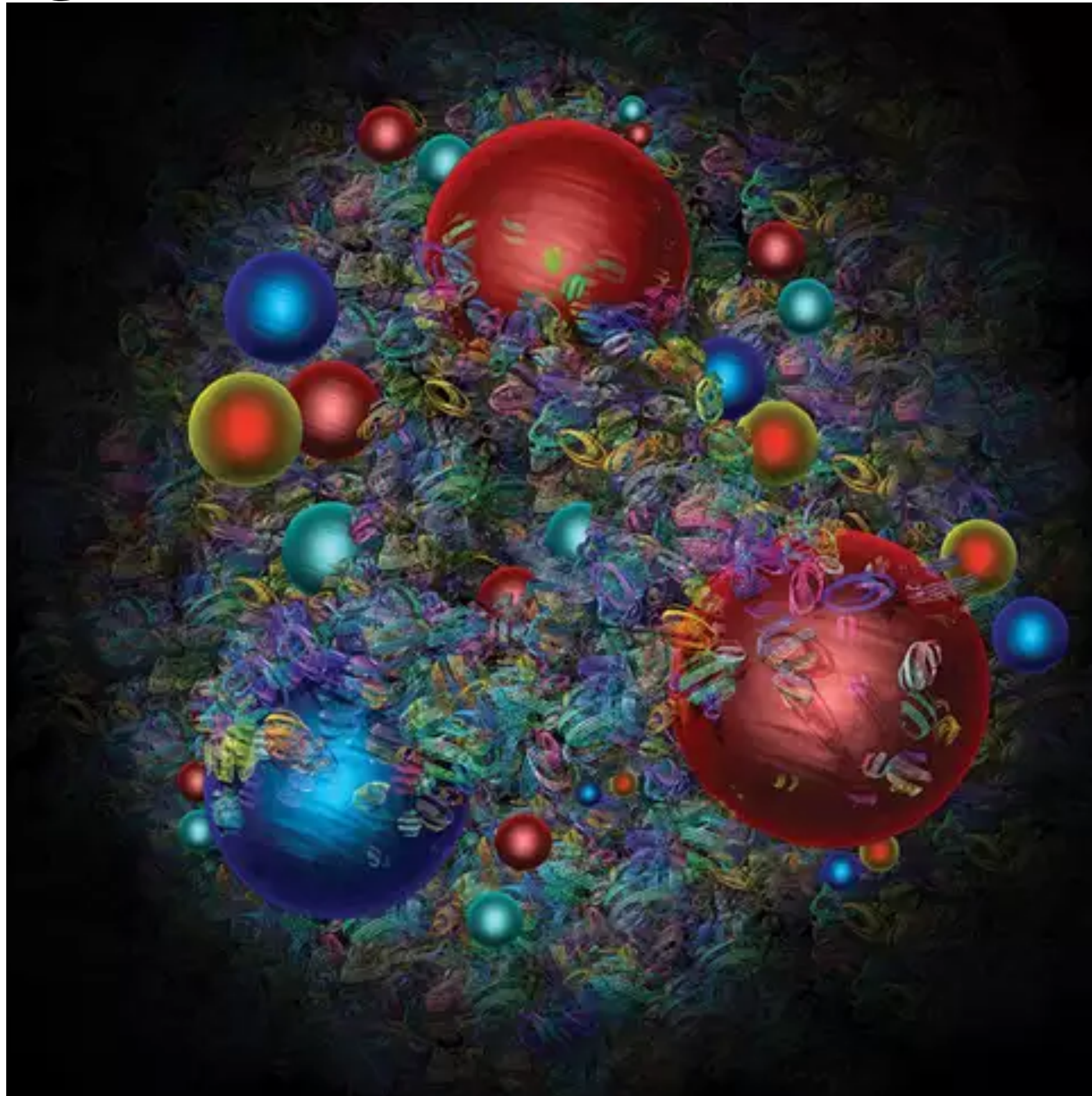
All bound together

- BSM models interact with quarks not hadrons. High precision is needed for HL-LHC
- Precision standard model parameters require precision understanding of quark structure

CDF Collab. *Science* (2022) 376 6589 170-176

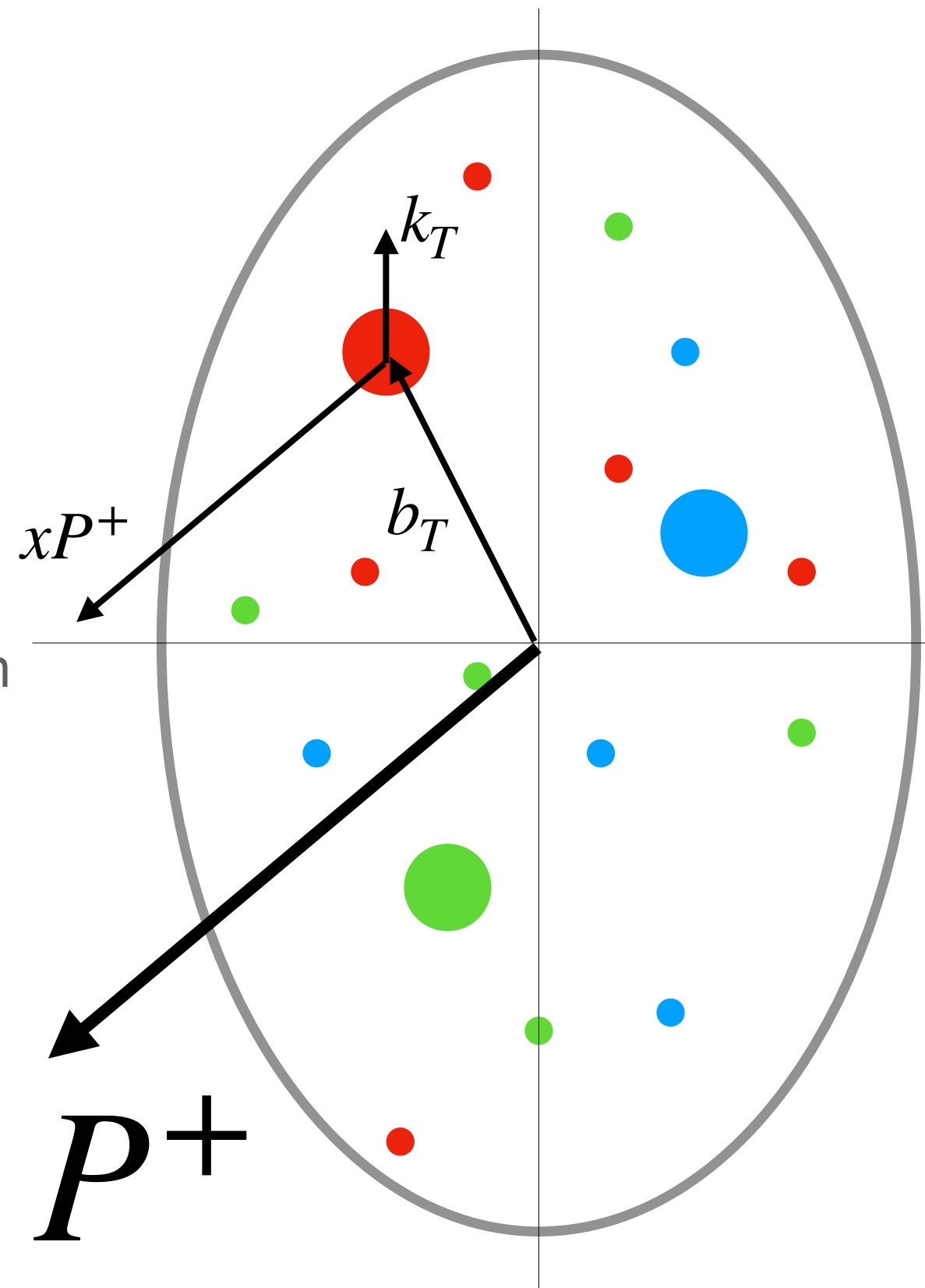
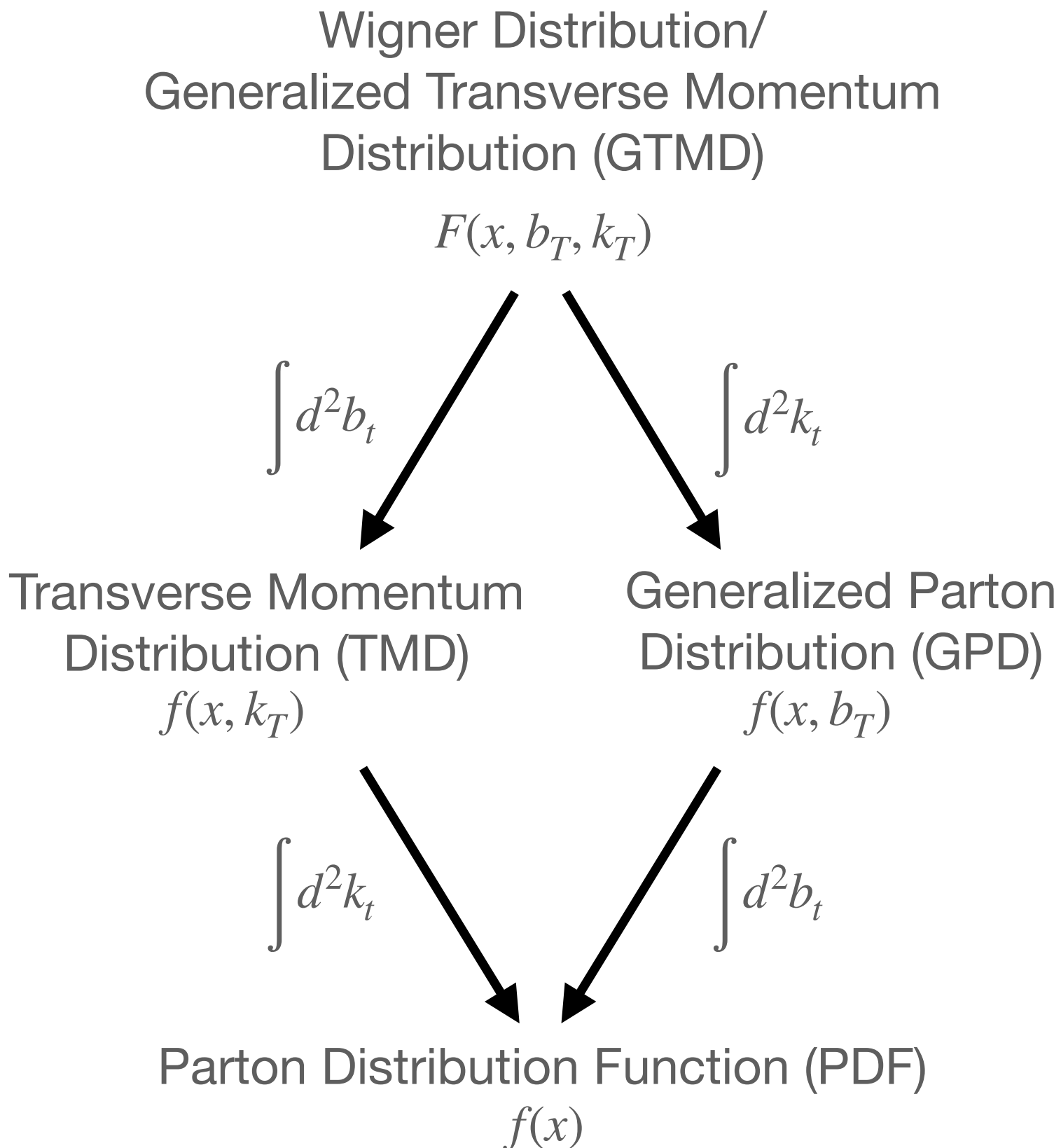
Table 2. Uncertainties on the combined M_W result.

Source	Uncertainty (MeV)
Lepton energy scale	3.0
Lepton energy resolution	1.2
Recoil energy scale	1.2
Recoil energy resolution	1.8
Lepton efficiency	0.4
Lepton removal	1.2
Backgrounds	3.3
p_T^Z model	1.8
p_T^W/p_T^Z model	1.3
Parton distributions	3.9
QED radiation	2.7
W boson statistics	6.4
Total	9.4



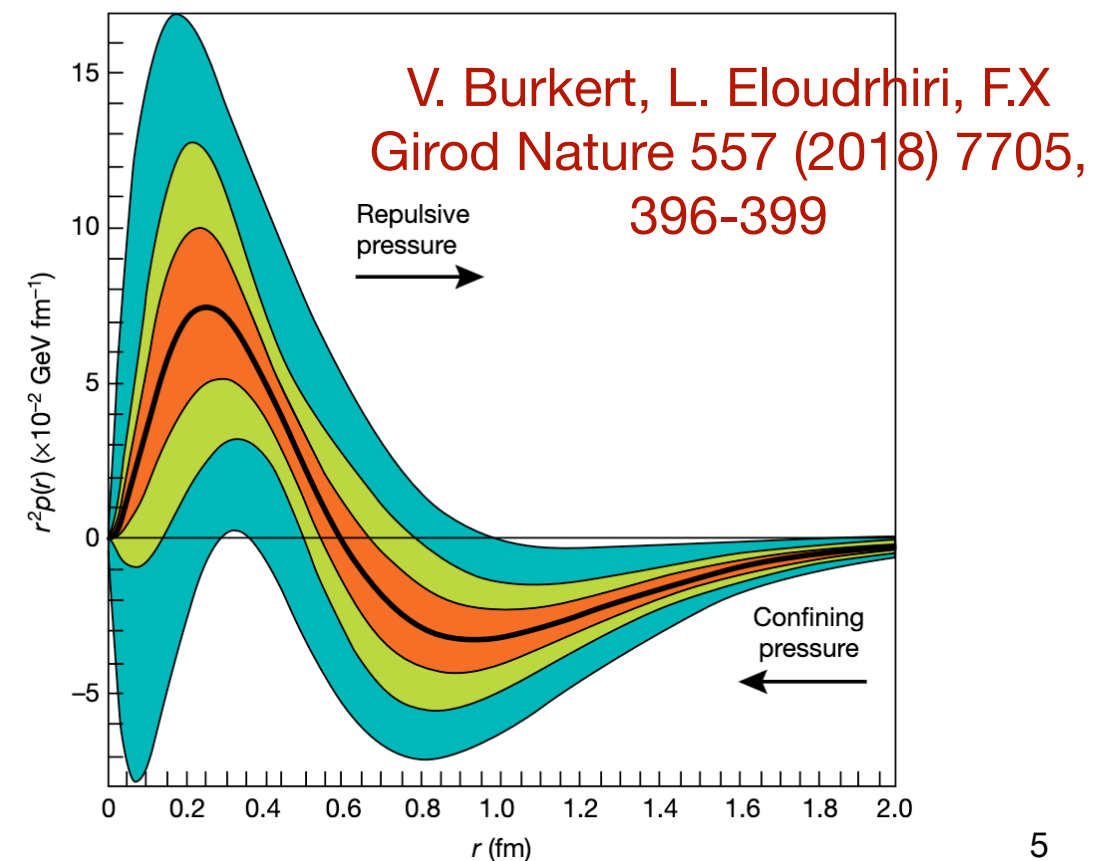
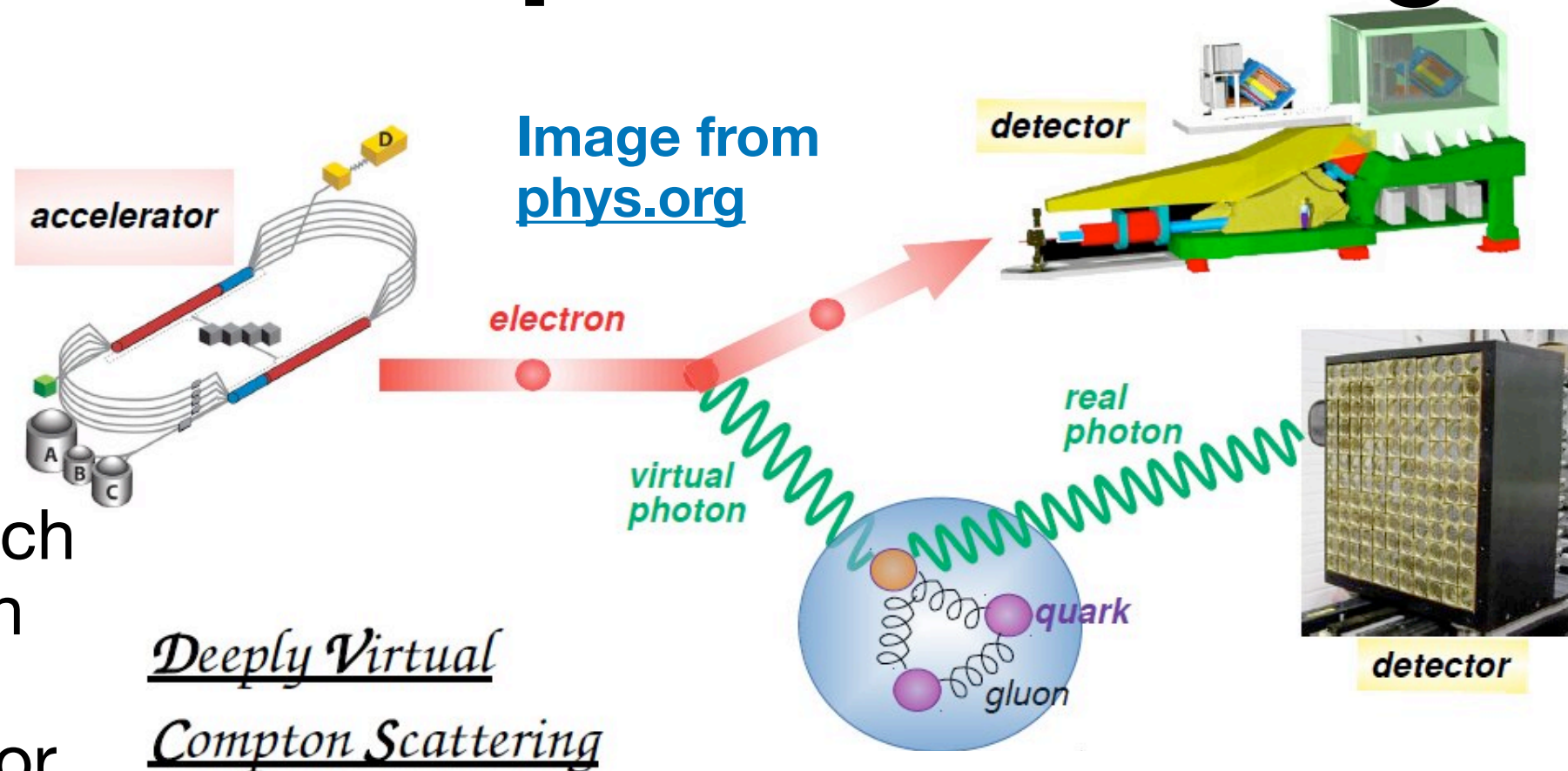
Parton Structure

