Moments of pion and kaon light-cone distribution amplitudes

from dynamical lattice QCD using the heavy-quark operator product expansion

Speaker: Alex Chang

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2026/07/27

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THE HOPE COLLABORATION

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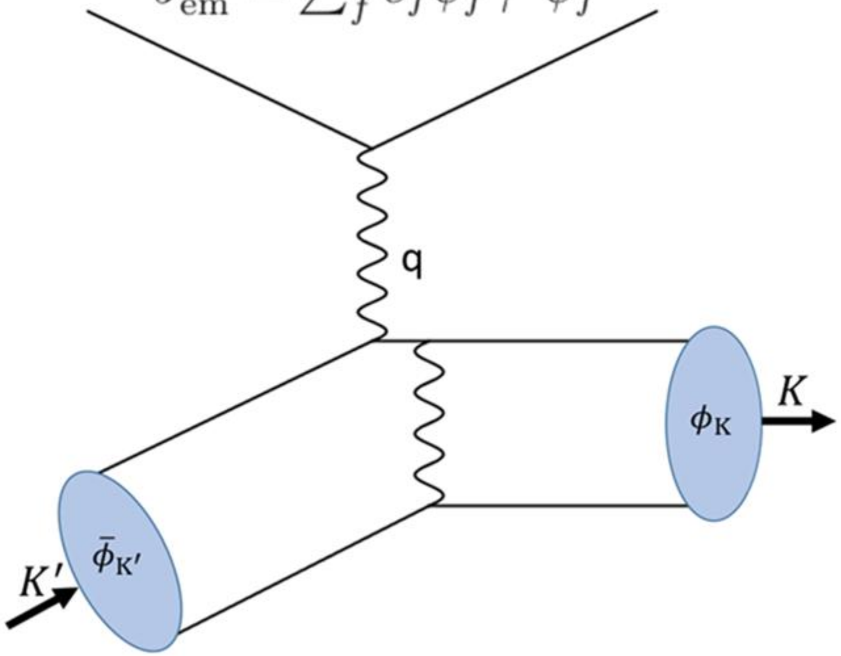
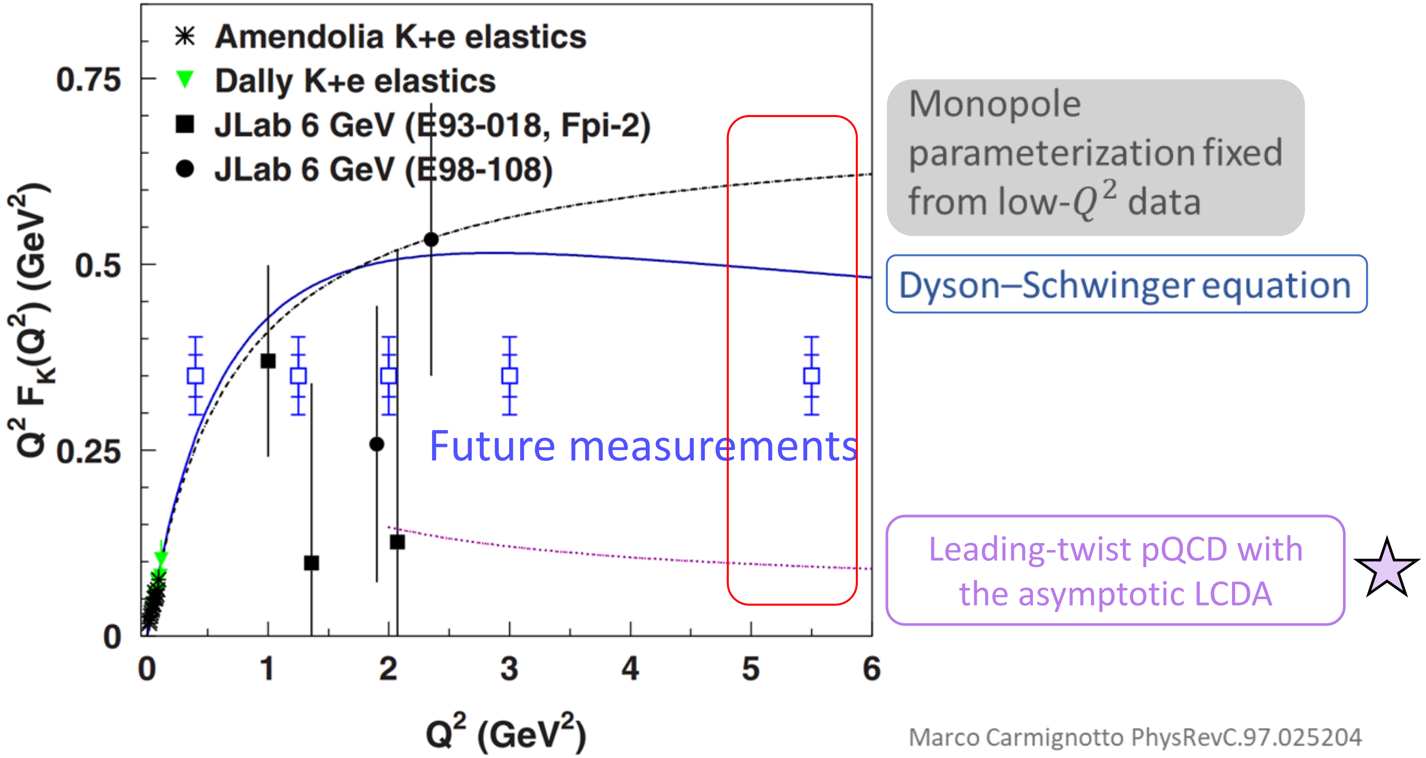
Kaon Electromagnetic Form Factors

$$\langle K^+(P')|J^\mu_{\rm em}|K^+(P)\rangle = F_K(Q^2)(P^\mu + P'^\mu)$$

$$J^\mu_{\rm em} = \sum_f e_f \bar{\psi}_f \gamma^\mu \psi_f$$



elastic electron-kaon scattering



Marco Carmignotto PhysRevC.97.025204

- Model-dependent.
- At asymptotically large Q^2 , the result should approach the conformal-limit prediction.
- High Q^2 experimental data are difficult to obtain.

Nonperturbative theory is important to know the Kaon Electromagnetic Form Factors

Kaon Electromagnetic Form Factors

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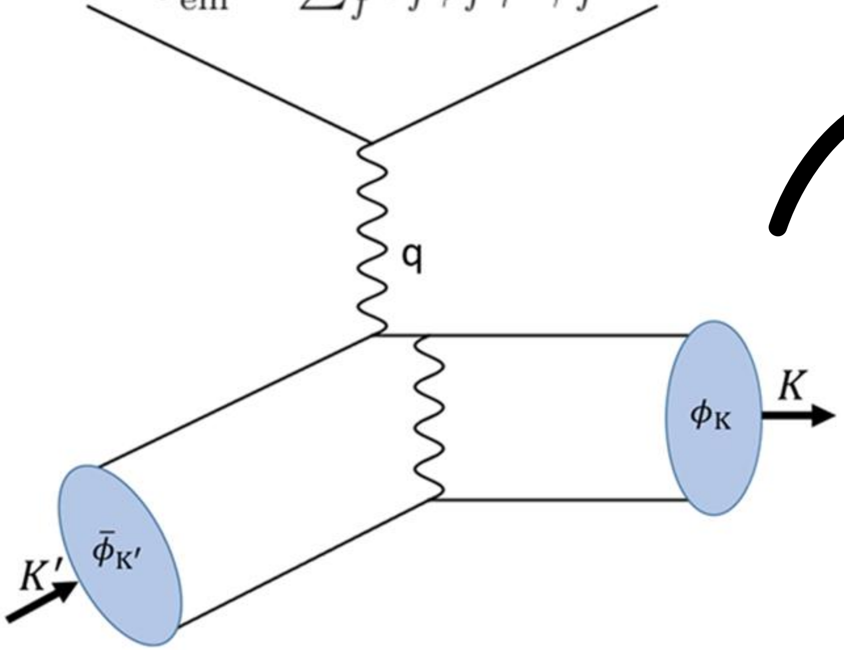
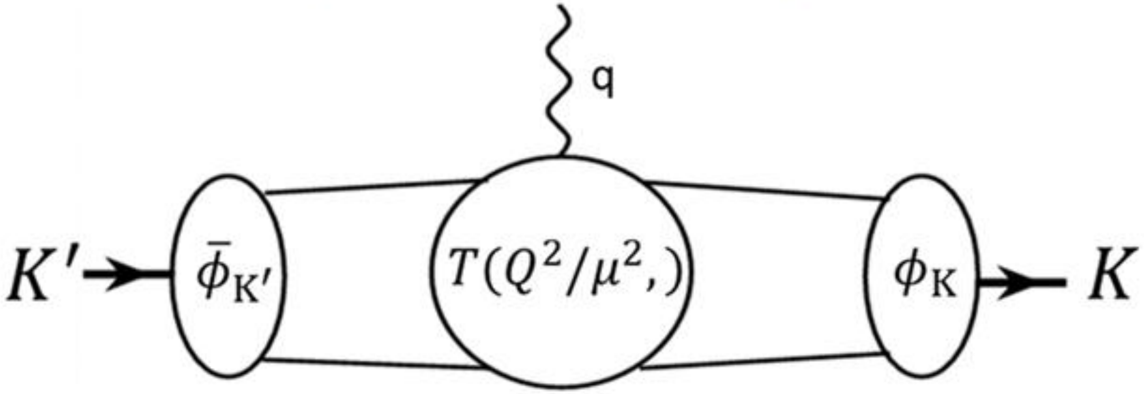


elastic electron-kaon scattering

QCD Factorization



$$F_K(Q^2) = \int dxdy\bar{\phi}_K(x,Q^2)T(x,y,Q^2)\phi_K(y,Q^2)$$

LCDA ϕ encoding the non-perturbative structure of the kaon.

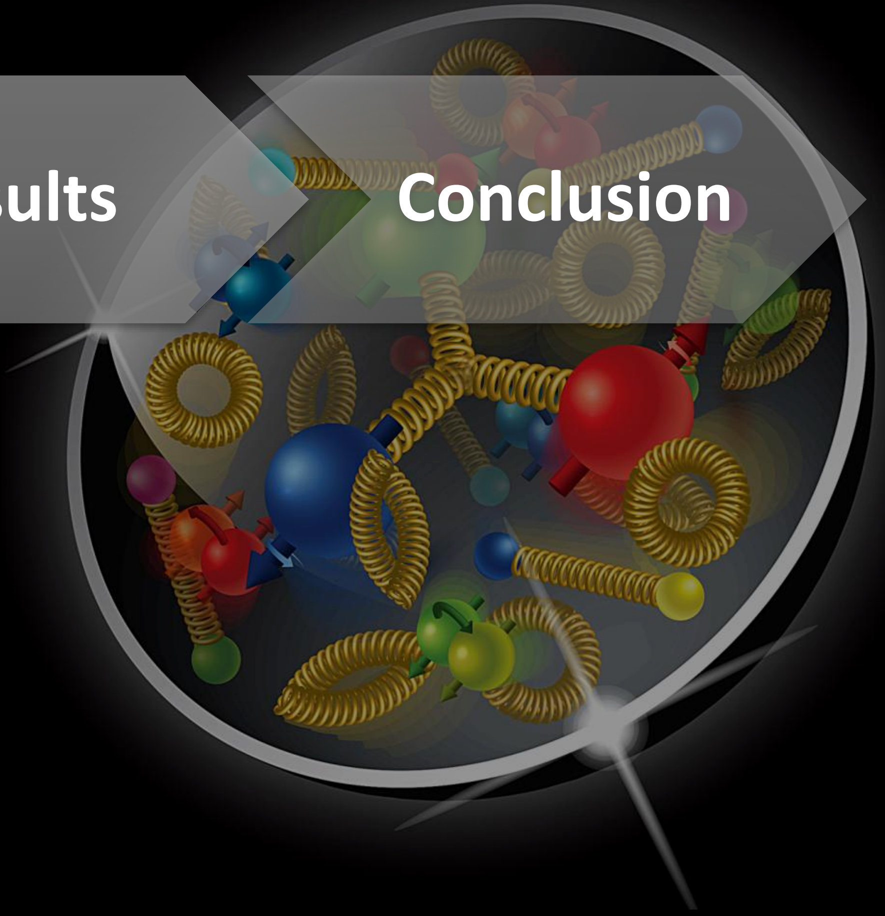
Lattice QCD

Outline

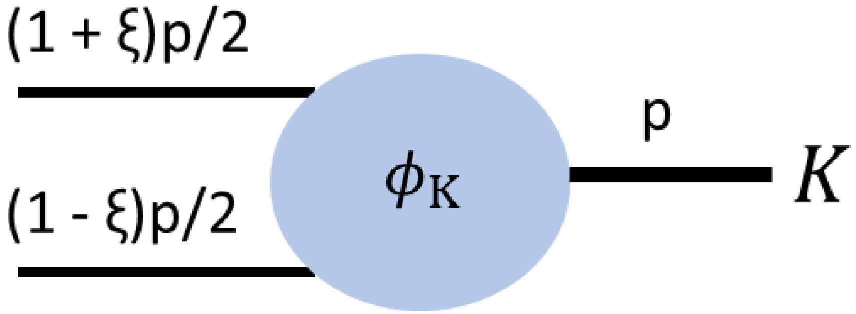
- Heavy-Quark Operator Product Expansion (HOPE Method)

Introduction → Method → Results → Conclusion

- Light-Cone Distribution Amplitudes
- Light-Cone OPE & Limitations on Lattice



Light-Cone Distribution Amplitudes (LCDAs)



$$\langle\Omega|\, \bar{\psi}_\alpha(z)\gamma_\mu\, \gamma_5\, W[z,-z]\psi_\beta(-z)|\mathrm{K}^+(p)\rangle$$
$$= i\, f_{\mathrm{K}}\, p_\mu \int_{-1}^{1} d\xi\, e^{-i\xi\, p\cdot z}\, \phi_{\mathrm{K}}(\xi,\mu^2)$$

- f_K is the pseudoscalar kaon decay constant and W is light-like Wilson line

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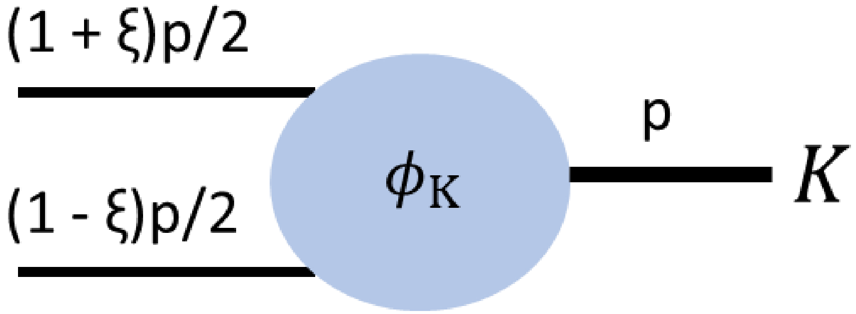

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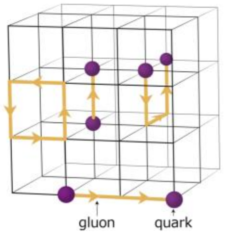
Direct calculation of light-cone objects is impossible on a Euclidean lattice



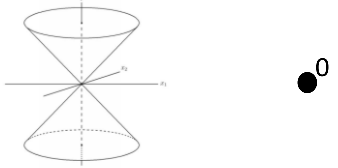
Lattice QCD

Directly

LCDAs

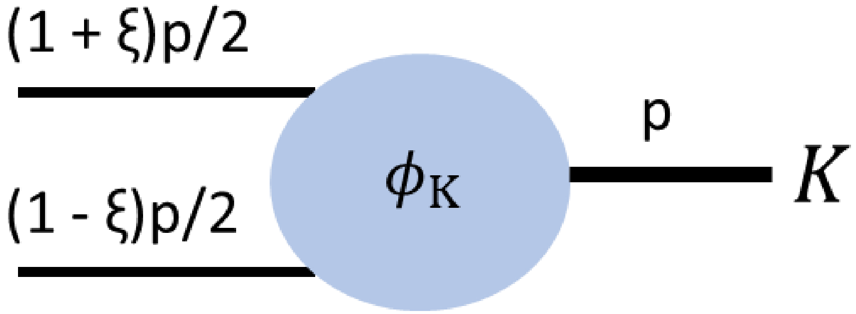
Wick rotation $t \to -i\tau \;\Rightarrow\;$ Minkowski $\to$ Euclidean

light cone,
defined by $z^2 = 0$



There is no notion of a light-cone in Euclidean space.

Light-Cone Distribution Amplitudes (LCDAs)



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- f_K is the pseudoscalar kaon decay constant and W is light-like Wilson line

◆ Light-Cone OPE

- Symmetric traceless projection selects the leading-twist part

$$\bar{\psi}(z)\gamma^\mu\gamma_5 W[z,-z]\psi(-z)=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}z_{\mu_1}\cdots z_{\mu_n}\,\bar{\psi}(0)\gamma^{\{\mu}\gamma_5(i\overleftrightarrow{D}^{\mu_1})\cdots(i\overleftrightarrow{D}^{\mu_n\}})\psi(0)\Big|_{\text{traceless}}$$

$O^{\mu_0\mu_1..\mu_n}$

$$\langle 0|\left[\,\bar{\psi}_\alpha\,\gamma^{\{\mu_0}\gamma_5\,(i\overleftrightarrow{D}^{\mu_1})\ldots(i\overleftrightarrow{D}^{\mu_n\}})\psi_\beta\;-\;trace\right]|K^+(p)\rangle$$

local & twist-two

twist = dim – spin

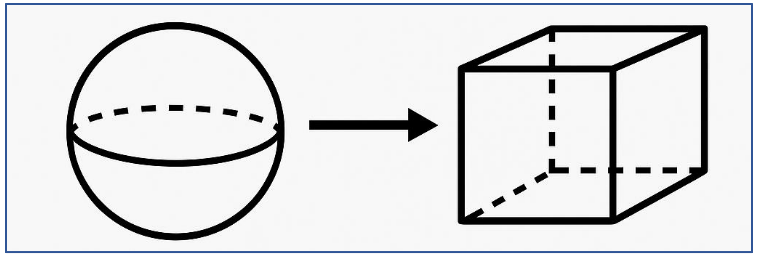
$$=\;f_K\,\langle\xi^n\rangle\,[\,p^{\mu_0}\,p^{\mu_1}\ldots\,p^{\mu_n}\;-\;trace]$$

◆ Mellin moments

$$\langle\xi^n\rangle=\int_{-1}^{1}d\xi\;\xi^n\phi_{\mathrm{K}}(\xi,\mu^2)$$

Challenges of Local-Operator OPE on the Lattice

The lattice regularization breaks SO(4) to hyper cubic group H(4)



Continuum Lattice

twist = dim – ~~spin~~

$$\mathcal{O}_i = \sum_j C_{ij}^{\mathrm{latt}}(a)\mathcal{O}_j^{\mathrm{latt}}$$

- **mix with lower dimension operators**
- **coefficients may contain power divergences**

Direct calculations become difficult for higher moments.

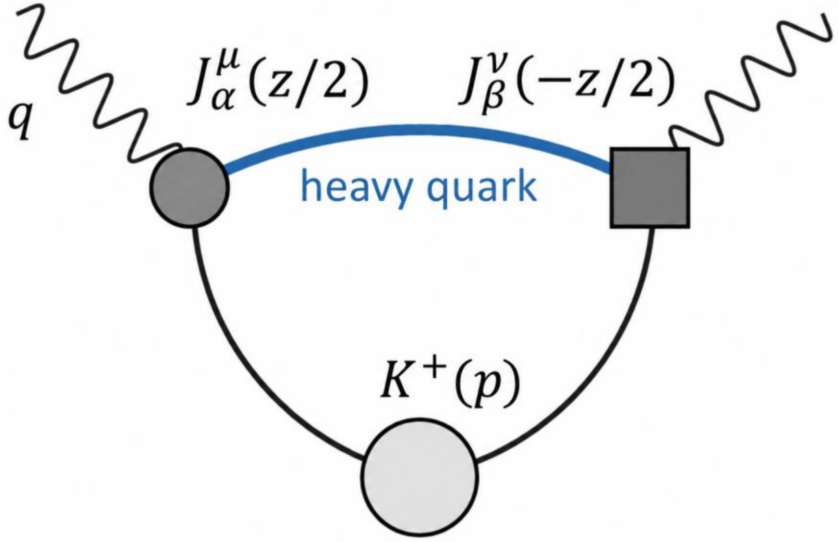
Recent alternative: gradient flow

A. Shindler, Phys. Rev. D 110, L051503 (2024)

HOPE→ **Avoid direct calculation of higher-spin local-operator matrix elements**

W. Detmold and C.-J. D. Lin, Phys. Rev. D 73, 014501 (2006)

HOPE Method



◆ Hadronic Tensor

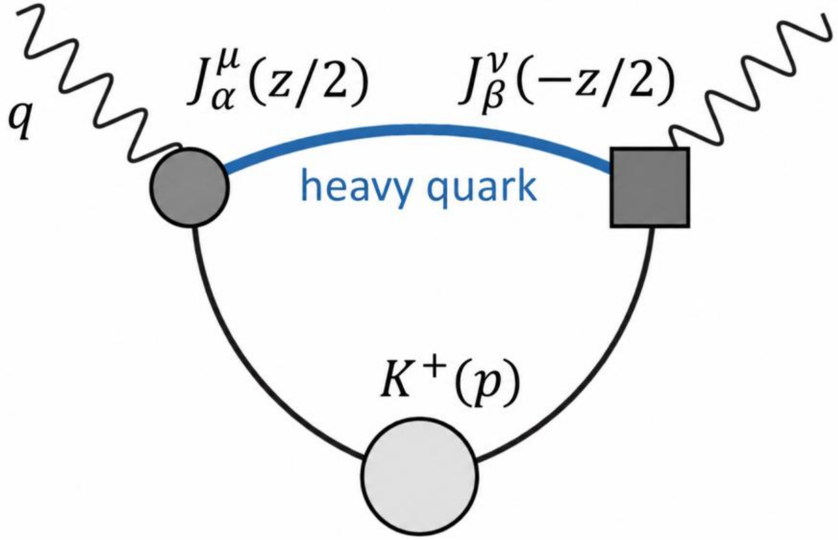
$$V^{\mu\nu}(q,p) = \int d^4z\, e^{iq\cdot z} \langle \Omega | T\{ J^{\mu}_{\alpha}(z/2) J^{\nu}_{\beta}(-z/2)\} | K^{+}(p) \rangle$$

$$J^{\mu}_{\alpha}(z) = \bar{\Psi}(z)\, \gamma^{\mu}\, \gamma_5\, \psi_{\alpha}(z) + \bar{\psi}_{\alpha}(z)\, \gamma^{\mu}\, \gamma_5\, \Psi(z)$$

Ψ is the fictitious valence heavy quark

The heavy quark is not a physical quark. It is introduced to provide a controllable hard scale.