For various flavors and spin combinations



Deeply Virtual Compton Scattering

- Transverse spatial structure studied by deeply virtual probe and real emission, such as a photon or meson
- Compton Form Factor described as integrals of Generalized Parton Distributions,
- x weighted integral of GPD, called Gravitational Form Factors, give access to energy momentum tensor, pressure stress, mass, angular momentum



Generalized Parton Distributions

- Generalized Ioffe time distributions (GITD)

“Ioffe time distributions instead of parton momentum distributions in description of DIS”

V. Braun, P. Gornicki, L. Mankiewicz

Phys Rev D 51 (1995) 6036-6051

$$I^\mu(p', p, z = z^-, \mu^2) = \frac{1}{2P^\mu} \langle p' | \bar{q}\left(-\frac{z^-}{2}\right) \gamma^\mu W\left(-\frac{z^-}{2}, \frac{z^-}{2}\right) q\left(\frac{z^-}{2}\right) | p \rangle_{\mu^2}$$

Ioffe Time

Momentum Transfer

squared

Skewness

$$\nu = \frac{p + p'}{2} \cdot z = P \cdot z$$

$$t = (p' - p)^2 = q^2$$

$$\xi = \frac{q \cdot z}{2P \cdot z}$$

- Generalized Parton Distributions

$$z_\mu I^\mu(\nu, t, \xi, \mu^2) = \int dx e^{i\nu x} \left[H(x, t, \xi) \bar{u}' z_\mu \gamma^\mu u + E(x, t, \xi) \bar{u}' \frac{iz_\mu \sigma^{\mu\nu} q_\nu}{2m} u \right]$$

Taylor expand $e^{i\nu x}$ to see GITD as series of Mellin moments of GPD

X. Ji PRL 78 (1997) 610-613

Double Distributions

- Generalized Ioffe time distributions (GITD)

“Ioffe time distributions instead of parton momentum distributions in description of DIS”

V. Braun, P. Gornicki, L. Mankiewicz

Phys Rev D 51 (1995) 6036-6051

$$I^\mu(p', p, z = z^-, \mu^2) = \frac{1}{2P^\mu} \langle p' | \bar{q} \left(-\frac{z^-}{2} \right) \gamma^\mu W \left(-\frac{z^-}{2}, \frac{z^-}{2} \right) q \left(\frac{z^-}{2} \right) | p \rangle_{\mu^2}$$

Ioffe Time

Momentum Transfer

squared

Skewness

$$\nu = \frac{p + p'}{2} \cdot z = P \cdot z$$

$$t = (p' - p)^2 = q^2$$

$$\xi = \frac{q \cdot z}{2P \cdot z}$$

- Double Distributions

$$z_\mu I^\mu(\nu, t, \xi, \mu^2) = \int d\beta d\alpha e^{i(\nu\beta + \xi\nu\alpha)} \left[h(\alpha, \beta, t) \bar{u}' z_\mu \gamma^\mu u + e(\alpha, \beta, t) \bar{u}' \frac{iz_\mu \sigma^{\mu\nu} q_\nu}{2m} u + \delta(\beta) D(\alpha, t) \xi\nu \bar{u}' u \right]$$

β is conjugate to $\nu = P \cdot z$ and α is conjugate to $\bar{\nu} = \xi\nu = q \cdot z/2$

Two Faced Distribution

GPDs:

x, ξ

Radon Transform

DDs:

β, α

$$f(x, \xi) = \int_{-1}^1 d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha \delta(x - \beta - \xi\alpha) \tilde{f}(\alpha, \beta)$$

- Interpretation: average/change in parton momentum fraction
- Mellin moments give Form Factors and Angular Momentum decomposition
- Complex interrelation of variables
 - ERBL/DGLAP regions and polynomiality

- Interpretation: Hybridize PDFs/DAs
- β acts like PDF x
- α acts like DA x
- GPD evolution and polynomiality arise naturally from parameterized DD



Statue of Janus Bifrons
(Wikipedia)

Many Good Lattice Cross Sections

GLCS are calculable matrix elements which factorize to light cone distributions

- **Non-local bilinears operators**

$$O_{WL}(x; z) = \bar{\psi}(x + z)\Gamma W(x + z; x)\psi(x)$$

- LaMET/quasi-PDF *X. Ji Phys. Rev. Lett. 110 (2013) 262002*

- **Pseudo-PDF** *A. Radyushkin Phys. Rev. D 96 (2017) 3, 034025*

- Coulomb Gauge Fixed Wilson Line-less

X. Gao et al Phys. Rev. D 109 (2024) 9, 094506

- Two current correlators

- Hadronic Tensor

K.-F. Liu et al Phys. Rev. Lett. 72 1790 (1994)

Phys. Rev. D 62 (2000) 074501

- HOPE

W. Detmold and C.-J. D. Lin, Phys. Rev. D 73 (2006) 014501

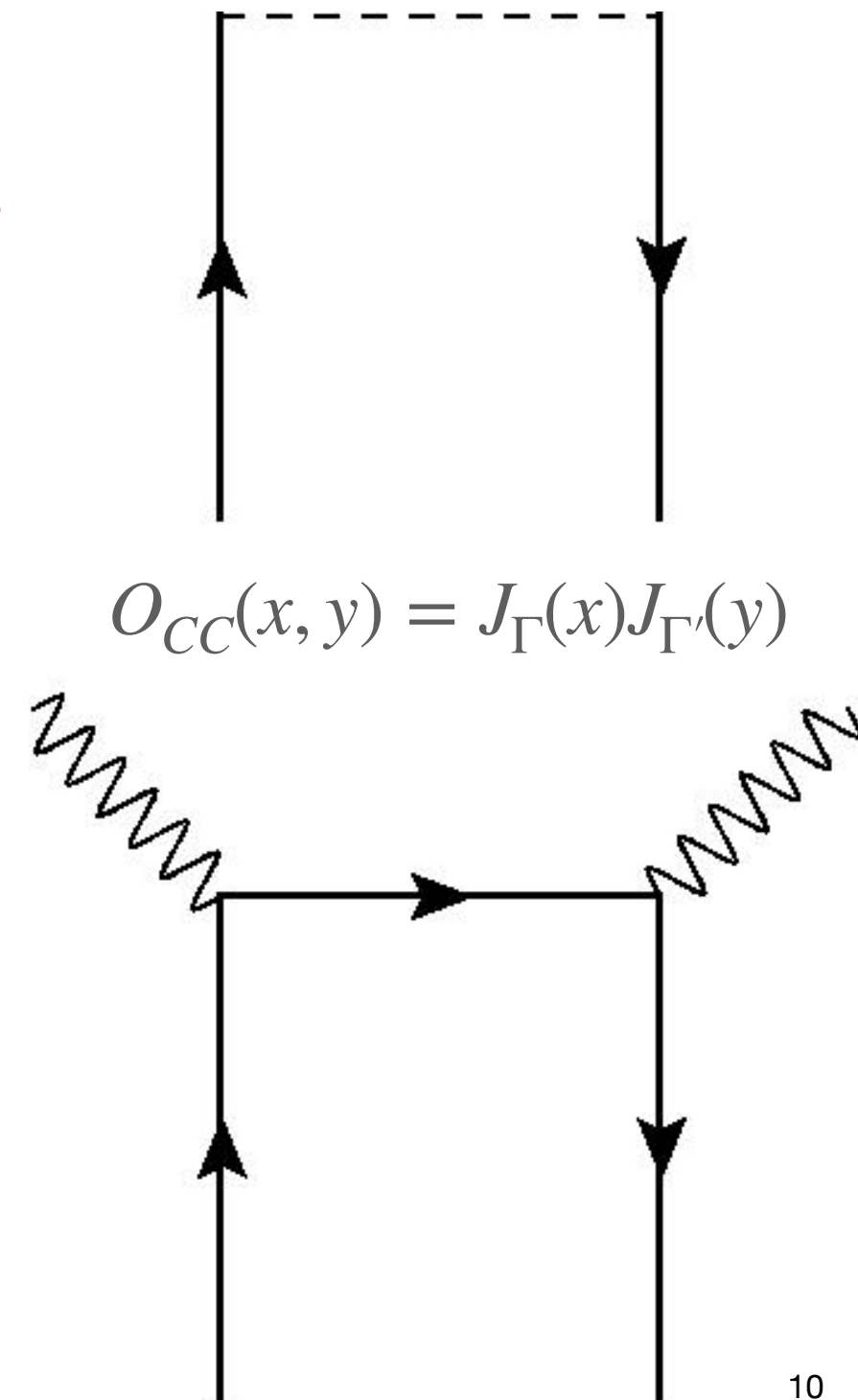
- Short distance OPE

V. Braun and D. Muller Eur. Phys. J. C 55 (2008) 349

Y.-Q. Ma and J.-W. Qiu Phys. Rev. Lett. 120 (2018) 2, 022003

- OPE-without-OPE

A. Chambers et al, Phys. Rev. Lett. 118 (2017) 242001



**We can calculate nice matrix elements
which can be fit with physics informed
models.**

**How much do we really trust the model
does not bias the result from the truth?**

Approaches to inverse problem

- **Parametric**

- Fit a phenomenologically motivated function
 - Method used by global fits
 - Systematic expansions (Jacobi polynomials or DFTs)
 - Potentially significant, but controllable model bias

- Fit to a neural network

S. Forte, L. Garrido, J. Latorre, A. Piccione (2002) 0204232

K. Cichy, L. Del Debbio, T. Giani (2019) 1907.06037

L. Del Debbio, T. Giani, JK, K. Orginos, A. Radyushkin, S. Zafeiropoulos (2020) 2010.03996

- **Non-Parametric**

For NN/BG/MEM/BR: JK, K. Orginos, A. Rothkopf, S. Zafeiropoulos (2019) 1901.05408

- Backus Gilbert

J. Liang, K-F. Liu, Y-B. Yang (2017) 1710.11145

- Maximum Entropy Method / Bayesian Reconstruction

Y. Burnier and A. Rothkopf (2013) 1307.6106, J. Liang et al (2019) 1906.05312

- Bayes-Gauss-Fourier transform

C. Alexandrou, G. Iannelli, K. Jansen, F. Manigrasso (2020) 2007.13800

- **Gaussian Process Regression**

A. Candido, L. Del Debbio, T. Giani, G. Petrillo (2024) 2404.07573

H. Dutrieux, JK, K. Orginos, S. Zafeiropoulos (2024) 2412.05227

EARLIER TODAY!

Y Cahuana Medrano, H Dutrieux, JK,

K Orginos, S Zafeiropoulos

arXiv:2510.21041

Gaussian Process Priors

Bayes's Theorem

$$P[q | M, I] = \frac{P[M | q] P[q | I]}{P[M | I]}$$

ETMC PRD 102 (2020) 9, 094508
A. Candido et al Eur. Phys. J. C 84 (2024) 7, 716
H. Dutrieux et al PRD 111 (2025) 3, 034515

EARLIER TODAY!