HOPE Method



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Ψ is the fictitious valence heavy quark

Short-distance & heavy scale

$$V^{[\mu\nu]}(q,p) = \frac{-2\, i\, \epsilon^{\mu\nu\rho\sigma}\, q_\rho\, p_\sigma}{\tilde{Q}^2} f_K \sum_{n=0}^{\infty} C_W^{(n)}(\tilde{Q}^2,\mu^2,m_\Psi)\langle\xi^n\rangle\left(\frac{\tilde{\omega}}{2}\right)^n$$

LQCD calculations:
extract hadronic tensor

Fitting

QCD perturbation theory:
One-loop Wilson coefficients

Fit parameters:
- $\langle\xi^n\rangle$ – Mellin moments
- f_K – kaon decay constant
- m_Ψ – heavy quark mass

$$\tilde{\omega} = (2\, q\cdot p)/\tilde{Q}^2$$

$$\tilde{Q}^2 = -q^2 + m_\Psi^2$$

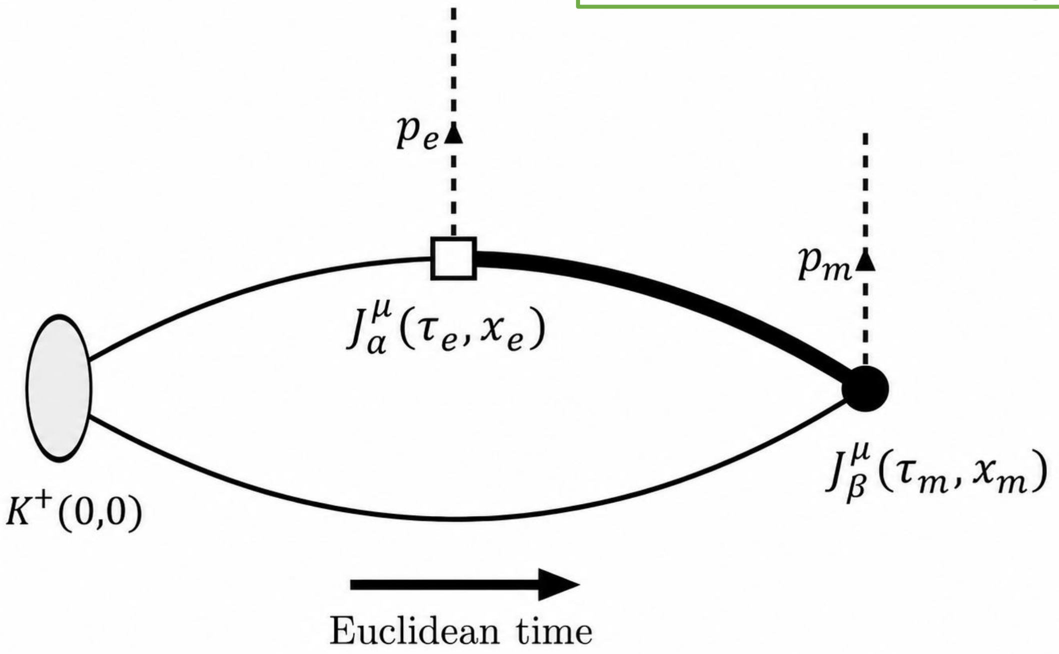
$$\Lambda_{\mathrm{QCD}} \ll m_\Psi \sim \sqrt{q^2} \ll \frac{1}{\hat{a}}$$

HOPE Method

3pt-function

$$C_3^{\mu\nu}(\tau_e, \tau_m; \mathrm{p_e}, \mathrm{p_m}) = \int d^3x_e\, d^3x_m\, e^{i\mathrm{p_e}\cdot x_e} e^{i\mathrm{p_m}\cdot x_m} \langle 0|\mathcal{T}\left[J_\alpha^\mu(\tau_e, x_e) J_\beta^\nu(\tau_m, x_m) \mathcal{O}_K^\dagger(0)\right]|0\rangle$$

$$\langle 0|T\{J_\alpha^\mu(\tau_e, \mathrm{p_e}) J_\beta^\nu(\tau_m, \mathrm{p_m}) K^+(0, \mathrm{p})\}|0\rangle$$



Fourier transform of Hadronic Tensor

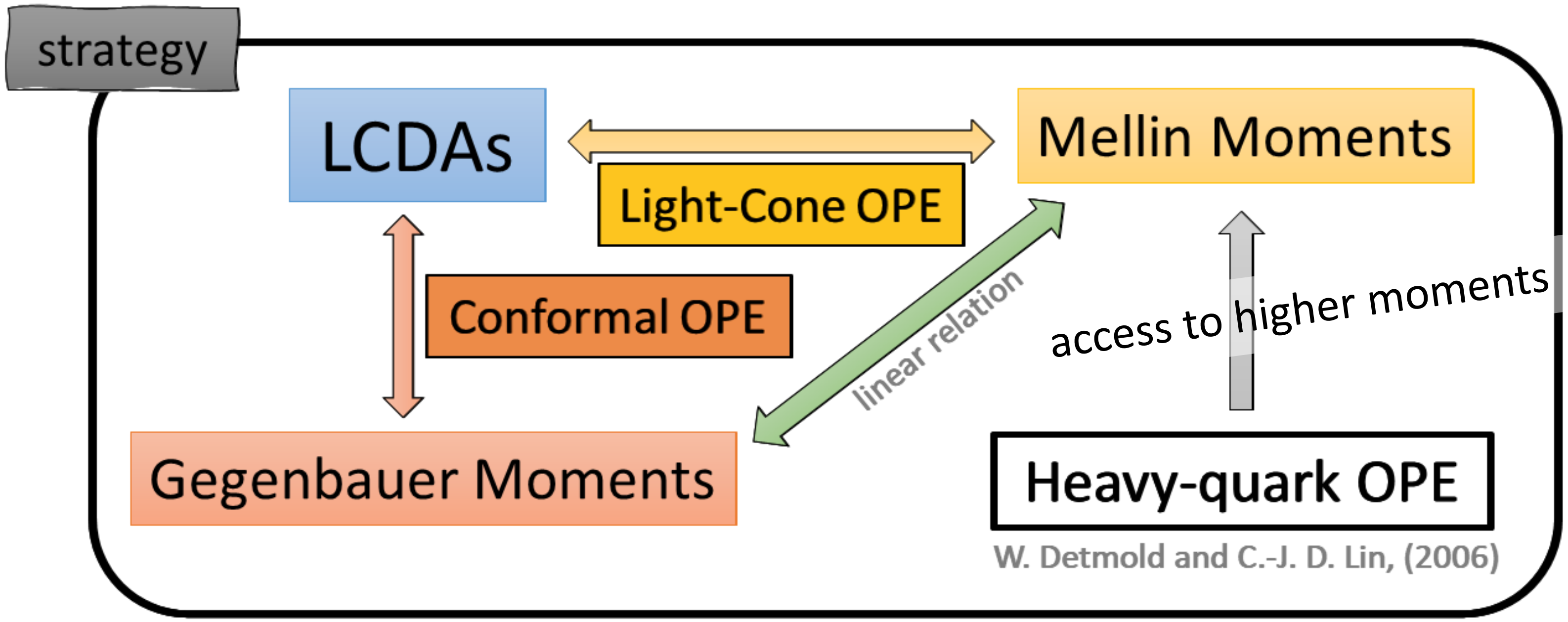
$$R^{\mu\nu}(\tau, \mathrm{q}, \mathrm{p}) = \langle 0|T\{J_\alpha^\mu(\tau_e, \mathrm{p}_e) J_\beta^\nu(\tau_m, \mathrm{p}_m)\}|K^+(p)\rangle$$

2pt-function

$$C_2(\tau, \mathrm{p}) = \int d^3x\, e^{i\mathrm{p}\cdot x} \langle 0|\mathcal{O}_K(\tau, x)\, \mathcal{O}_K^\dagger(0,0)|0\rangle$$

$$\mathrm{p} = \mathrm{p_e} + \mathrm{p_m}$$
$$\mathrm{q} = (\mathrm{p_e} - \mathrm{p_m})/2$$
$$\tau = \tau_e - \tau_m$$

$$R^{\mu\nu}(\tau; \mathrm{q}, \mathrm{p}) \sim C_3^{\mu\nu}(\tau_e, \tau_m; \mathrm{p_e}, \mathrm{p_m}) \frac{2\, E_K(\mathrm{p})}{Z_K(\mathrm{p})} e^{-E_K(\mathrm{p})(\tau_e + \tau_m)/2}$$



LCDA(x)

Quasi-DA / LaMET

Xiangdong Ji,
Phys.Rev.Lett. 110 (2013)

Pseudo-DA

Anatoly Radyushkin,
Phys.Lett. B767 (2017)

Moments

Braun–Müller approach

Two currents separated by space-like distance
V. M. Braun and D. Müller,
Eur. Phys. J. C 55, 349 (2008).

- ◆ Constructing the full x-dependence is difficult near the end points.
- ◆ Moments provide observables that do not require constructing the full x-dependence.

Effect of Moments on the Shape of the LCDA

◆ Gegenbauer OPE:

$$\phi_K(\xi,\mu^2) = \frac{3}{4}(1-\xi^2)\sum_{n=0}^{\infty} \phi_n(\mu^2)\, C_n^{3/2}(\xi)$$

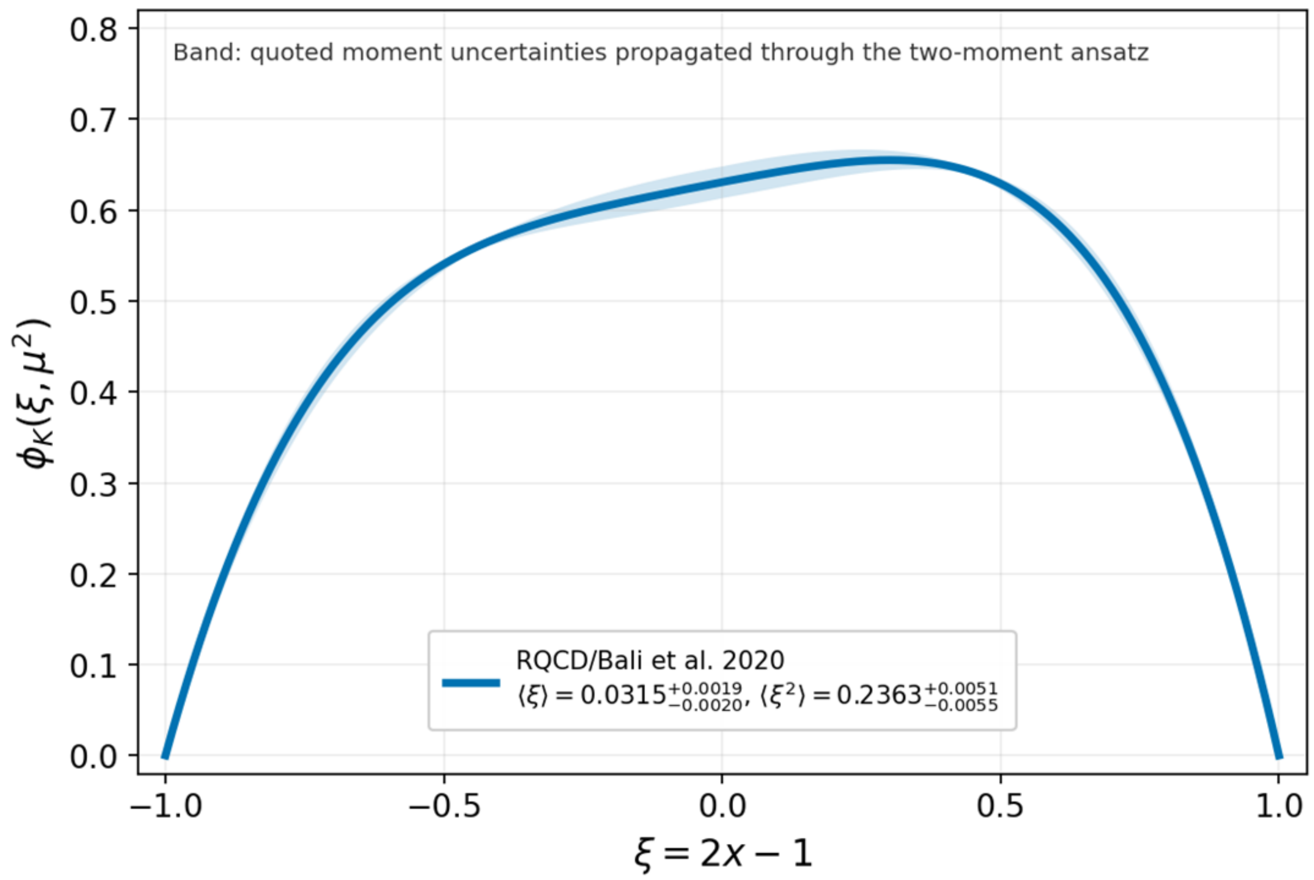
Gegenbauer moments *Gegenbauer polynomials*

$$\phi_0 = \langle \xi^0 \rangle = 1, \qquad \phi_1 = \frac{5}{3}\langle \xi \rangle, \qquad \phi_2 = \frac{7}{12}\left(5\langle \xi^2 \rangle - \langle \xi^0 \rangle\right)$$