Y Cahuana Medrano, H Dutrieux, JK,
K Orginos, S Zafeiropoulos arXiv:2510.21041

- Given a theory: $M(\nu) = \int dx B(\nu, x) q(x)$
- Assume data-likelihood is **Gaussian** $P[M | q] \propto \exp[-\frac{1}{2} \chi^2]$
- Assume the PDF has a **Gaussian** prior with given mean $g(x)$ and covariance kernel $K(x, x')$

$$P[q | I] \propto \exp \left[-\frac{1}{2} \int dx dx' (q(x) - g(x)) K^{-1}(x, x') (q(x') - g(x')) \right]$$

- The posterior is **Gaussian**

$$P[q | M, I] \propto \exp \left[-\frac{1}{2} \int dx dx' (q(x) - \bar{q}(x)) H^{-1}(x, x') (q(x') - \bar{q}(x')) + \dots \right]$$

 **Final Mean**
 **Final Covariance**
 **$q(x)$ independent terms**

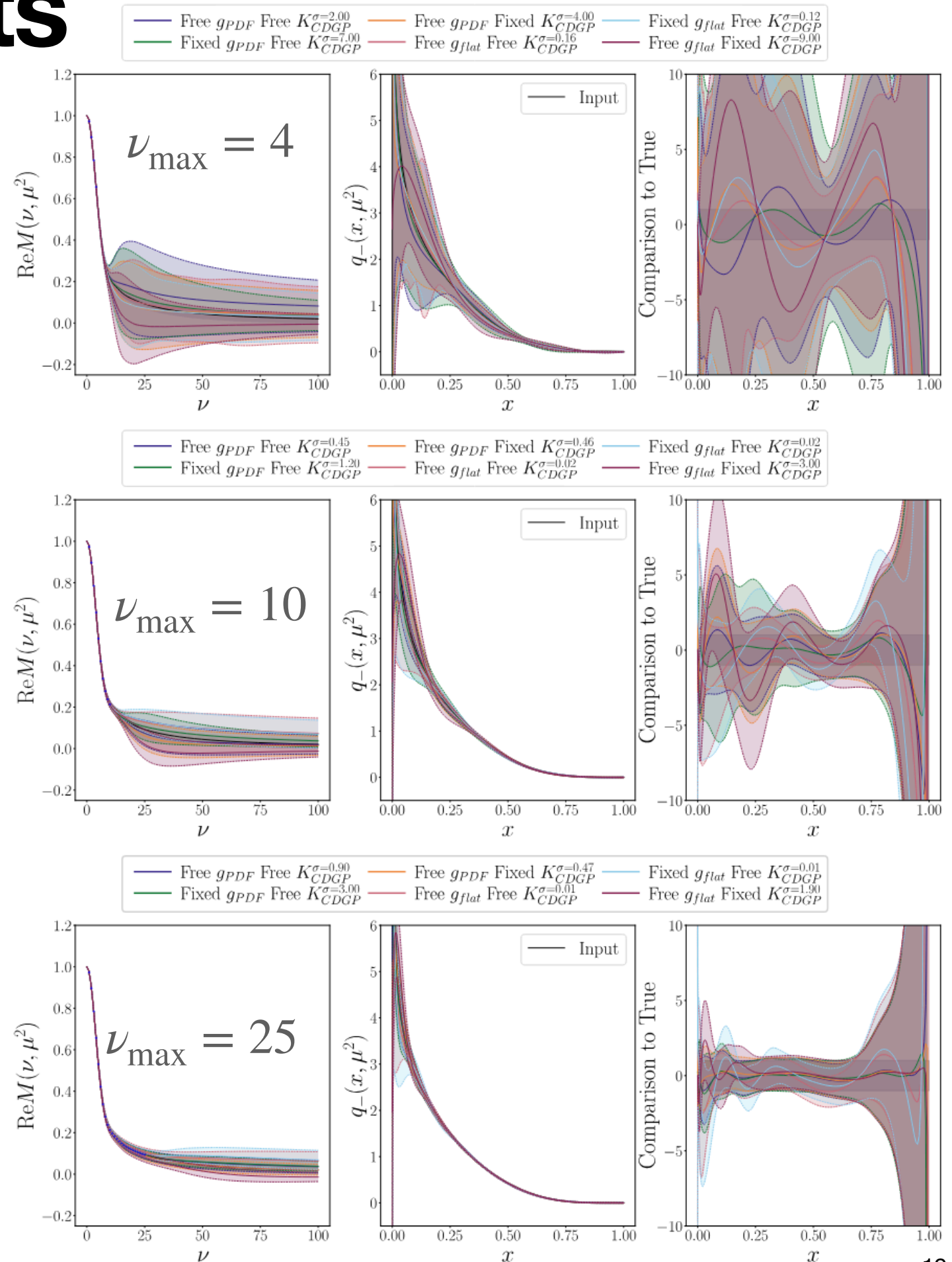
- $K(x, x')$ and $g(x)$ may have hyperparameters to tune or integrate over

Mock data results

- Many options of prior covariances/means and ways to treat their parameters
- Closure test to study the systematic improvement over different ν range
- All give highly consistent results in agreement with data
- Use information criteria to select models or take weighted average

EARLIER TODAY!

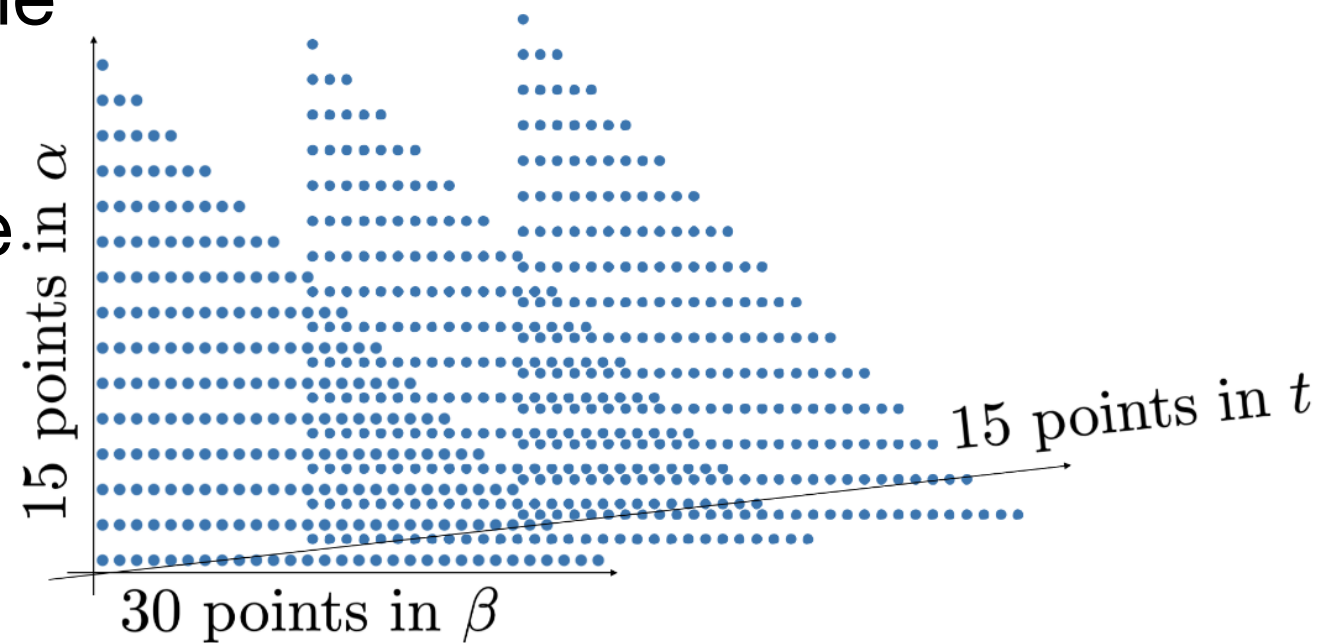
Y Cahuana Medrano, H Dutrieux, JK,
K Orginos, S Zafeiropoulos arXiv:2510.21041



Three Dimensional GPR

HadStruc (2026) 2604.21476

- *CP* Symmetry to bring DD into one quadrant
- Integrals approximated with Finite Element Method on grid of points interpolated by cubic splines
- Mean function is 0 and the 3D kernel to allow certain features
- Future work will study new kernels and model averaging like PDFs



$$K_{ij} = K[(\beta_i, \alpha_i, t_i), (\beta_j, \alpha_j, t_j)],$$

$$= \sigma^2 (\beta_i \beta_j)^{-\gamma} ((1 - \beta_i)(1 - \beta_j))^{\delta} ((1 - \alpha_i - \beta_i)(1 - \alpha_j - \beta_j))^{\eta}$$

$$\times \exp\left(-\frac{(\ln(\beta_i) - \ln(\beta_j))^2}{2l_{\beta}^2}\right) \exp\left(-\frac{(\beta_i \beta_j)^{-\gamma} (\alpha_i - \alpha_j)^2}{2l_{\alpha}^2}\right) \exp\left(-\frac{(t_i - t_j)^2}{2l_t^2}\right)$$

Converge as power of $(1 - \beta)$ as $\beta \rightarrow 1$

Converge on boundary

Diverge as $\beta \rightarrow 0$

Decorrelate small and large β

Less correlation in α when β is small

Kinematical coverage

HadStruc (2026) 2604.21476

249 initial and final momenta configurations

ID	a (fm)	m_π 3(MeV)	β	$m_\pi L$	$L^3 \times N_T$	N_{cfg}	N_{srcs}	rk ($\mathcal{D}$)
a094m358	0.094(1)	358(3)	6.3	5.4	$32^3 \times 64$	348 ([44])	4	64
						1490 (new)		

Ioffe Time

Momentum Transfer

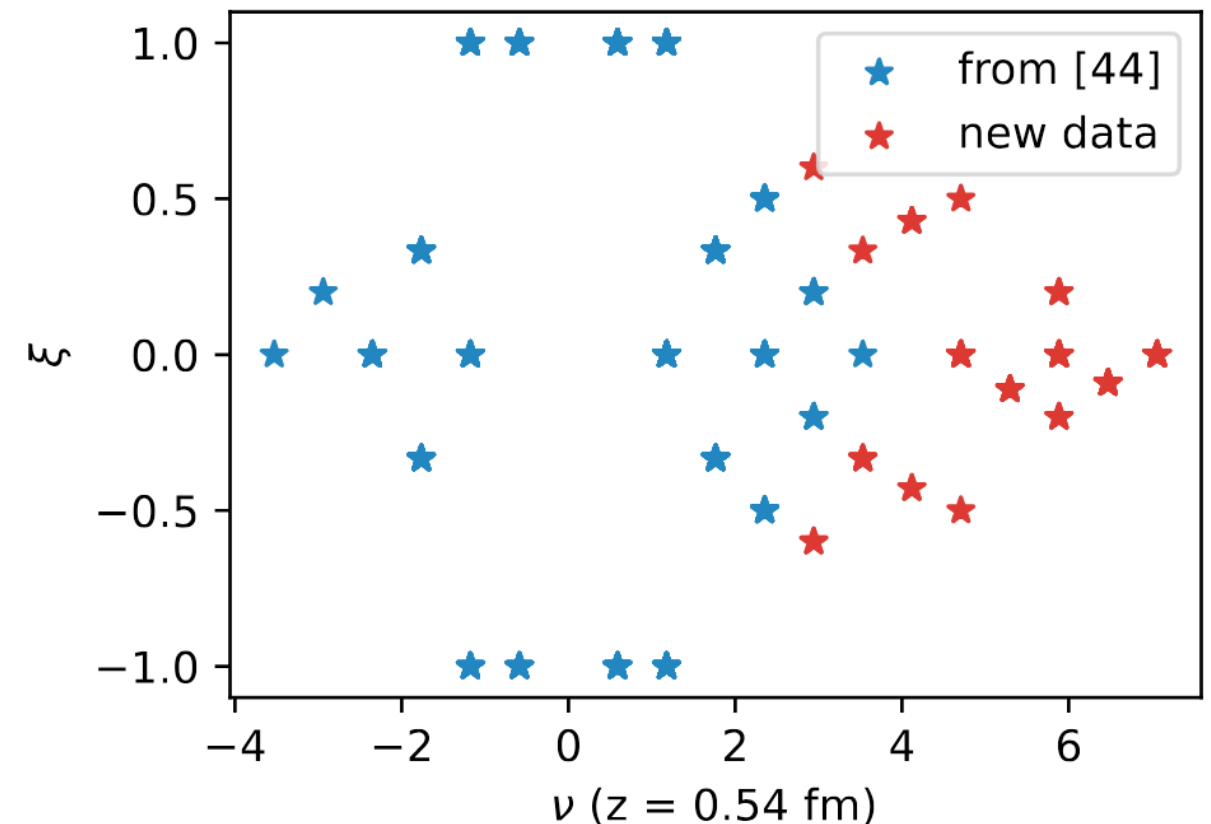
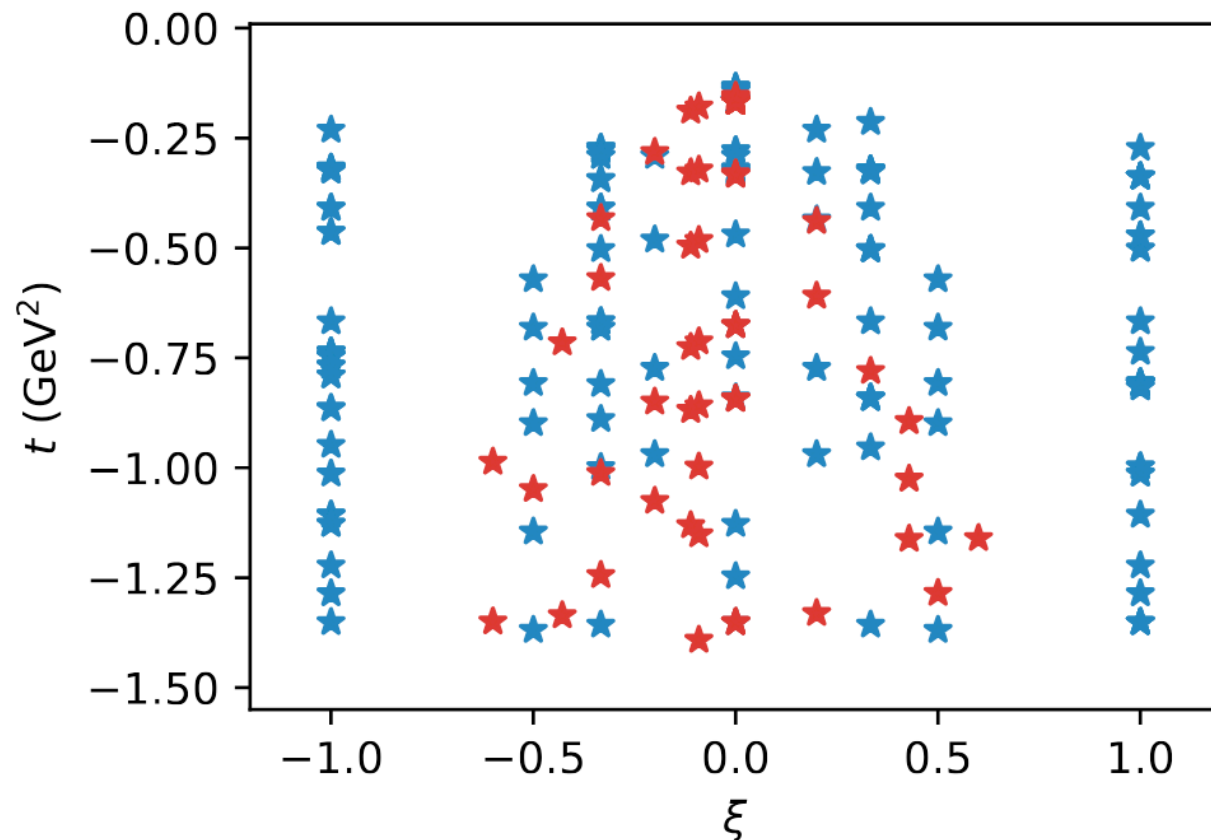
squared

Skewness

$$\nu = \frac{p + p'}{2} \cdot z = P \cdot z$$

$$t = (p' - p)^2 = q^2$$

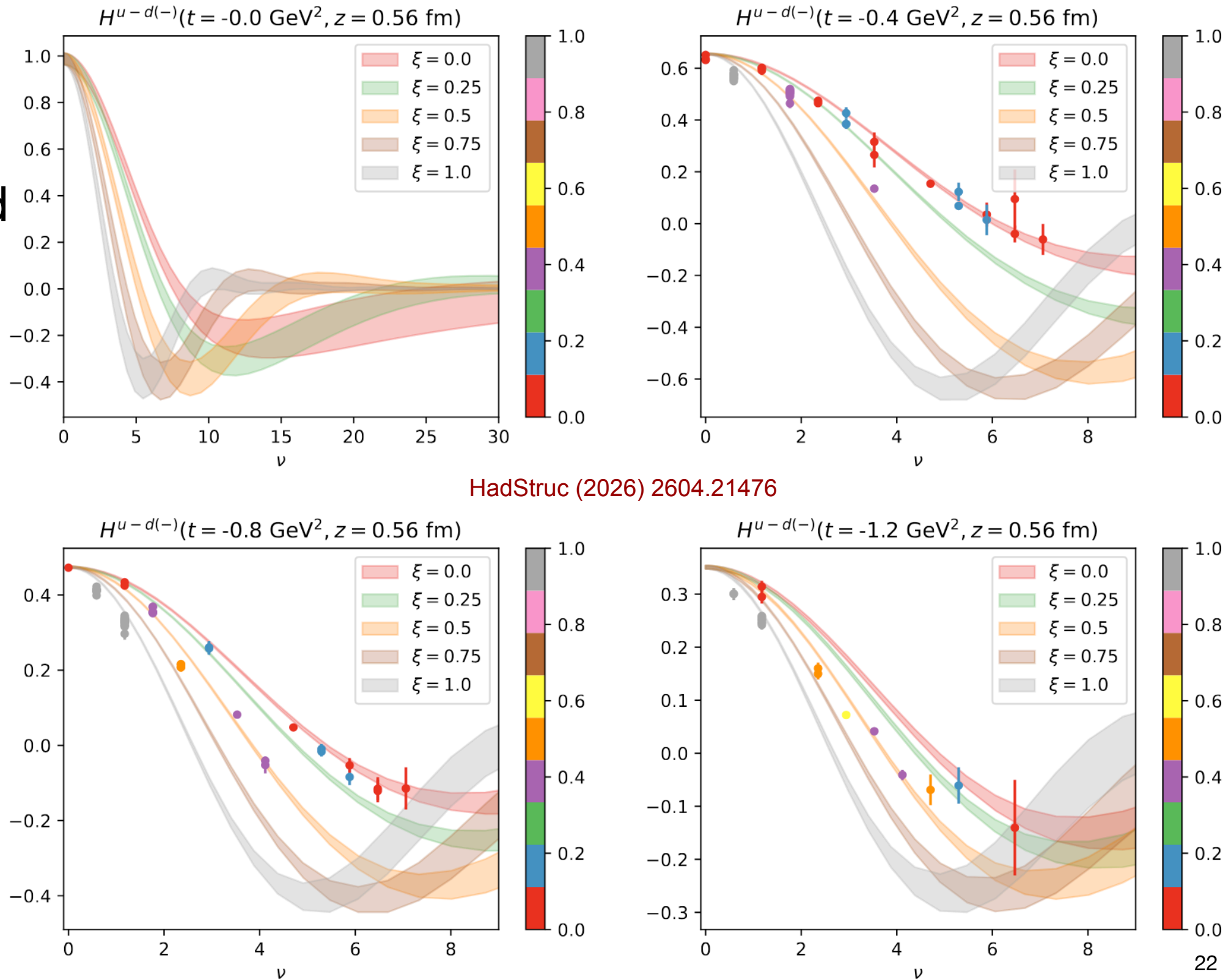
$$\xi = \frac{q \cdot z}{P \cdot z}$$



loffe time pseudo-distributions

Iso-vector valence H pseudo-GITD

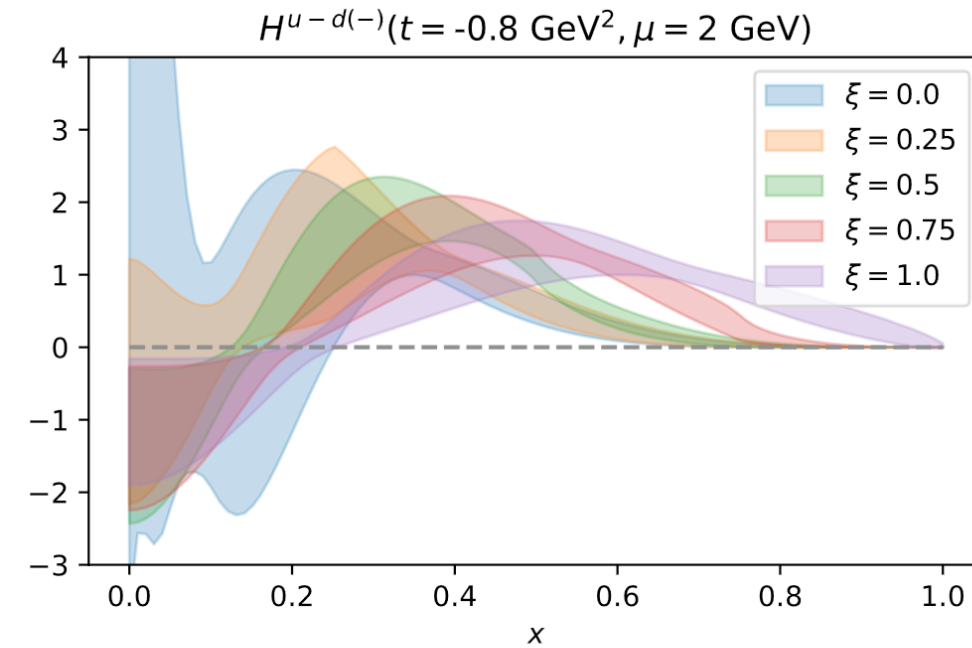
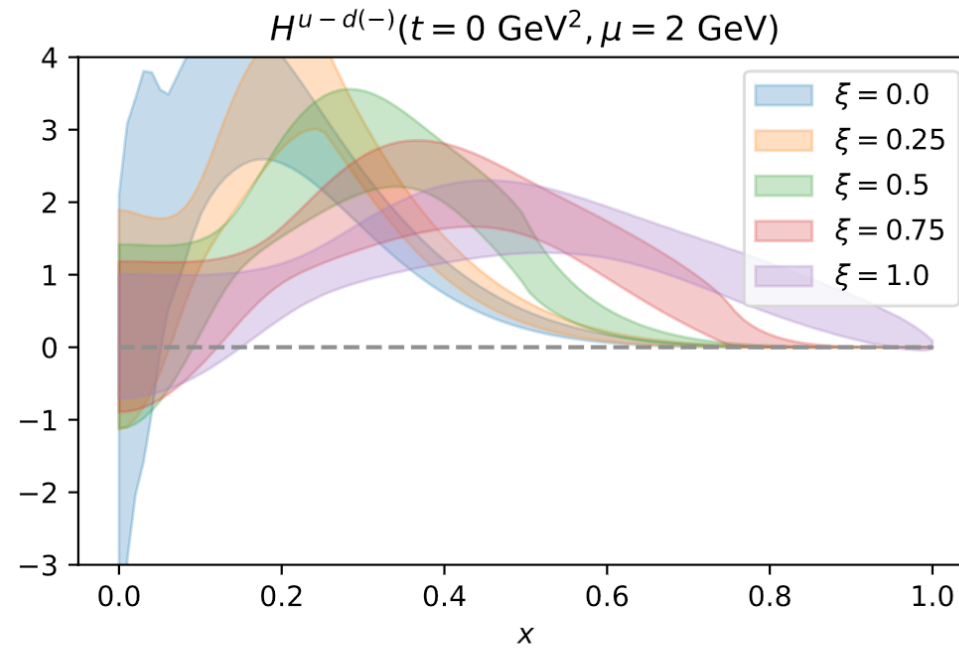
- Model DD and reconstruct GITD at wanted t, ξ
- Skewness dependence extrapolated to $t = 0$



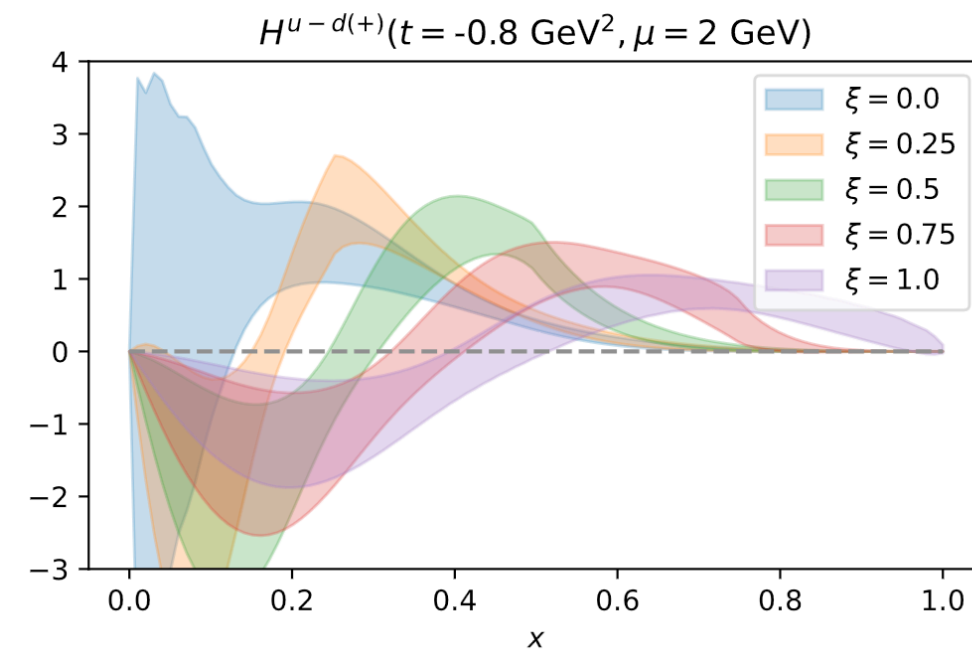
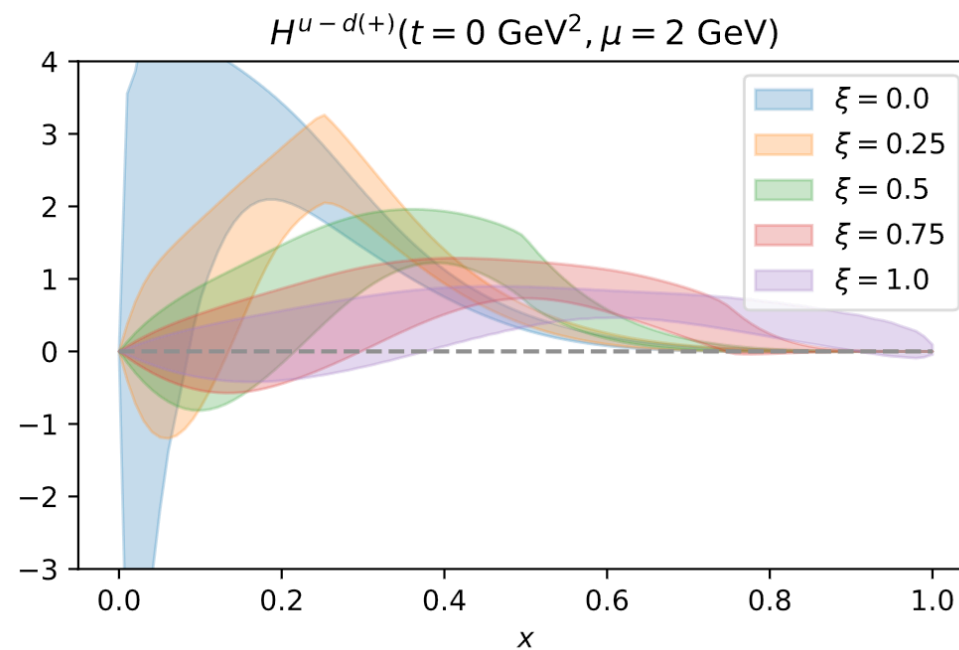
Generalized pseudo-distributions

Iso-vector valence H pseudo-GPD

- Model DD and reconstruct GPD at wanted t, ξ
- Skewness dependence extrapolated to $t = 0$
- Low β, α are not constrained by the data which leads to uncertainty at low x



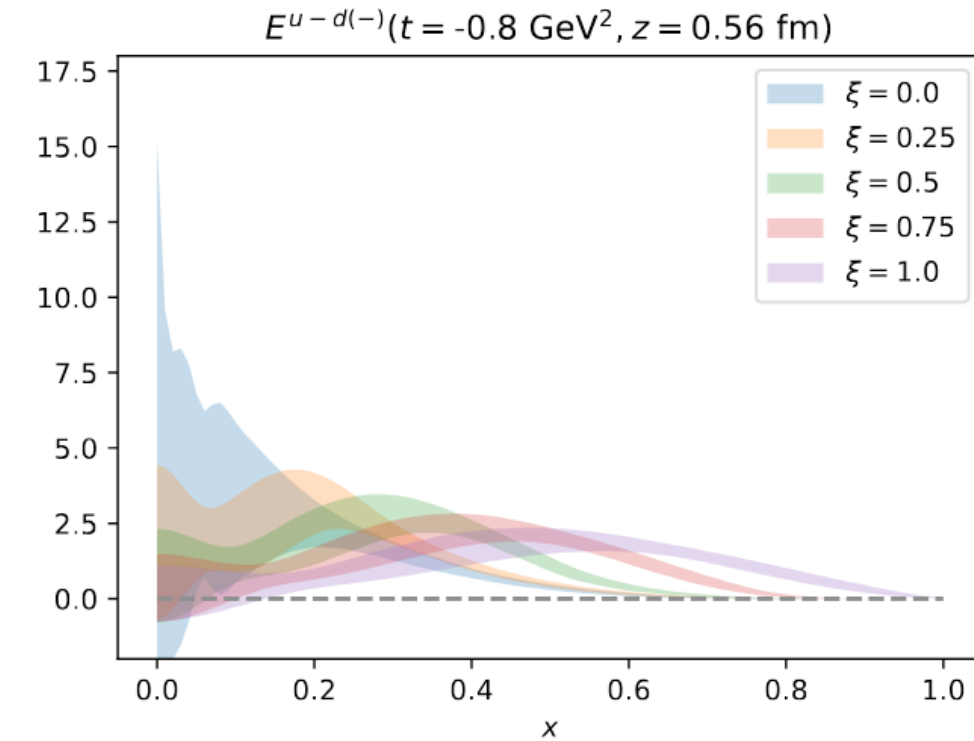
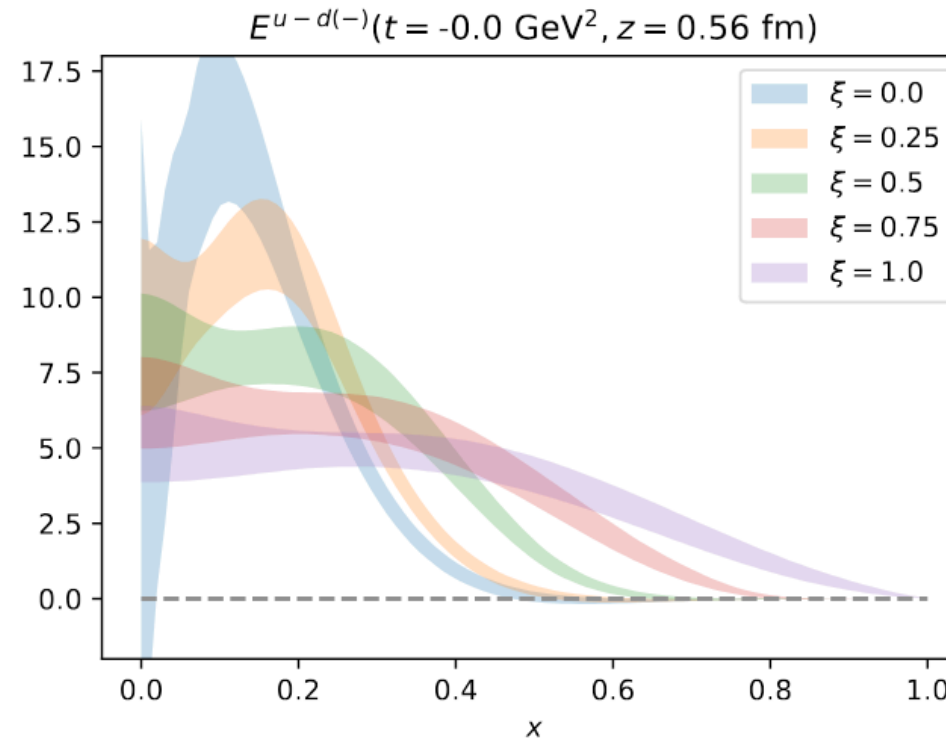
HadStruc (2026) 2604.21476