Non zero odd moment:

the strange and light quarks share longitudinal momentum unequally

Kaon LCDA



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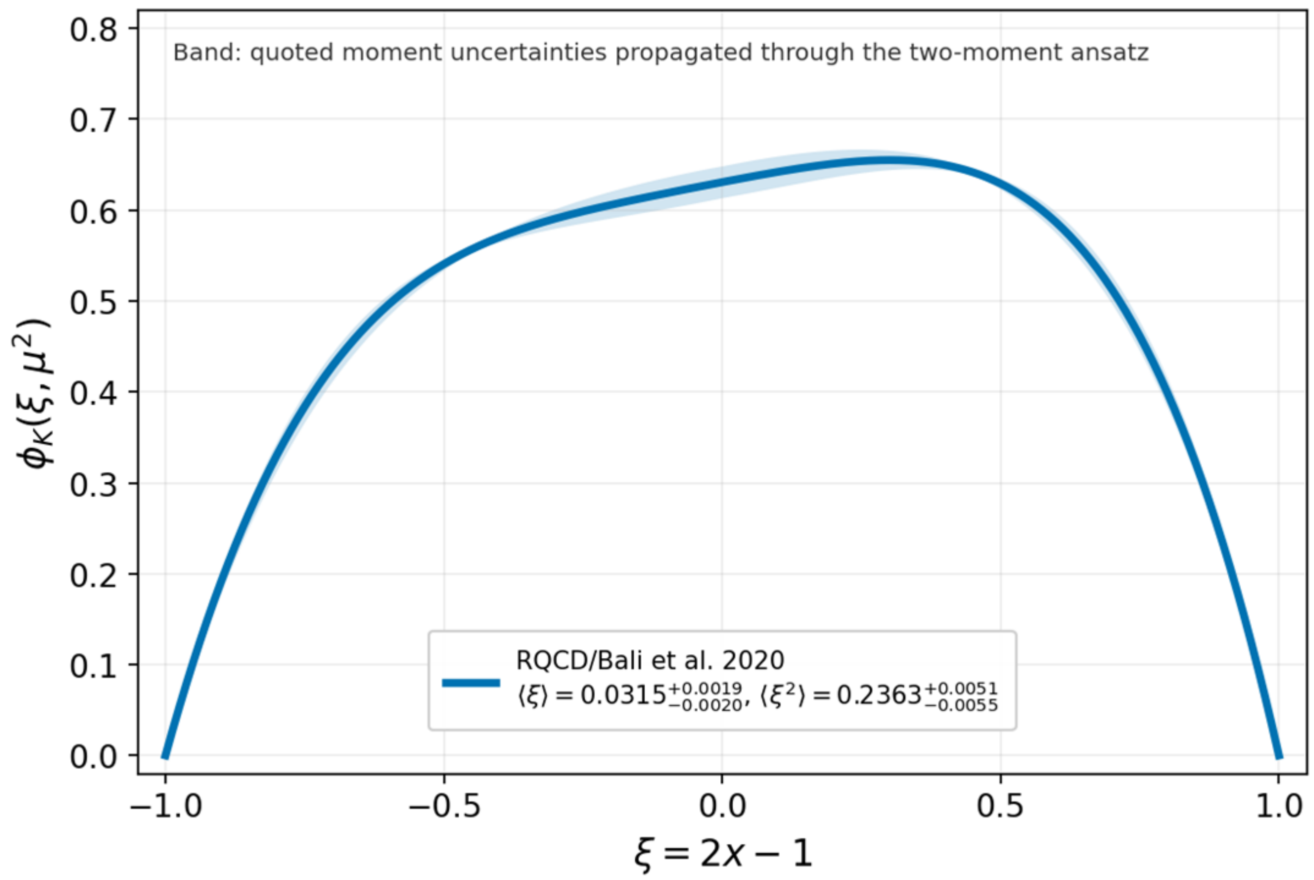
What about the third moment?

Kaon LCDA



HOPE Method for LCDA

$$V^{[\mu\,\nu]}(q,p) = \frac{-2\,i\,\epsilon^{\mu\nu\rho\sigma}\,q_\rho\,p_\sigma}{\tilde{Q}^2} f_K \sum_{n=0}^{\infty} C_W^{(n)}(\tilde{Q}^2,\mu^2,m_\Psi)\langle\xi^n\rangle\left(\frac{\tilde{\omega}}{2}\right)^n$$

$$\tilde{\omega} = (2\,q\cdot p)/\tilde{Q}^2$$
$$\tilde{Q}^2 = -q^2 + m_\Psi^2$$

$$q \to -q$$

$$\left(\frac{\tilde{\omega}}{2}\right)^{2n+1} \to -\left(\frac{\tilde{\omega}}{2}\right)^{2n+1}$$

$$\left(\frac{\tilde{\omega}}{2}\right)^{2n} \to \left(\frac{\tilde{\omega}}{2}\right)^{2n}$$

◆ Separated by **even** and **odd** Mellin moments

- Antisymmetric (*q → −q*)

$$V_{even}^{[\mu\,\nu]}(q,p) = \frac{1}{2}\left(V^{[\mu\,\nu]}(q,p) - V^{[\mu\,\nu]}(-q,p)\right)$$

– Even Mellin Moments

- Symmetric (*q → −q*)

$$V_{odd}^{[\mu\,\nu]}(q,p) = \frac{1}{2}\left(V^{[\mu\,\nu]}(q,p) + V^{[\mu\,\nu]}(-q,p)\right)$$

– Odd Mellin Moments

HOPE Method for Kaon LCDA

$$V^{[\mu\,\nu]}(q,p) = \frac{-2\,i\,\epsilon^{\mu\nu\rho\sigma}\,q_\rho\,p_\sigma}{\tilde{Q}^2} f_K \sum_{n=0}^{\infty} C_W^{(n)}(\tilde{Q}^2,\mu^2,m_\Psi)\langle\xi^n\rangle\left(\frac{\tilde{\omega}}{2}\right)^n$$

$$\tilde{\omega} = (2\,q\cdot p)/\tilde{Q}^2$$
$$\tilde{Q}^2 = -q^2 + m_\Psi^2$$

Fitting parameters in each channel:

	Imag part	Real part
$V_{even}^{[\mu\,\nu]}$	f_K 、 m_Ψ	$\langle\xi^2\rangle$
$V_{odd}^{[\mu\,\nu]}$	$\langle\xi^1\rangle$ 、 $\langle\xi^3\rangle$	$\langle\xi^1\rangle$ 、 $\langle\xi^3\rangle$

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$V^{[\mu\,\nu]}_{odd}$	$\langle\xi^1\rangle$ 、 $\langle\xi^3\rangle$	$\langle\xi^1\rangle$ 、 $\langle\xi^3\rangle$

The n=0 term does not have a real component.

Special kinematics chosen

$$i\,\epsilon^{\mu\nu\rho\sigma}\,q_\rho\,p_\sigma = i(q_0p_3 - p_0q_3) = -q_4p_3 - iE_kq_3$$

chose $\mathrm{p} = (1,0,0) \rightarrow \mathrm{p}_3 = 0$

$$i\,\epsilon^{\mu\nu\rho\sigma}\,q_\rho\,p_\sigma = -iE_kq_3$$ pure imaginary

Fourier transform of HOPE

$$\boxed{V^{[\mu\,\nu]}(q,p)} = \int_{-\infty}^{\infty} d\tau\, e^{i\tau q_4} \boxed{\mathrm{R}^{[\mu\,\nu]}(\tau;\mathrm{q},\mathrm{p})}$$