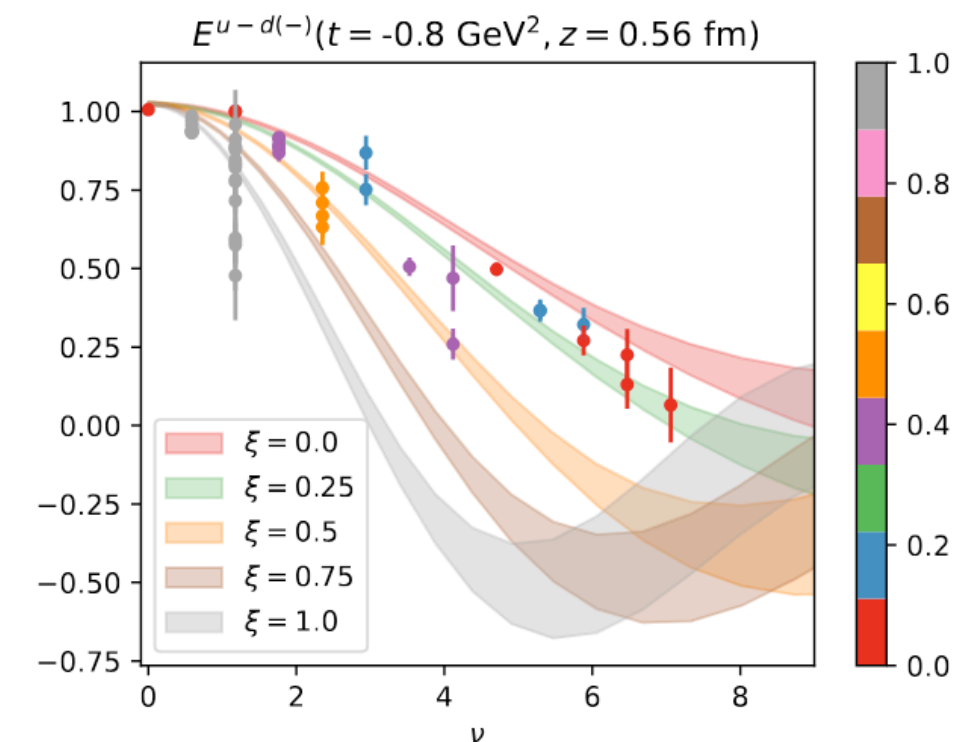
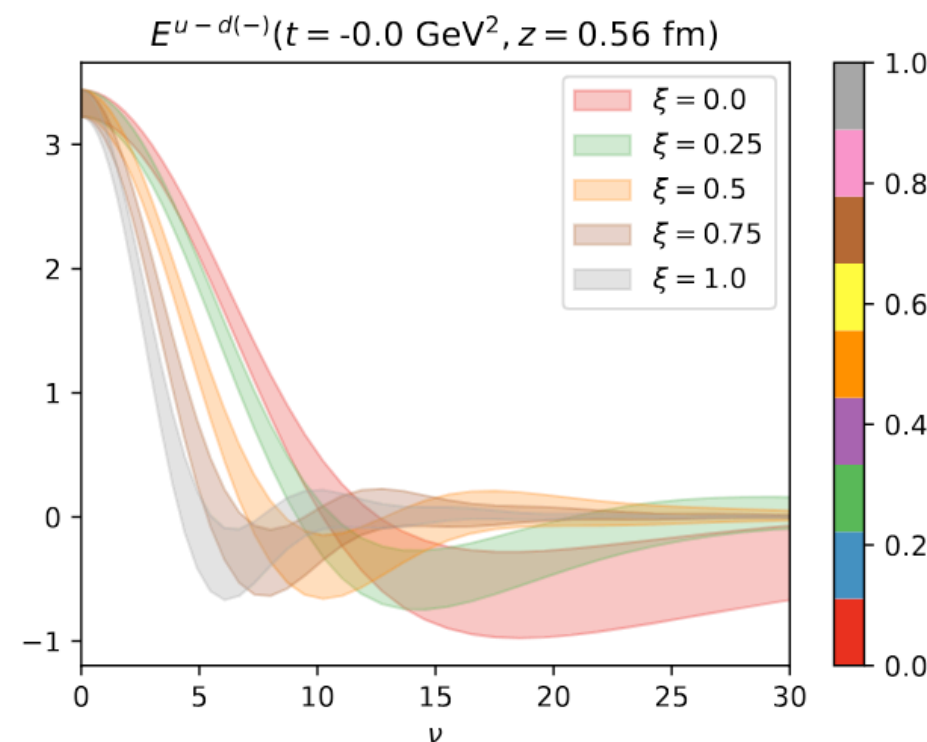
loffe time pseudo-distributions

Iso-vector valence E pseudo-GITD

- Model DD and reconstruct GITD/GPD at wanted t, ξ
- Skewness dependence extrapolated to $t = 0$
- Low β, α are not constrained by the data which leads to uncertainty at low x



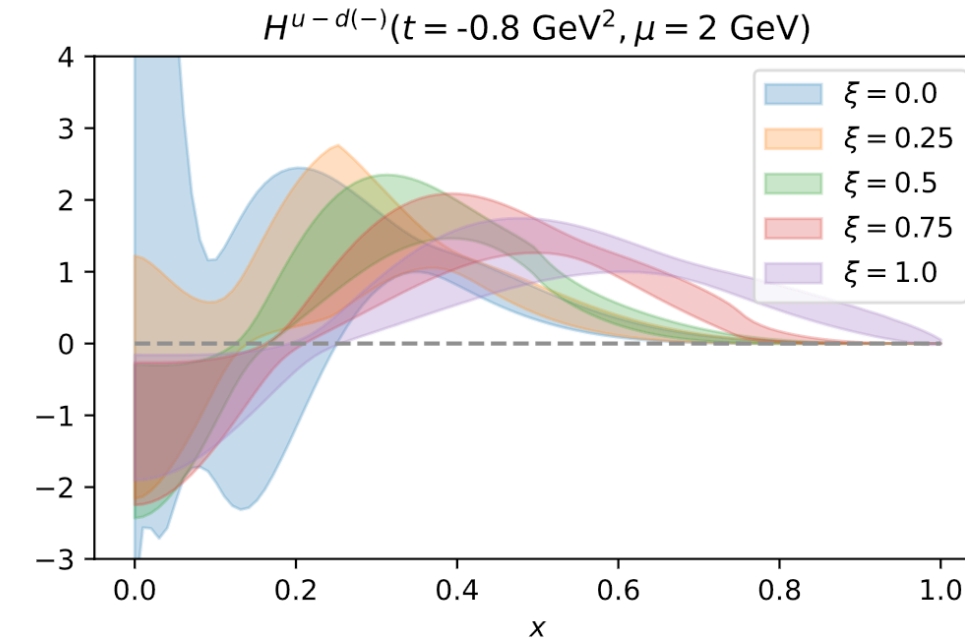
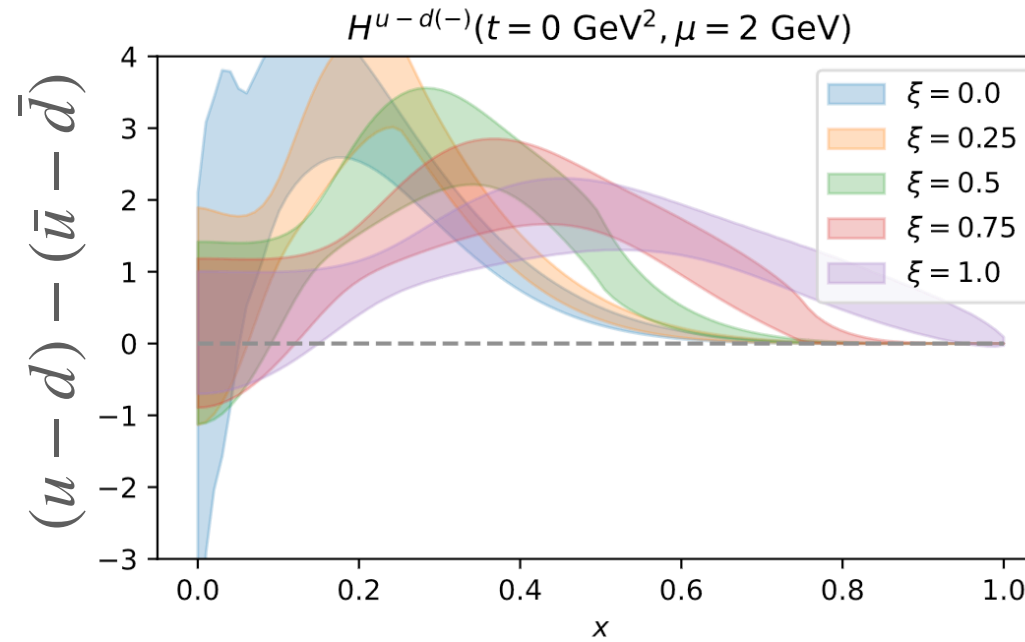
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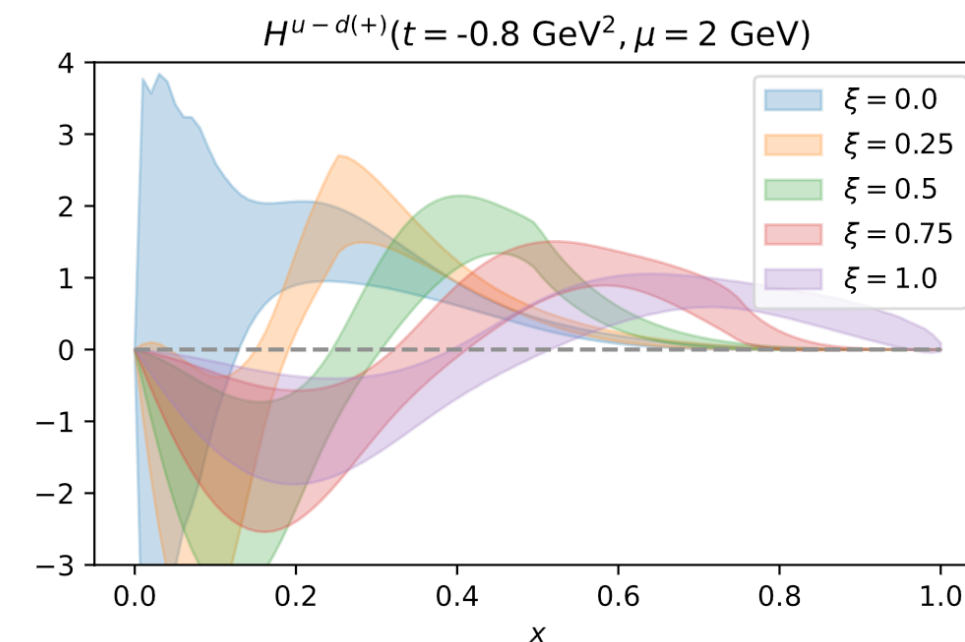
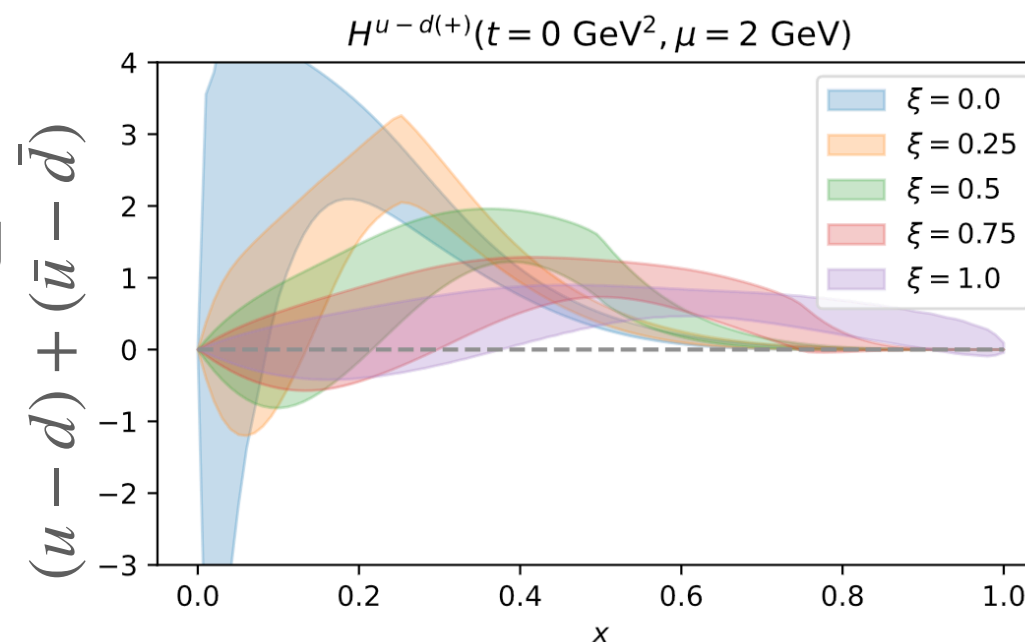
Matching to the $\overline{\text{MS}}$ scheme

LO (1 loop) Matching to H GPD

- Matched at $\mu_0^2 = 1/z^2$ then RGE evolved to $\mu^2 = 4 \text{ GeV}^2$
- Uncertainty is statistical and excited state contamination
- Varying matching scale by $\sqrt{2}$ and changing hyperparameters were smaller errors



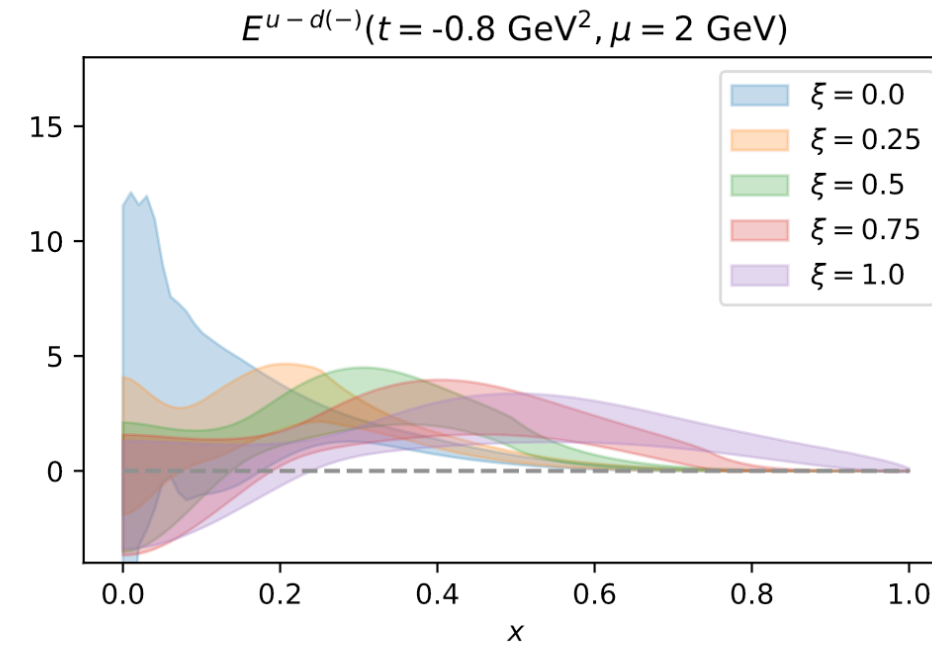
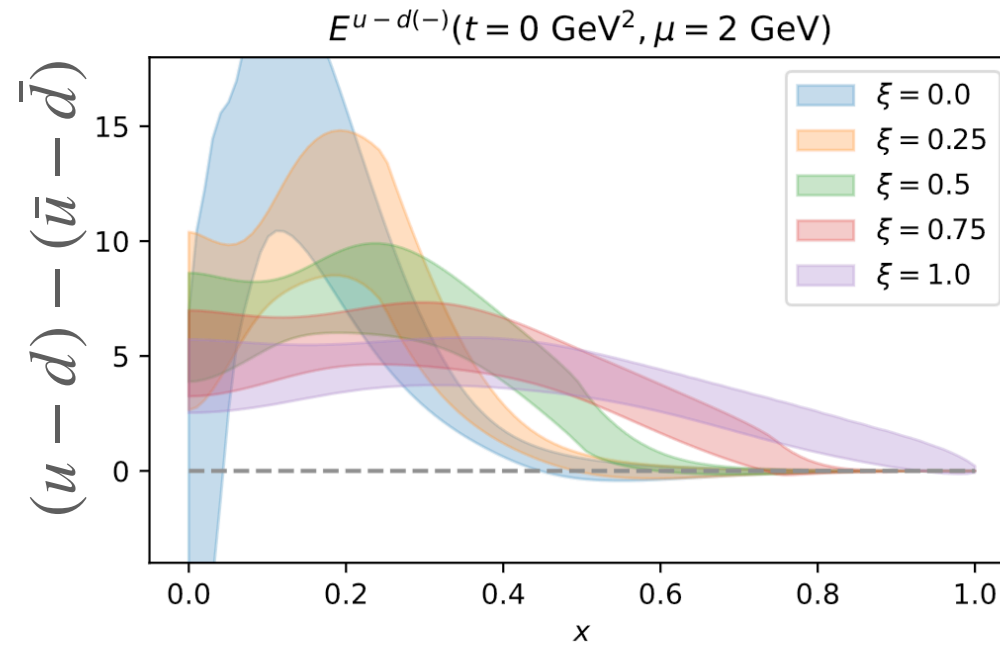
HadStruc (2026) 2604.21476