HOPE formula

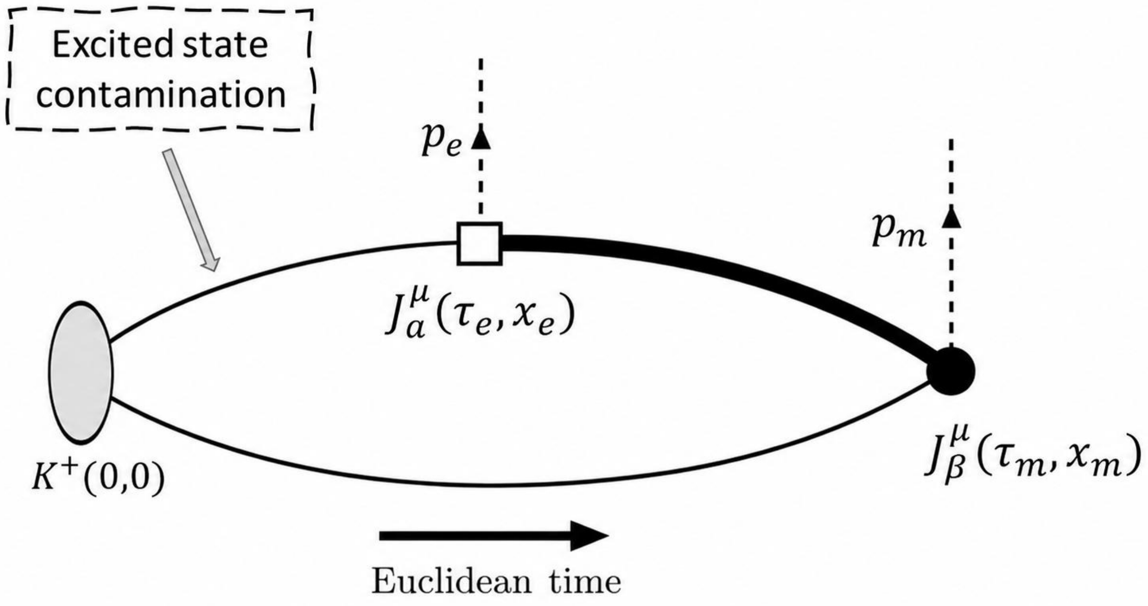
Construct ratio from correlators

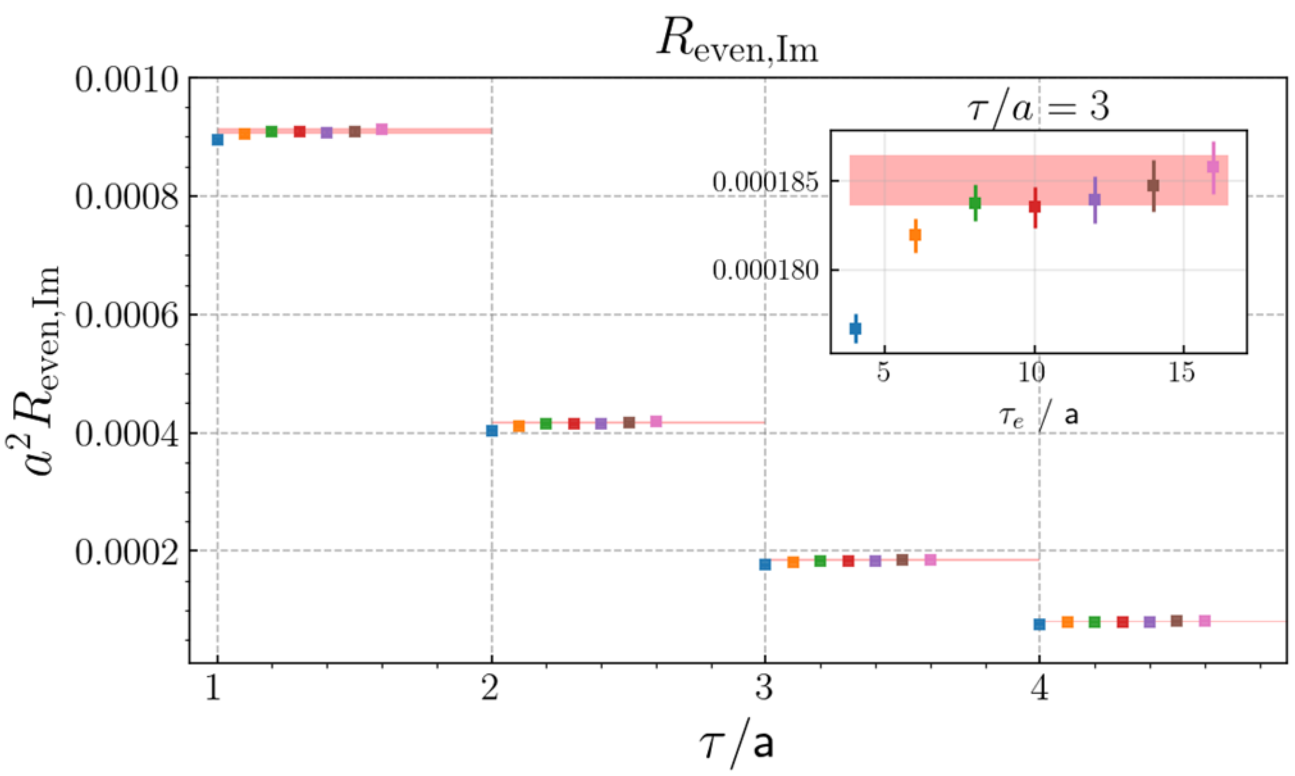
Fourier transform of HOPE formula

$$R^{[\mu\,\nu]}(\tau;\mathrm{q},\mathrm{p})$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dq_4 e^{-i\tau q_4} \frac{-2\,i\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{q}_\rho\,\mathrm{p}_\sigma}{\tilde{Q}^2} f_K \sum_{n=0}^{\infty} C_W^{(n)}(\tilde{Q}^2,\mu^2,m_\Psi)\langle\xi^n\rangle\left(\frac{\tilde{\omega}}{2}\right)^n$$

→ numerical Fourier transform

Excited state contamination





$$\mathrm{p} = \mathrm{p_e} + \mathrm{p_m}$$
$$\mathrm{q} = (\mathrm{p_e} - \mathrm{p_m})/2$$
$$\tau = \tau_e - \tau_m$$

fitting form:

$$R_{even,Im}(\tau_e, \tau_m; \mathrm{p_e}, \mathrm{p_m}) = A\, e^{-B\tau_e} + R_{even,Im}^{\tau_e \to \infty}$$

$A, B$and $R_{even,Im}^{\tau_e \to \infty}$ are the fit parameters

Resources

the Fugaku project id: hp220312, hp230466.

HPCI High Performance Computing Infrastructure

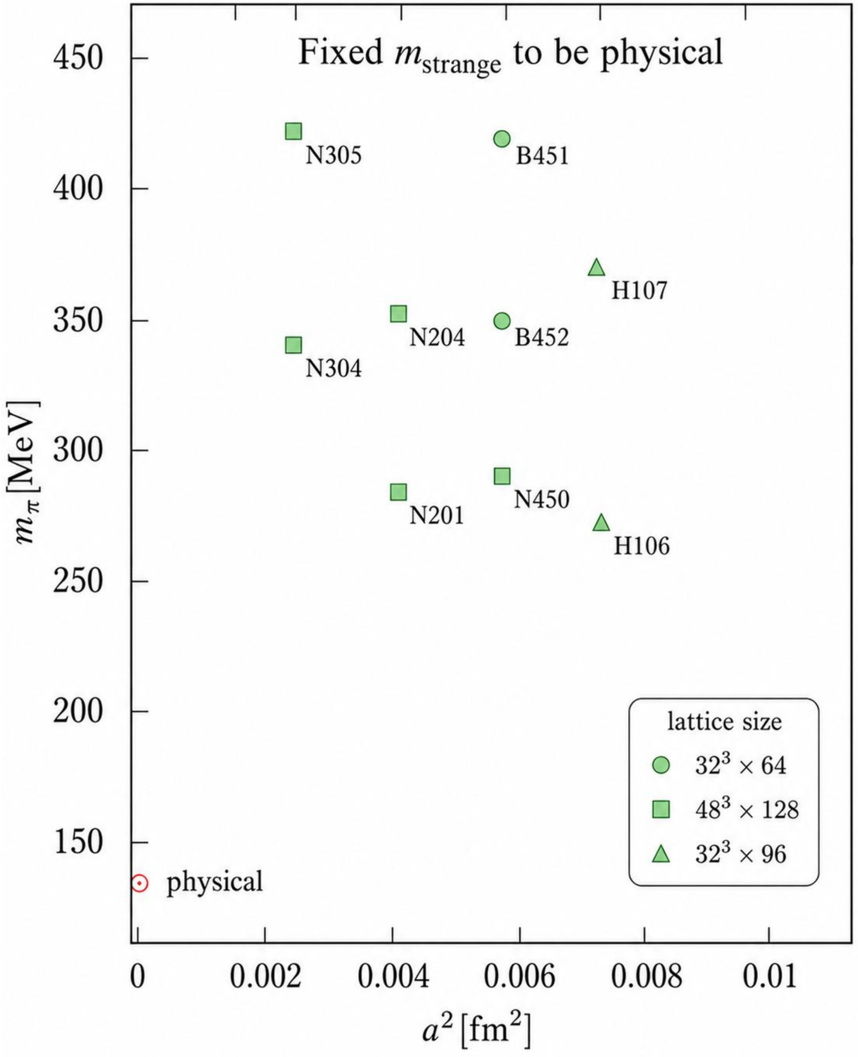
RIKEN R-CCS RIKEN Center for Computational Science