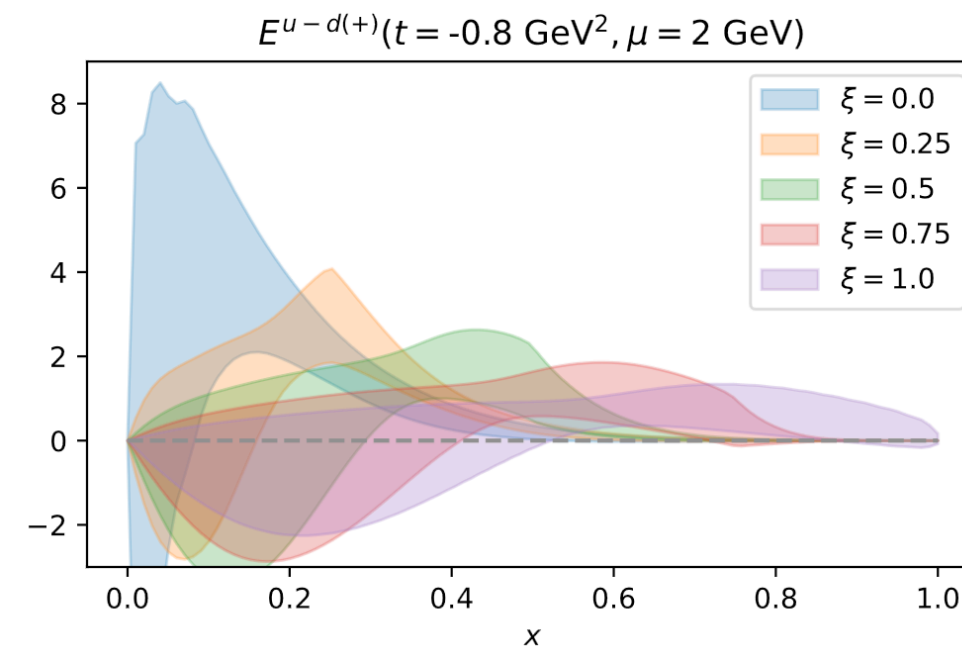
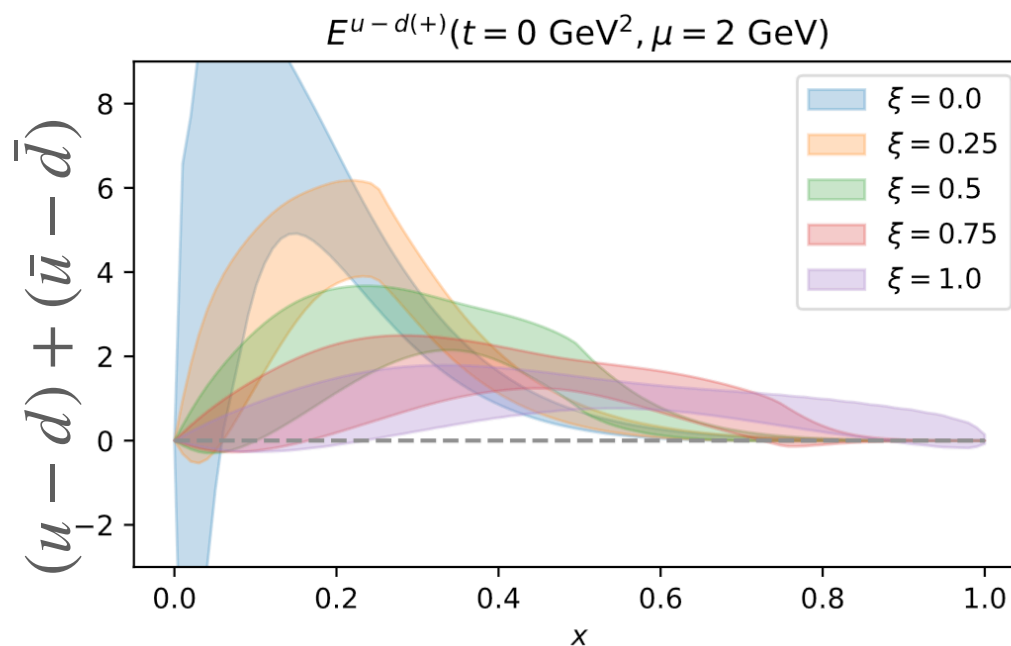
Matching to the $\overline{\text{MS}}$ scheme

LO (1 loop) Matching of E GPD

- Matched at $\mu_0^2 = 1/z^2$ then RGE evolved to $\mu^2 = 4 \text{ GeV}^2$
- Uncertainty is statistical and excited state contamination
- Varying matching scale by $\sqrt{2}$ and changing hyperparameters were smaller errors



HadStruc (2026) 2604.21476



Future Outlook

Calculation of GPDs full 3D structure from lattice QCD is possible!

Lattice Direction

- Lattice systematics of volume, lattice spacing, and pion mass must be varied
- Higher order perturbation theory and control over higher twist needs precision study
- Extension to gluon and singlet underway

Compare and Confront Phenomenology

- Pseudo-GPDs have finite power corrections allowing comparison at $x = \xi$
- Combined analysis may give information at ξ where lattice does not constrain and $x \neq \xi$ which many experiments do not constrain

Back up slides



**Wait, didn't you say Lattice have
many errors to control?**

**How do we know when lattice
data are ready?**

Window Observables instead of x space comparison

- Lattice QCD data is more precise in the low ν (wide, slow Fourier modes) and lack any information at high ν (narrow, fast Fourier modes)
- The lattice results have this precision hiding in the covariance of the PDF

Window Moments

$$a_n(x_-, x_+) = \int_{x_-}^{x_+} dx x^n f(x)$$

“Truncated moments” ($x_+ = 1$)
have known evolution

Gaussian windows

$$g_n(x_-, x_+) = \int_{-1}^1 dx \sqrt{\frac{n^2}{2\pi x_d^2}} \exp\left[-n^2 \frac{(x - x_0)^2}{2x_d^2}\right] f(x)$$

$$x_d = x_+ - x_- \quad x_0 = \frac{x_+ + x_-}{2}$$

Window Observables instead of x space comparison

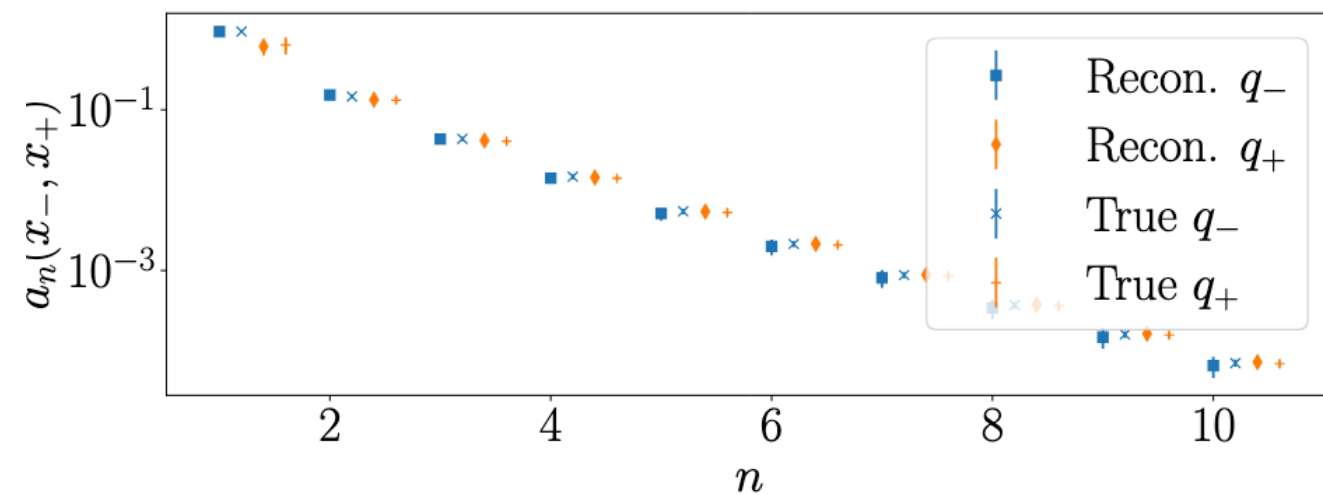
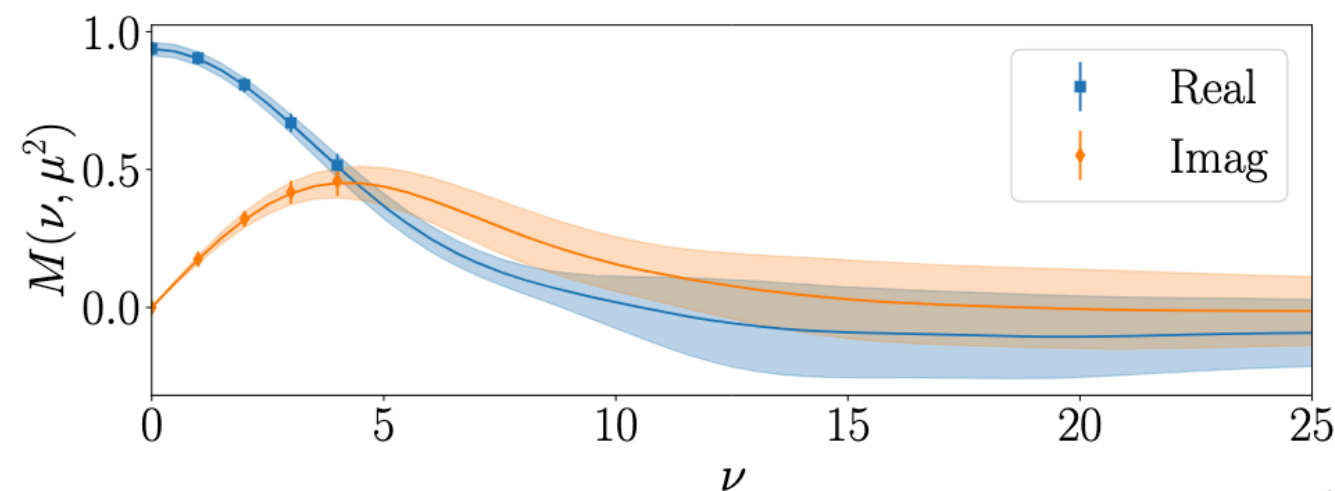
- GP reconstruction of mock data from JAM3D* transversity u-d PDFs

L. Gamberg et al arXiv:2205.00999

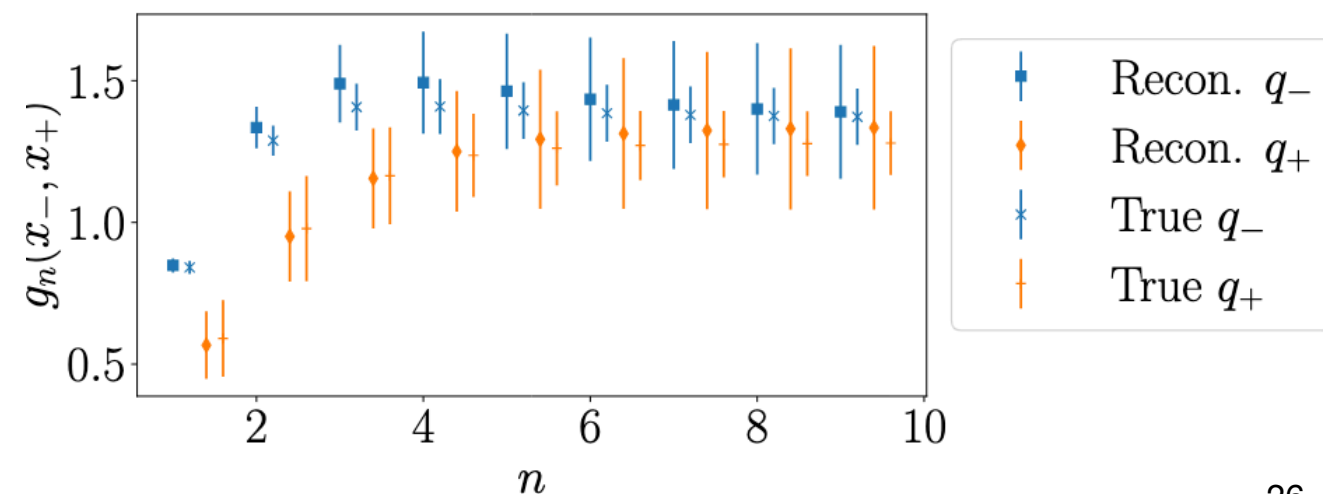
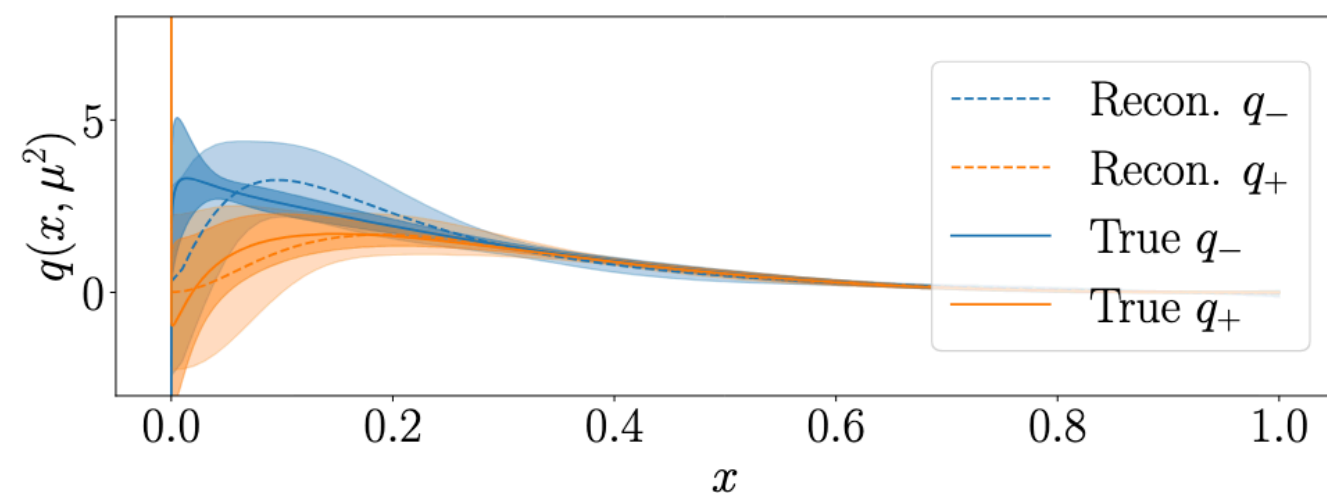
C. Cocuzza et al arXiv:2306.12998

- Precision of the low n (widest windows) is hiding in the covariance of the data

JK et al arXiv:2505.22795



$$x_- = 0.1 \quad x_+ = 0.5$$



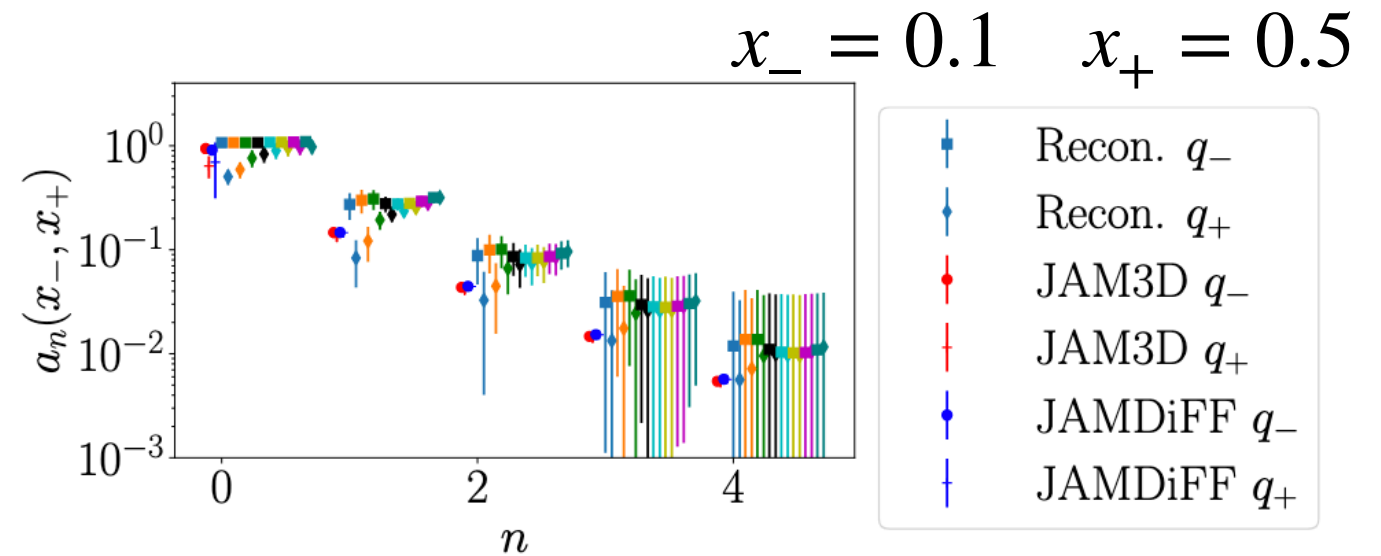
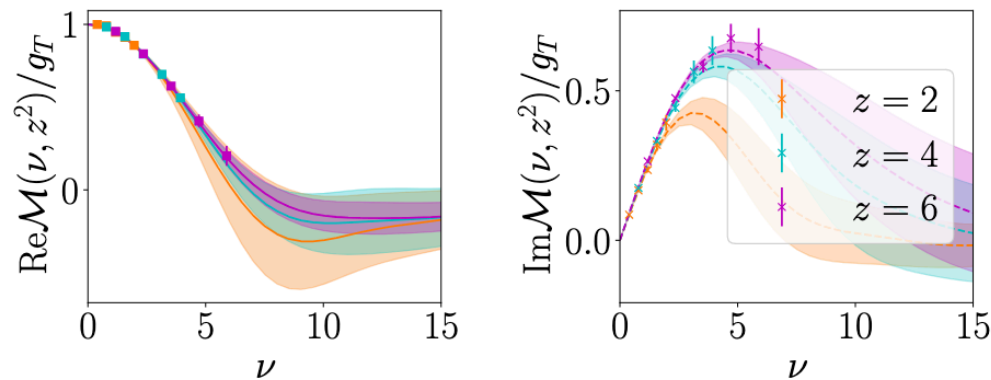
Window Observables instead of x space comparison

- GP reconstruction of HadStruc transversity u-d PDFs

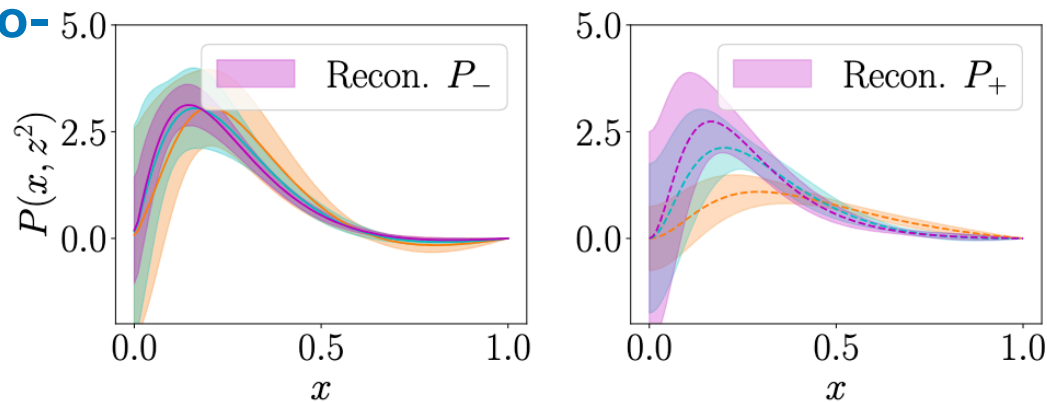
C. Egerer et al arXiv:2111.01808

- Precision of the low n (widest windows) is hiding in the covariance of the data

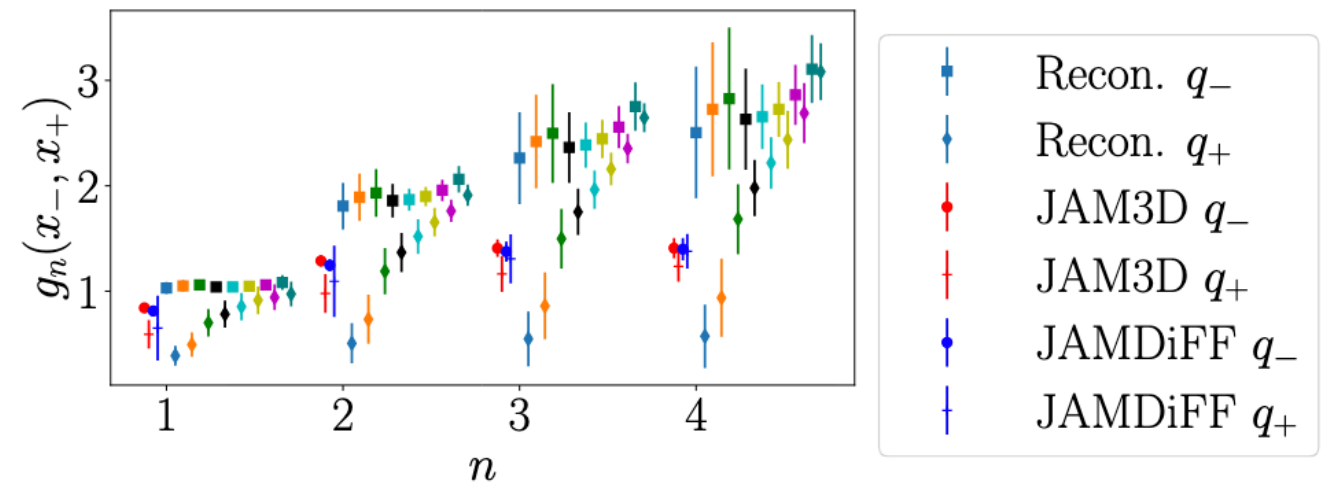
ITD



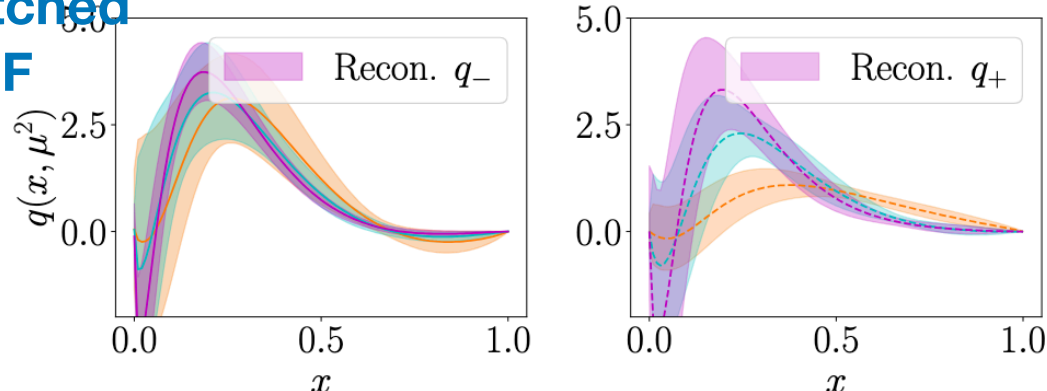
pseudo-PDF



$a = 0.094\text{fm}$ $m_\pi = 358\text{MeV}$



LL matched PDF

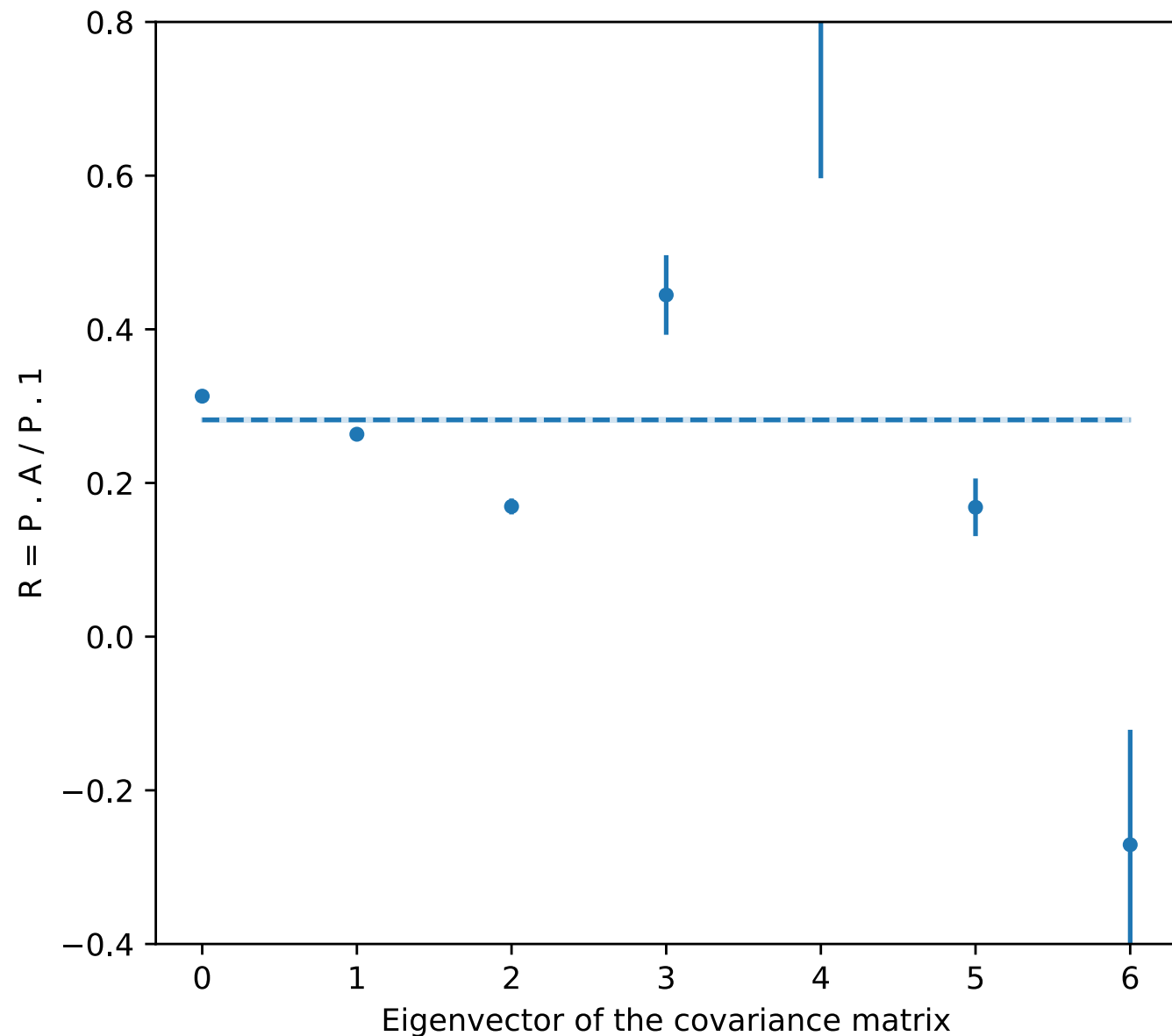
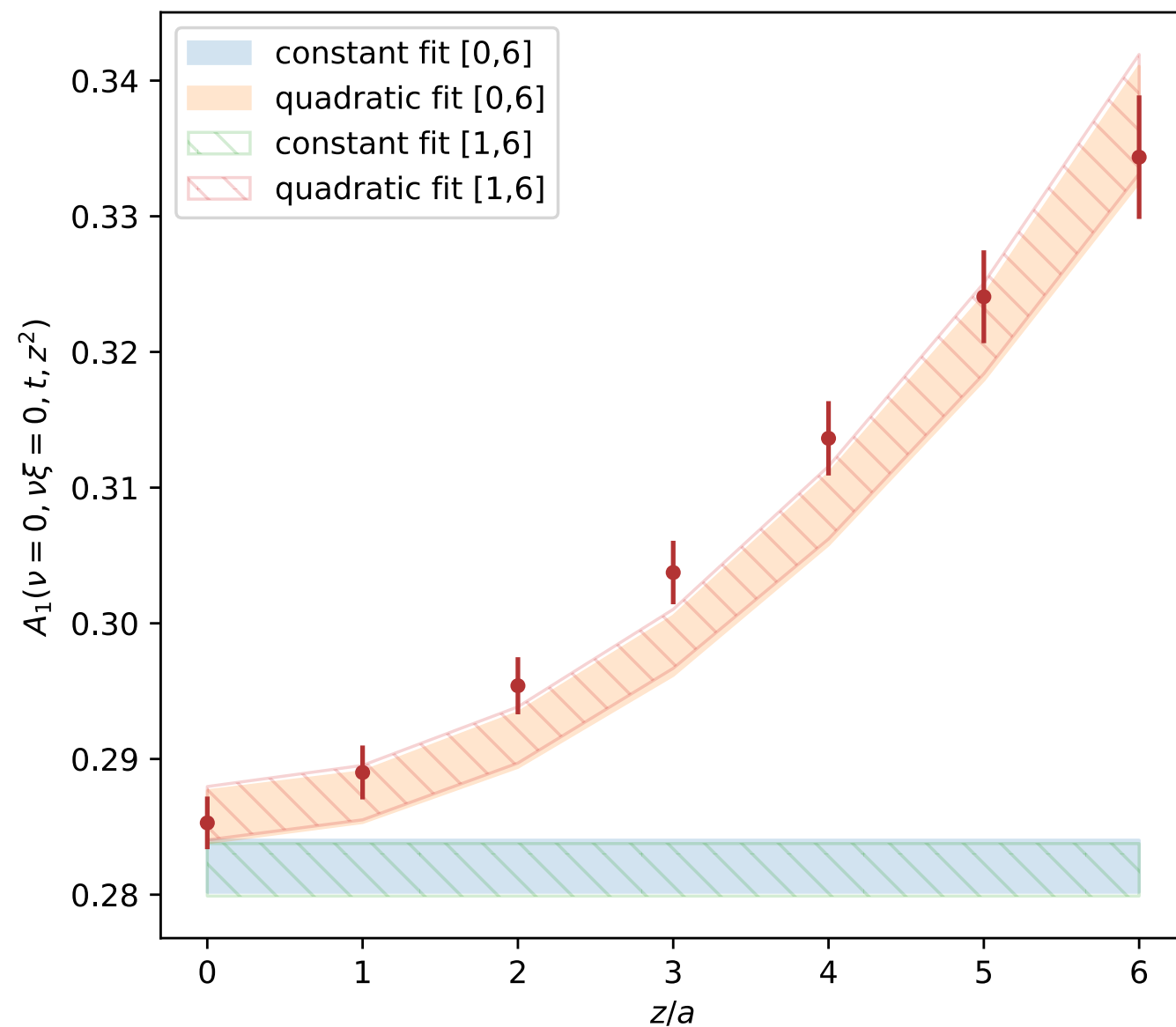


- Lowest z s give bad reconstruction, but errors are promising

JK et al arXiv:2505.22795

There is less than you think

- Lattice data are highly correlated and you have less information than you think
- What is the average of these 7 data points?