DiCOS
Distributed Cloud Operating System

Academia Sinica

BRIDGE++

S. Ueda et al., J.Phys.Conf.Ser. 523 (2014) 012046

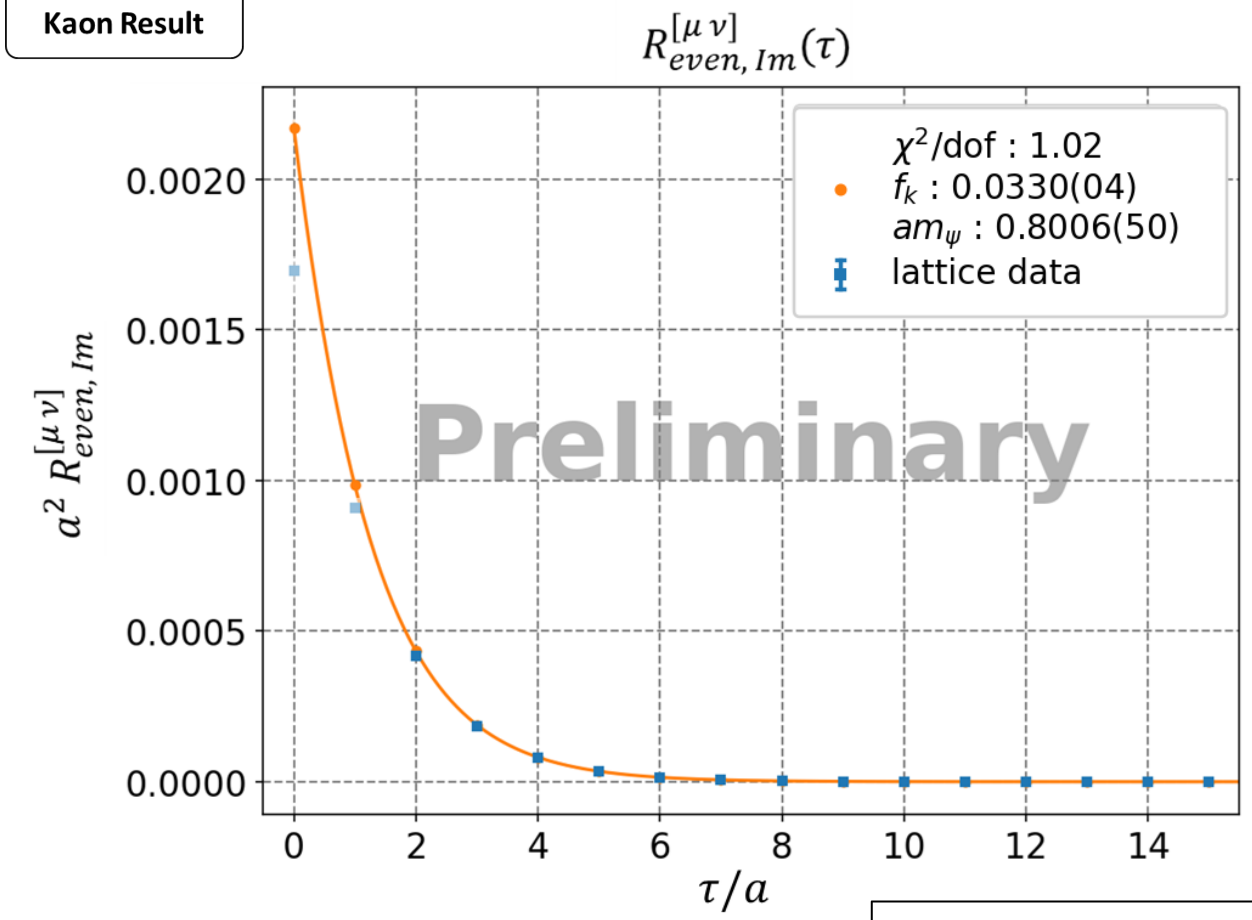


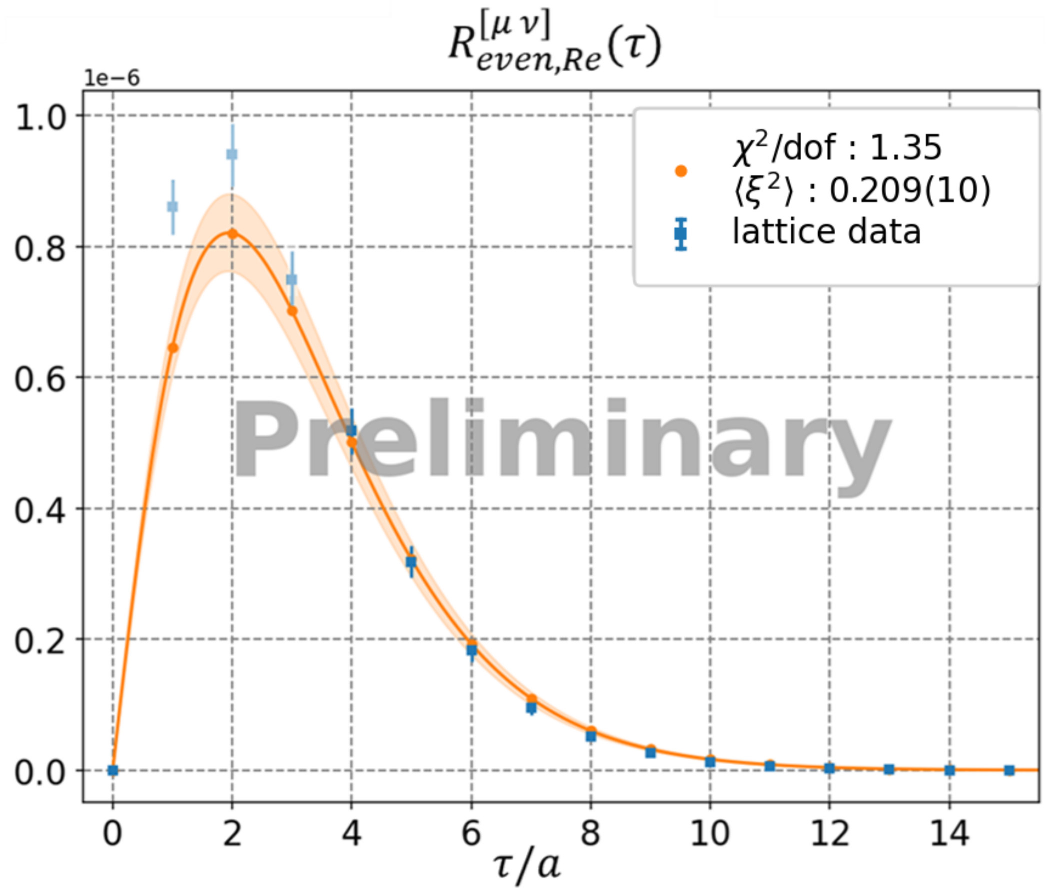
CLS ensembles

Example HOPE Fit

$(L/a)^3$ *T/a	(a) (fm)	Configurations Used	Sources/Config	M_π(MeV)	M_k(MeV)	$\|p\|$(MeV)
32^3 *64	0.0750	400	16	352.8(1.2)	551.1(0.9)	517

Kaon Result





data show significant lattice artifacts

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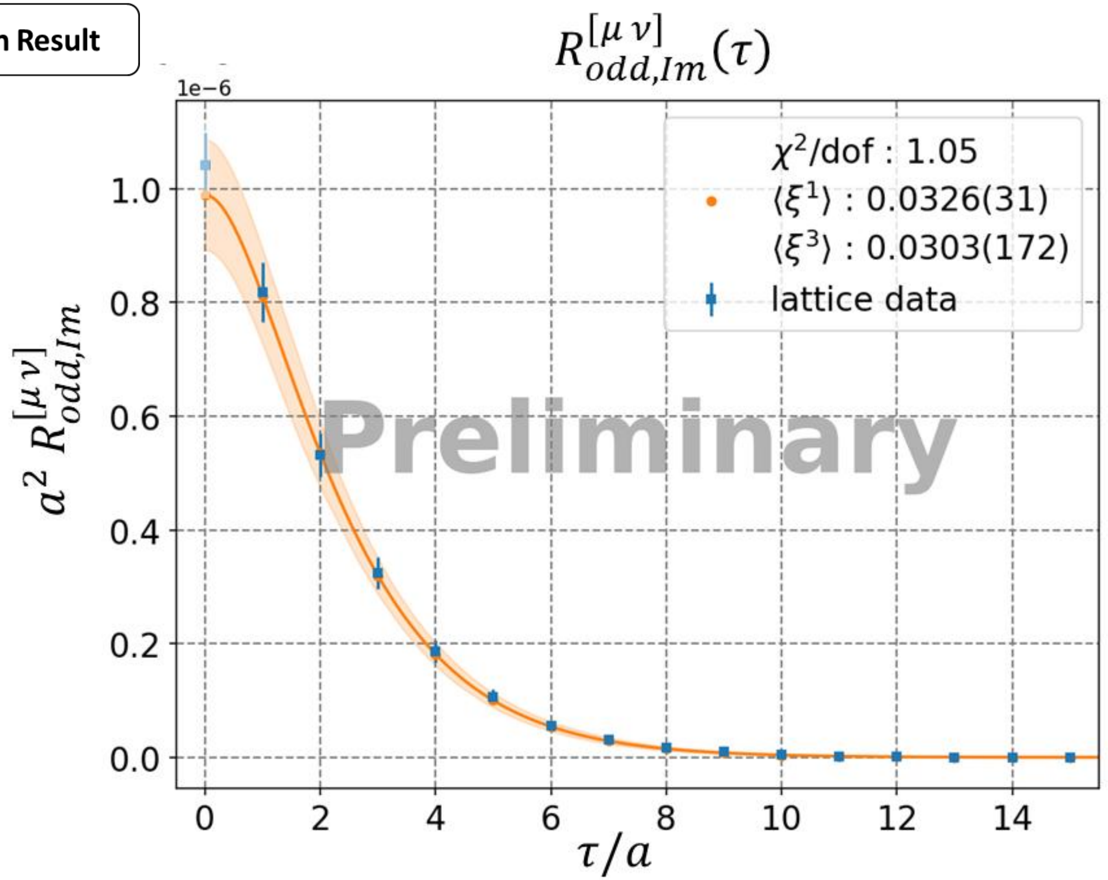

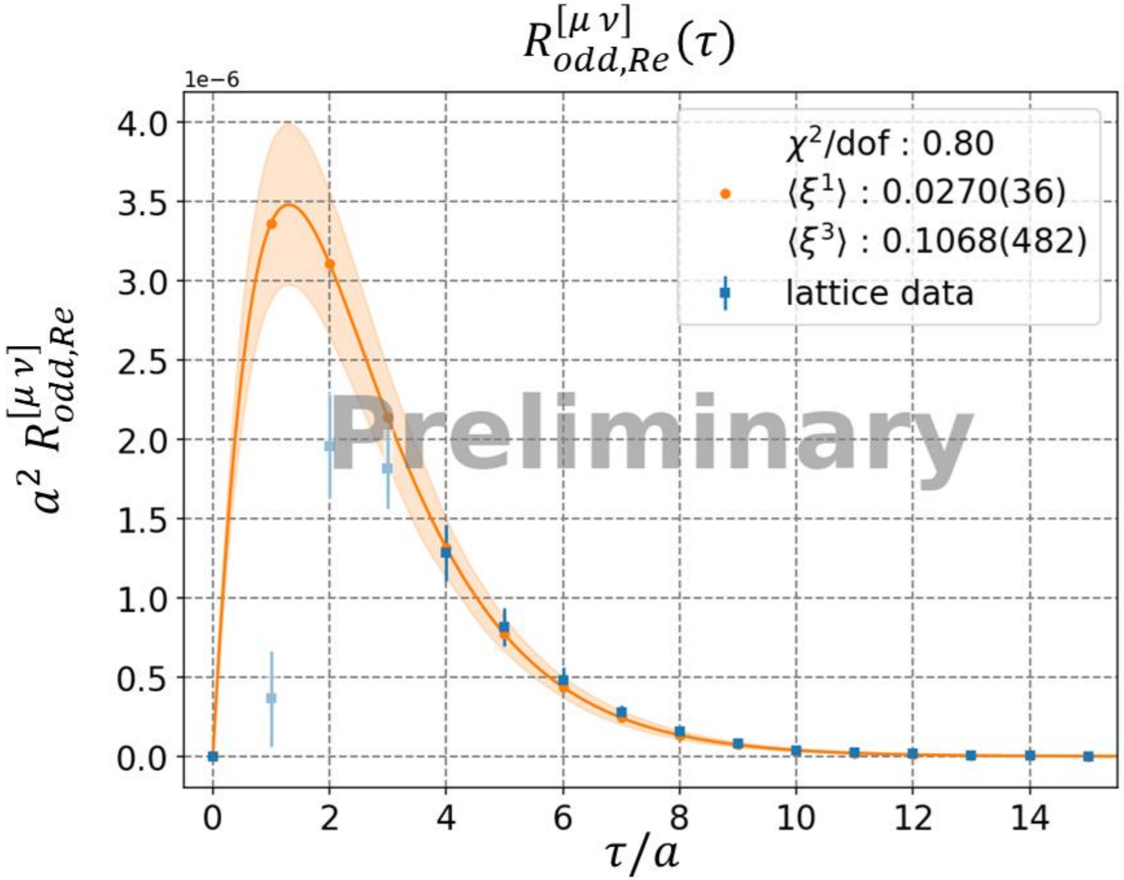

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Chiral–Continuum–Heavy-mass Extrapolation

$$\bar{m}^2 = \frac{2m_K^2 + m_\pi^2}{3} \qquad \delta m^2 = m_K^2 - m_\pi^2$$

odd moments vanish as $\delta m^2 \to 0$

Even moments

$$\langle \xi^{(2n)} \rangle (a, m_\Psi, m) = X_{even}^{(2n)} (\bar{m}^2, \delta m^2) \left[1 + D_{even}^{(2n)} (a, m_\Psi)\right]$$

$$X_{even}^{(2n)} = \langle \xi^{(2n)} \rangle_0 + \bar{A}^{(2n)} \bar{m}^2 + \delta A^{(2n)} \delta m^2$$

9 fit parameters

Odd moments

$$\langle \xi^{(2n+1)} \rangle (a, m_\Psi, m) = \delta A^{(2n+1)} \delta m^2 \left[1 + D_{odd}^{(2n+1)} (a, m_\Psi)\right]$$

SU(3)-symmetric limit:

$$\delta m^2 = 0 \Rightarrow \langle \xi^{(2n+1)} \rangle = 0$$

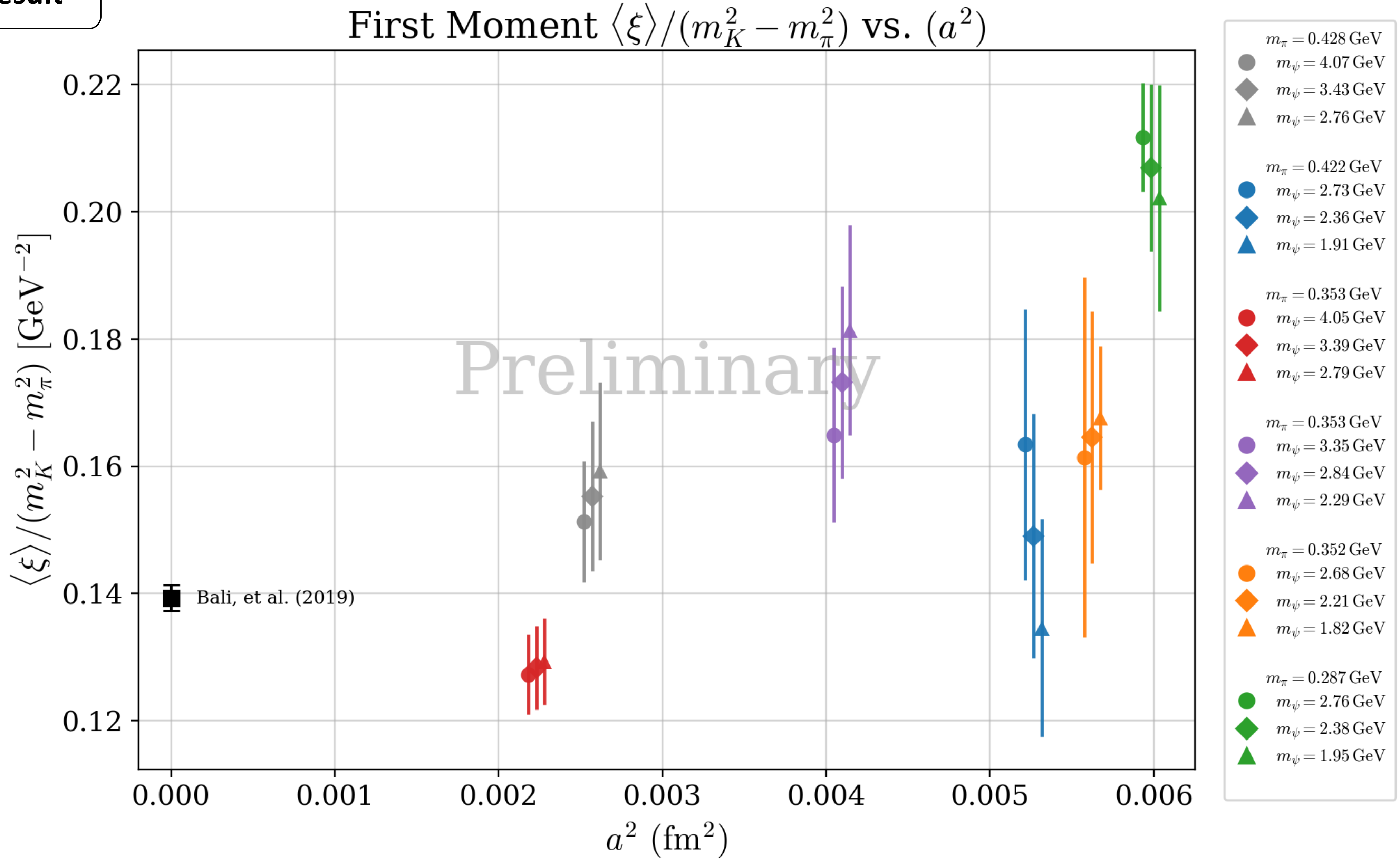
7 fit parameters

$$D^{(n)} = d_0^{(n)} a^2 + \bar{d}^{(n)} \bar{m}^2 a^2 + \delta d^{(n)} \delta m^2 a^2 + d_1^{(n)} a^2 m_\Psi + d_2^{(n)} a^2 m_\Psi^2 + \frac{h_0^{(n)}}{m_\Psi}$$

$$\boxed{a \to 0, \quad m_\Psi \to \infty, \quad a m_\Psi \to 0 \quad \Longrightarrow \quad D^{(n)} \to 0}$$

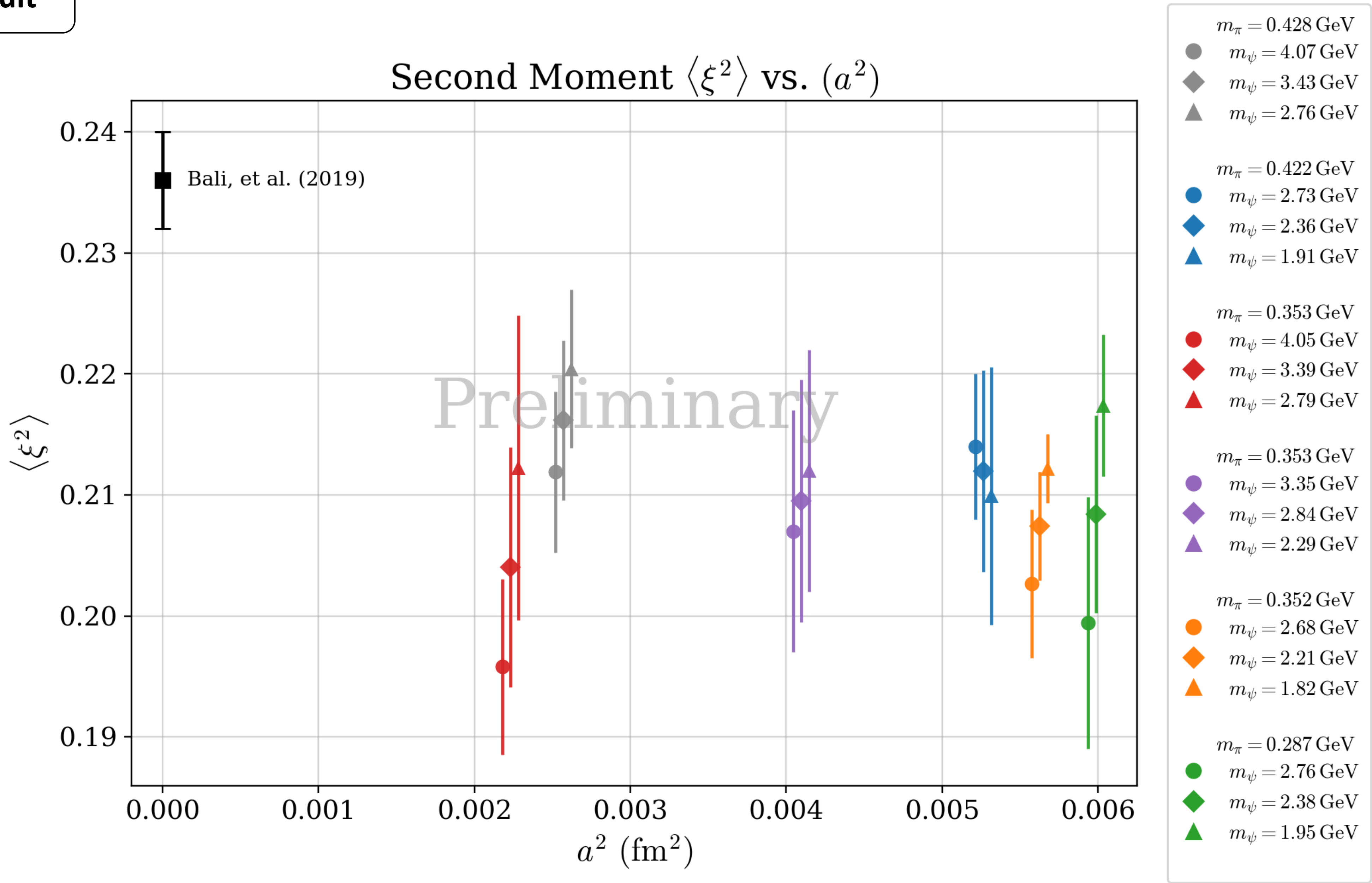
Even moments allow generic chiral dependence; odd moments are constrained to vanish in the SU(3)-symmetric limit.

Kaon Result



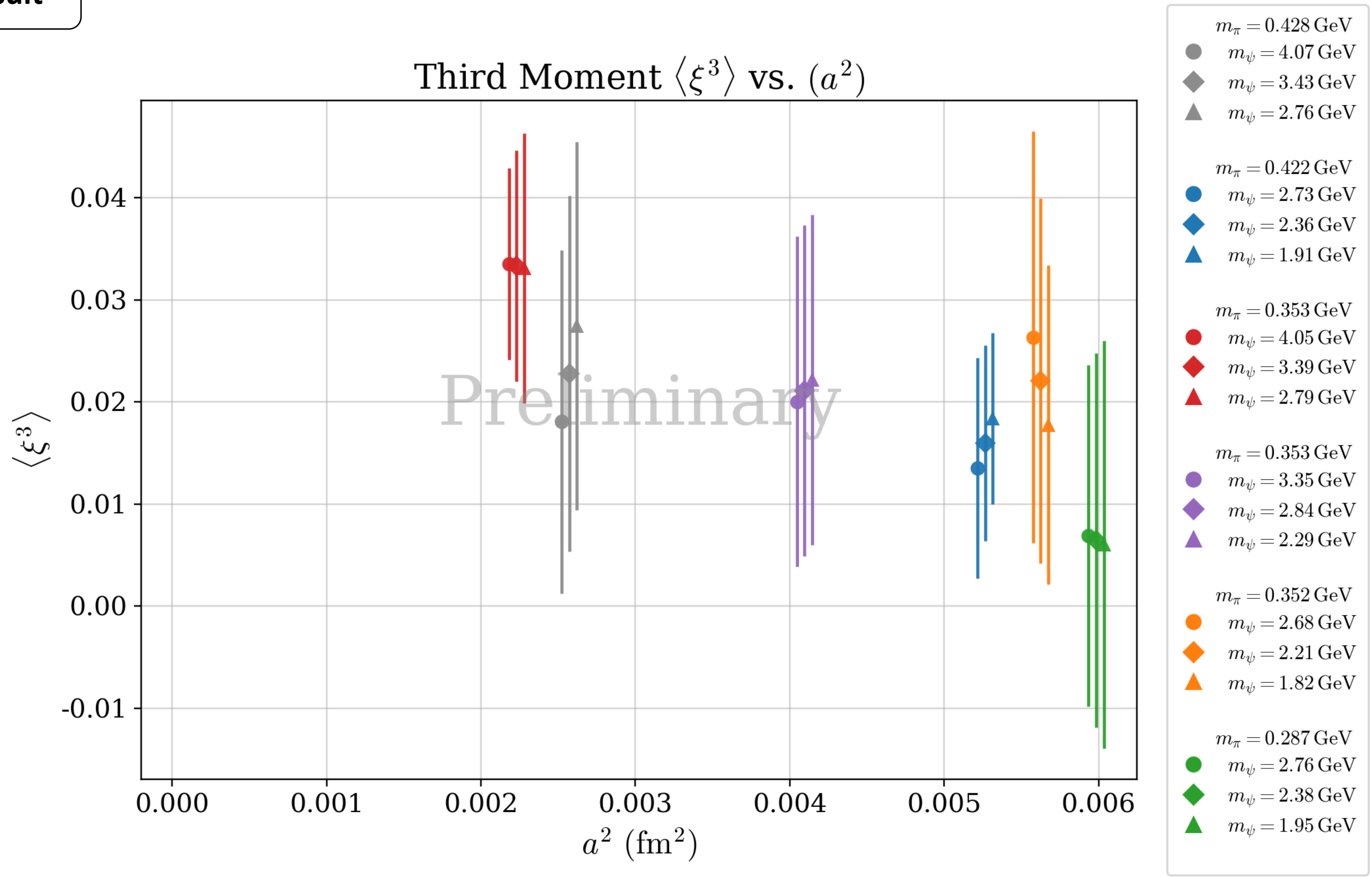
First Moment $\langle\xi\rangle/(m_K^2 - m_\pi^2)$ vs. (a^2)
$\langle\xi\rangle/(m_K^2 - m_\pi^2)$ [GeV^{-2}]
0.22
0.20
0.18
0.16
0.14
0.12
0.000
0.001
0.002
0.003
0.004
0.005
0.006
a^2 (fm^2)
Preliminary
Bali, et al. (2019)
$m_\pi = 0.428$ GeV
$m_\psi = 4.07$ GeV
$m_\psi = 3.43$ GeV
$m_\psi = 2.76$ GeV
$m_\pi = 0.422$ GeV
$m_\psi = 2.73$ GeV
$m_\psi = 2.36$ GeV
$m_\psi = 1.91$ GeV
$m_\pi = 0.353$ GeV
$m_\psi = 4.05$ GeV
$m_\psi = 3.39$ GeV
$m_\psi = 2.79$ GeV
$m_\pi = 0.353$ GeV
$m_\psi = 3.35$ GeV
$m_\psi = 2.84$ GeV
$m_\psi = 2.29$ GeV
$m_\pi = 0.352$ GeV
$m_\psi = 2.68$ GeV
$m_\psi = 2.21$ GeV
$m_\psi = 1.82$ GeV
$m_\pi = 0.287$ GeV
$m_\psi = 2.76$ GeV
$m_\psi = 2.38$ GeV
$m_\psi = 1.95$ GeV