Other Faces of WL Matrix Element

- Some are Lorentz invariant interpretations
- These interpretations nor the functions' bounds require small z^2 , only relation to light cone PDF with $z^2 = 0$ and some other regulation

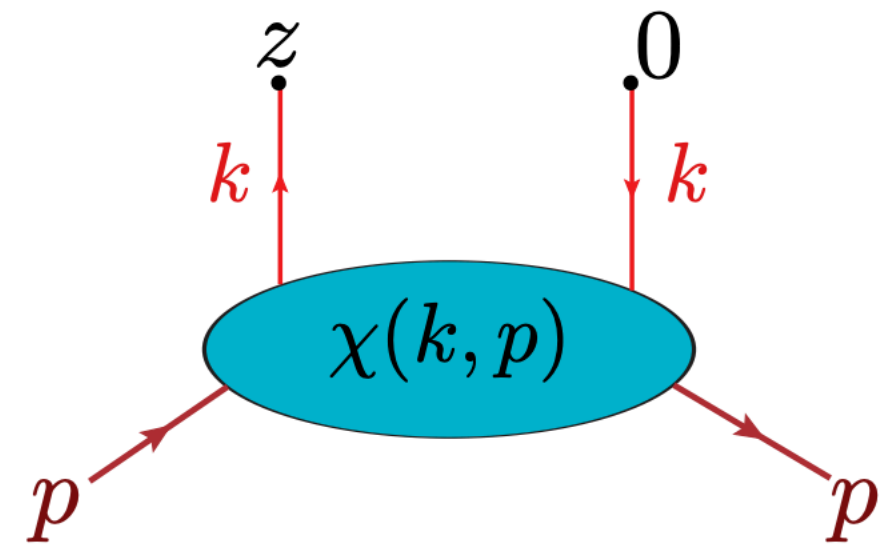
VDF: A. Radyushkin (2016) PRD 93 056002

$$i\chi_{d_i}(k, p) = i^l \frac{P(\text{c.c.})}{(4\pi i)^{2L}} \int_0^\infty \prod_{j=1}^l d\alpha_j [D(\alpha)]^{-2} \\ \times \exp \left\{ ik^2 \frac{A(\alpha)}{D(\alpha)} + i \frac{(p-k)^2 B_s(\alpha) + (p+k)^2 B_u(\alpha)}{D(\alpha)} \right\} \\ \times \exp \left\{ ip^2 \frac{C(\alpha)}{D(\alpha)} - i \sum_j \alpha_j (m_j^2 - i\epsilon) \right\}, \quad i\chi(k, p) = \int_0^\infty d\sigma \int_{-1}^1 dx e^{i\sigma[k^2 - 2x(k \cdot p) + i\epsilon]} V(x, \sigma)$$

$$\sigma_{d_i} = \frac{A_{d_i}(\alpha) + B_{s_{d_i}}(\alpha) + B_{u_{d_i}}(\alpha)}{D_{d_i}(\alpha)}$$

$$x_{d_i} = \frac{B_{s_{d_i}}(\alpha) - B_{u_{d_i}}(\alpha)}{A_{d_i}(\alpha) + B_{s_{d_i}}(\alpha) + B_{u_{d_i}}(\alpha)}$$

α_j are positive numbers
and A, B_u, B_s, C, D are
sums of products of α_j



Fourier transform to
position space

$$\mathcal{M}(\nu, z^2) = \int \frac{d^4 k}{(2\pi)^4} e^{ik \cdot z} \chi(k, p) = \int_{-1}^1 dx e^{i\nu x} \int_0^\infty e^{-i\sigma(z^2 - \epsilon)} V(x, \sigma)$$

Other Faces of WL Matrix Element

- Some are Lorentz invariant interpretations
- These interpretations nor the functions' bounds require small z^2 , only relation to light cone PDF with $z^2 = 0$ and some other regulation

VDF: A. Radyushkin (2016) PRD 93 056002

Virtuality Distribution Function

Lorentz invariant picture

σ^{-1} pole gives $\log z^2$

Limits from nature of Feynman diagrams

$$\mathcal{M}(\nu, z^2) = \int_{-1}^1 dx e^{i\nu x} \int_0^\infty d\sigma e^{-i\sigma(z^2 - \epsilon)} V(x, \sigma)$$

pseudo-PDF

Lorentz invariant picture

$\log z^2$ divergence from poles of TMD/VDF

$$\mathcal{M}(\nu, z^2) = \int_{-1}^1 dx e^{i\nu x} P(x, z^2)$$

$$\tilde{q}(y, p_z^2) = \int dz \int_{-1}^1 dx e^{ip_z z(x-y)} P(x, z^2)$$

Musch, Hagler, Negele, Schafer PRD 83 (2011) 094507

Straight Link / Primordial TMD

Frame dependent picture with nice interpretation

$1/k_T^2$ pole gives $\log z^2$

$$z = (0, z^-, z_T) \quad p = (p^+, \frac{m^2}{p^+}, 0_T)$$

$$\mathcal{M}(\nu, z^2) = \int_{-1}^1 dx e^{i\nu x} \int d^2 k_T e^{ik_T \cdot z_T} F(x, k_T^2)$$

Light cone PDF from regulated integral of TMD

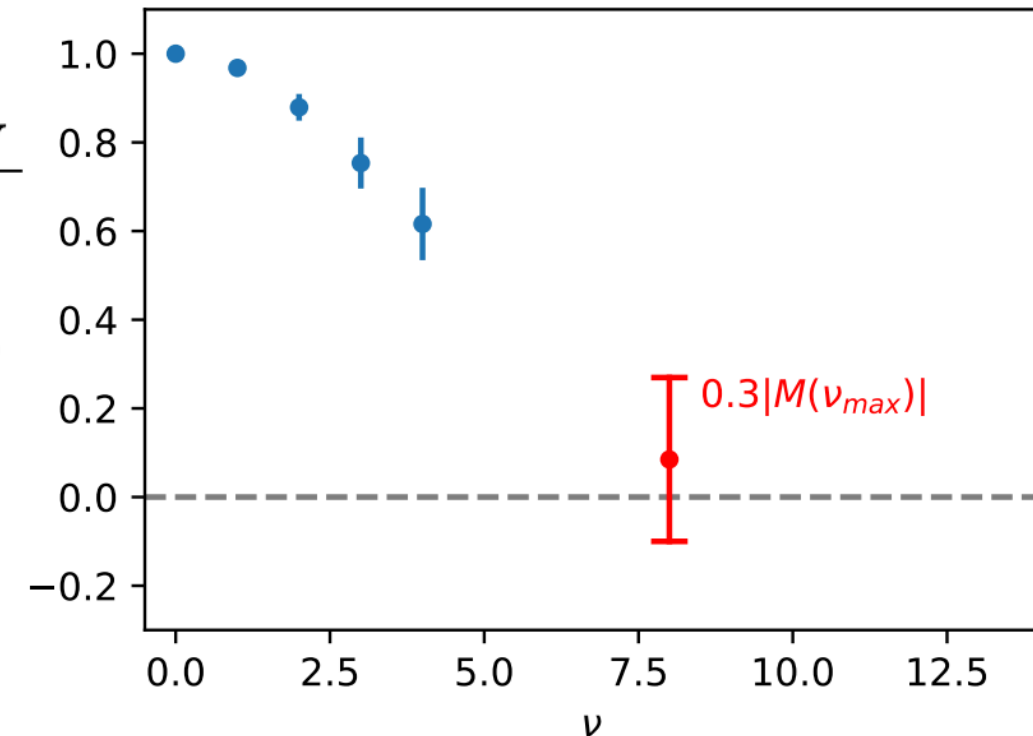
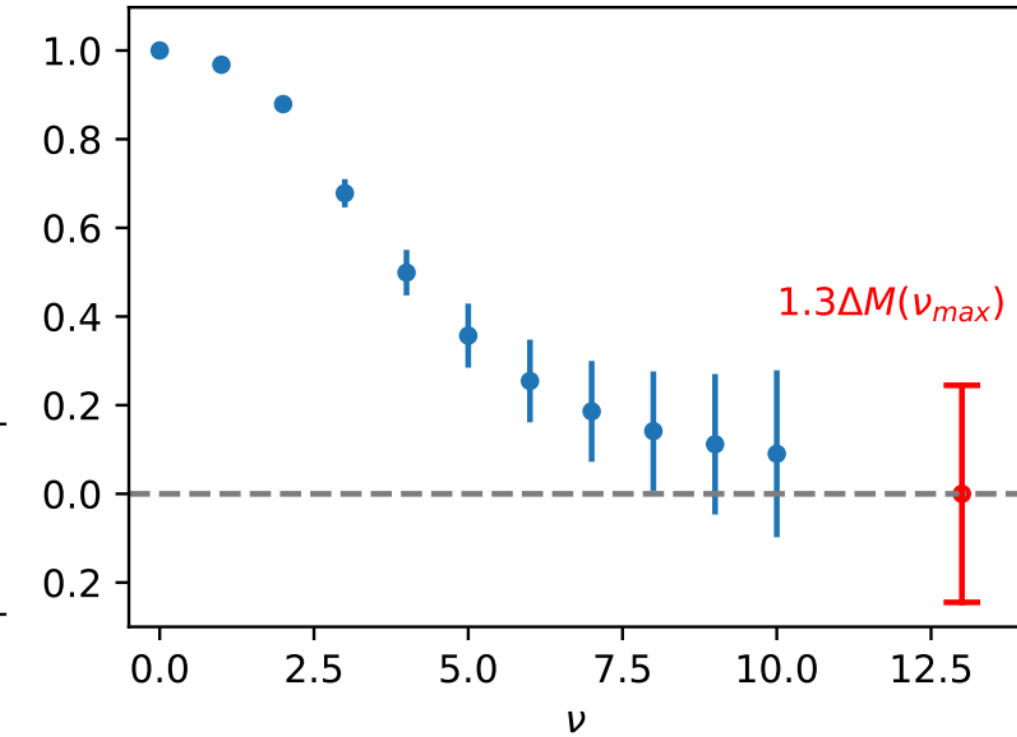
Relate to the Lorentz invariant VDF

$1/k_T^2$ or σ^{-1} poles generate $\log \mu^2$ divergence

$$f(x, \mu^2) = \int^{\mu^2} d^2 k_T F(x, k_T^2) = \int_0^\infty d\sigma \left[1 - e^{-\frac{i}{\sigma}(\mu^2 - i\epsilon)} \right] V(x, \sigma)$$

Choosing GPR Hyperparameters

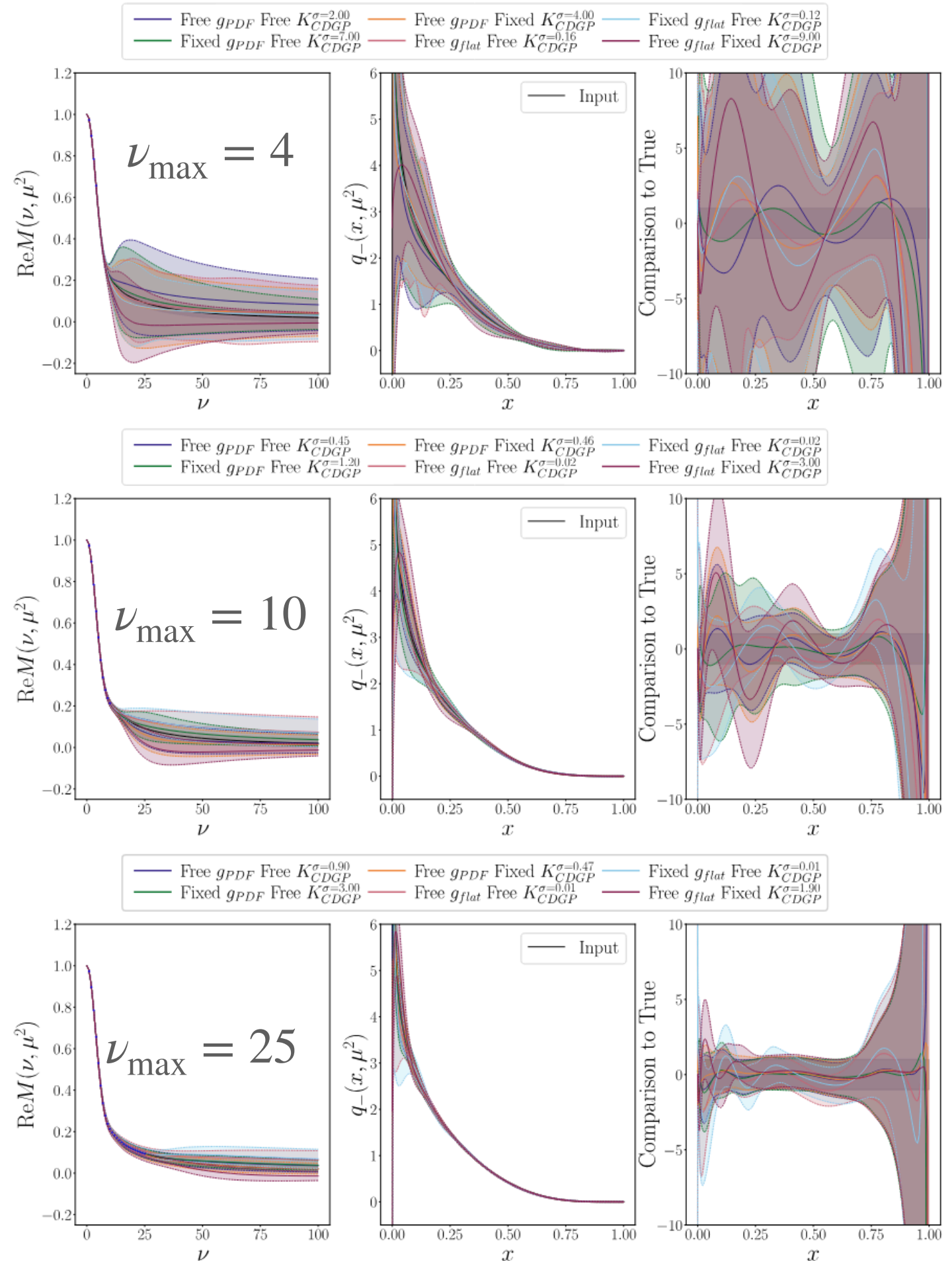
Choice	Motivation
$K(x, x')$ eq. (11)	Enforce smoothness in x while decorrelating small x from large x
$l = \ln(2) \approx 0.693$	Set the flexibility of the x -dependence
$g(x) = \sigma$	Set the uncertainty to 100% at $x = 0$ and loosely enforce positivity
σ^2 such that the worst uncertainty in Fourier space is $\max(1.3\Delta M(\nu_{max}), 0.3 M(\nu_{max}))$	Control the size of uncertainty in the extrapolation region in Fourier space



Closure tests

- Many options of prior covariances/means and ways to treat their parameters
- Made mock data for different ranges to study the systematic improvement
- Closure test on mean and central values
- All give highly consistent results in agreement with data

Y Cahuana Medrano, H Dutrieux, JK,
K Orginos, S Zafeiropoulos arXiv:2510.21041
Accepted in JHEP



Which Model is the best?

- “Information criteria” are different approximations of difference between your posterior and the true posterior
- IC define weights to average different models
- Largely give consistent central values with slightly different error estimates