Kaon Result



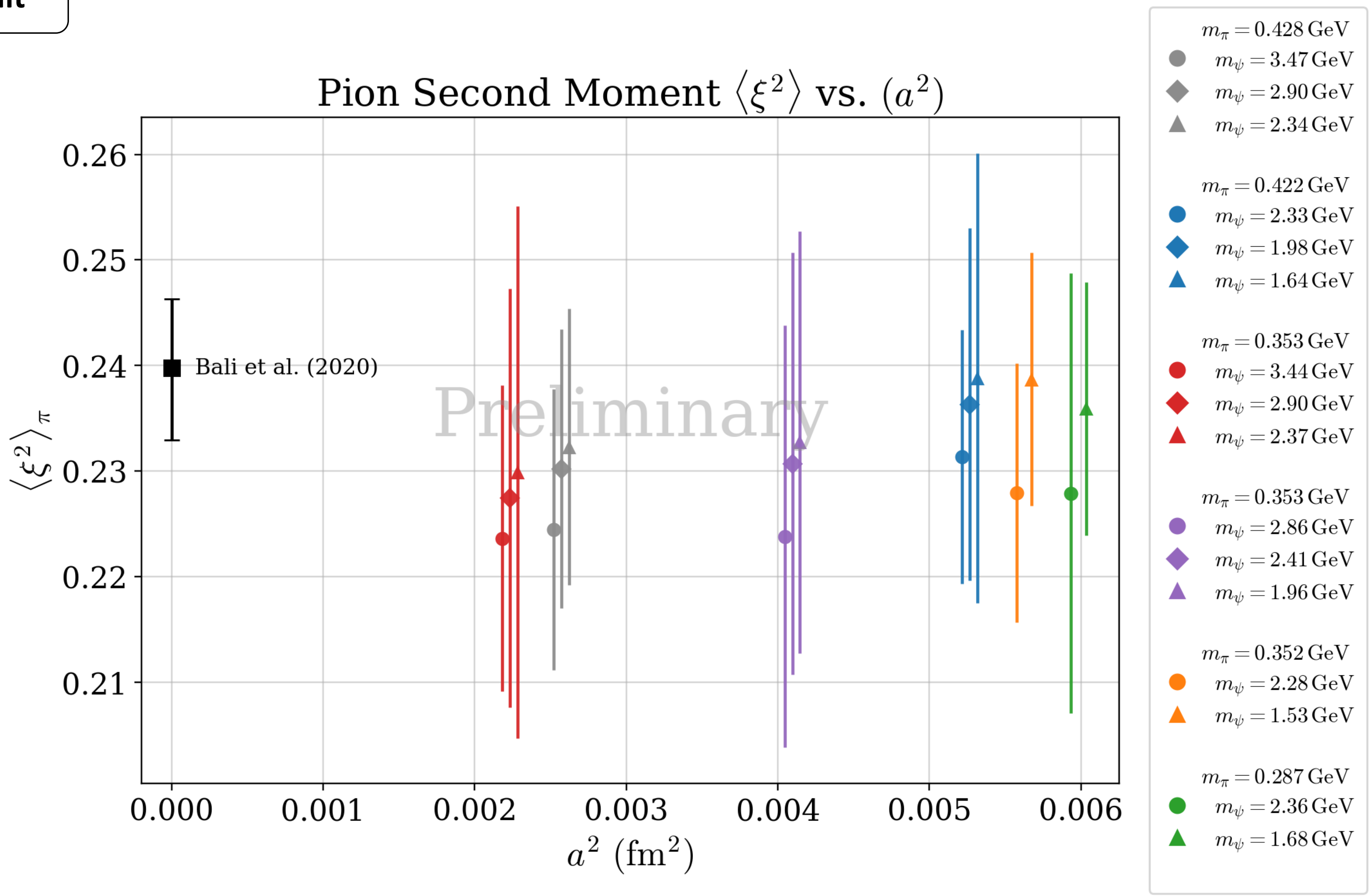
Second Moment $\langle\xi^2\rangle$ vs. (a^2)
Bali, et al. (2019)
Preliminary
$\langle\xi^2\rangle$
0.24
0.23
0.22
0.21
0.20
0.19
0.000
0.001
0.002
0.003
0.004
0.005
0.006
a^2 (fm^2)
$m_\pi = 0.428$ GeV
$m_\psi = 4.07$ GeV
$m_\psi = 3.43$ GeV
$m_\psi = 2.76$ GeV
$m_\pi = 0.422$ GeV
$m_\psi = 2.73$ GeV
$m_\psi = 2.36$ GeV
$m_\psi = 1.91$ GeV
$m_\pi = 0.353$ GeV
$m_\psi = 4.05$ GeV
$m_\psi = 3.39$ GeV
$m_\psi = 2.79$ GeV
$m_\pi = 0.353$ GeV
$m_\psi = 3.35$ GeV
$m_\psi = 2.84$ GeV
$m_\psi = 2.29$ GeV
$m_\pi = 0.352$ GeV
$m_\psi = 2.68$ GeV
$m_\psi = 2.21$ GeV
$m_\psi = 1.82$ GeV
$m_\pi = 0.287$ GeV
$m_\psi = 2.76$ GeV
$m_\psi = 2.38$ GeV
$m_\psi = 1.95$ GeV

Kaon Result



Third Moment $\langle \xi^3 \rangle$ vs. (a^2)
Preliminary
$\langle \xi^3 \rangle$
0.04
0.03
0.02
0.01
0.00
-0.01
0.000
0.001
0.002
0.003
0.004
0.005
0.006
a^2 (fm^2)
$m_\pi = 0.428$ GeV
$m_\psi = 4.07$ GeV
$m_\psi = 3.43$ GeV
$m_\psi = 2.76$ GeV
$m_\pi = 0.422$ GeV
$m_\psi = 2.73$ GeV
$m_\psi = 2.36$ GeV
$m_\psi = 1.91$ GeV
$m_\pi = 0.353$ GeV
$m_\psi = 4.05$ GeV
$m_\psi = 3.39$ GeV
$m_\psi = 2.79$ GeV
$m_\pi = 0.353$ GeV
$m_\psi = 3.35$ GeV
$m_\psi = 2.84$ GeV
$m_\psi = 2.29$ GeV
$m_\pi = 0.352$ GeV
$m_\psi = 2.68$ GeV
$m_\psi = 2.21$ GeV
$m_\psi = 1.82$ GeV
$m_\pi = 0.287$ GeV
$m_\psi = 2.76$ GeV
$m_\psi = 2.38$ GeV
$m_\psi = 1.95$ GeV

Pion Result



Pion Second Moment $\langle \xi^2 \rangle$ vs. (a^2)
0.26
0.25
0.24
0.23
0.22
0.21
$\langle \xi^2 \rangle_\pi$
Bali et al. (2020)
Preliminary
0.000
0.001
0.002
0.003
0.004
0.005
0.006
a^2 (fm^2)
$m_\pi = 0.428$ GeV
$m_\psi = 3.47$ GeV
$m_\psi = 2.90$ GeV
$m_\psi = 2.34$ GeV
$m_\pi = 0.422$ GeV
$m_\psi = 2.33$ GeV
$m_\psi = 1.98$ GeV
$m_\psi = 1.64$ GeV
$m_\pi = 0.353$ GeV
$m_\psi = 3.44$ GeV
$m_\psi = 2.90$ GeV
$m_\psi = 2.37$ GeV
$m_\pi = 0.353$ GeV
$m_\psi = 2.86$ GeV
$m_\psi = 2.41$ GeV
$m_\psi = 1.96$ GeV
$m_\pi = 0.352$ GeV
$m_\psi = 2.28$ GeV
$m_\psi = 1.53$ GeV
$m_\pi = 0.287$ GeV
$m_\psi = 2.36$ GeV
$m_\psi = 1.68$ GeV

Summary

Summary

- We use the **Heavy-Quark Operator Product Expansion (HOPE)** to access the **moments of pion and kaon LCDAs** from **Lattice QCD** calculations.

First, Second and Third Moments of the Kaon **(dynamical)**

Second and Fourth Moments of the Pion **(dynamical)**

Future Work

- Incorporate **more lattice data (different heavy quark mass)** to improve **Chiral--Continuum--Heavy-Mass Extrapolation.**

THE HOPE COLLABORATION

HOPE Method for LCDA

$$V^{[\mu\,\nu]}(q,p) = \frac{-2\,i\,\epsilon^{\mu\nu\rho\sigma}\,q_\rho\,p_\sigma}{\tilde{Q}^2} f_K \sum_{n=0}^{\infty} C_W^{(n)}(\tilde{Q}^2,\mu^2,m_\Psi)\langle\xi^n\rangle\left(\frac{\tilde{\omega}}{2}\right)^n$$

$$\tilde{\omega} = (2\,q\cdot p)/\tilde{Q}^2$$
$$\tilde{Q}^2 = -q^2 + m_\Psi^2$$

Fitting parameters in separate channels

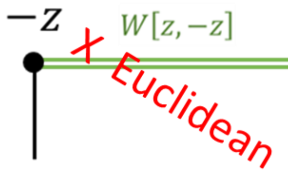
Pion

	Imag part	Real part
$V^{[\mu\,\nu]}_{even,\pi}$	f_π , m_Ψ	$\langle\xi^2\rangle_\pi$
$V^{[\mu\,\nu]}_{odd,\pi}$	vanishes (isospin limit)	vanishes (isospin limit)

Kaon

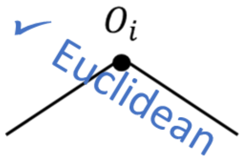
	Imag part	Real part
$V^{[\mu\,\nu]}_{even,K}$	f_K , m_Ψ	$\langle\xi^2\rangle_K$
$V^{[\mu\,\nu]}_{odd,K}$	$\langle\xi^1\rangle_K$, $\langle\xi^3\rangle_K$	$\langle\xi^1\rangle_K$, $\langle\xi^3\rangle_K$

LCDA



Light-Cone OPE

$$\sum_{i=0}^{\infty} C_W^{(i)}(z)$$



operators mixing & power divergence

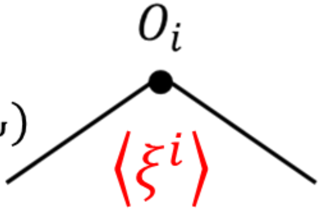
HOPE

fictitious valence heavy quark

Short-distance
& heavy scale

$\Lambda_{\mathrm{QCD}} \ll m_\Psi \sim \sqrt{q^2} \ll \frac{1}{\hat{a}}$

$$\sum_{i=0}^{\infty} C_{HOPE}^{(i)}(z, m_\Psi)$$



✓ higher-twist effects are suppressed by $1/\tilde{Q}^n$

$$t^{\mu\nu}_{\Psi,\psi} = \overline{\psi}\gamma^\mu \frac{-i\left(i\overleftrightarrow{\not{D}} + \not{q}\right) + m_\Psi}{(i\overleftrightarrow{D} + q)^2 + m_\Psi^2}\gamma^\nu\psi$$

Taylor expansion

$$\frac{-i\left(i\overleftrightarrow{\not{D}} + \not{q}\right) + m_\Psi}{(i\overleftrightarrow{D} + q)^2 + m_\Psi^2} = -\frac{-i\left(i\overleftrightarrow{\not{D}} + \not{q}\right) + m_\Psi}{Q^2 + \overleftrightarrow{D}^2 - m_\Psi^2}\sum_{n=0}^{\infty}\left(\frac{-2i\, q\cdot\overleftrightarrow{D}}{Q^2 + \overleftrightarrow{D}^2 - m_\Psi^2}\right)^n$$

$$G_3^{\mu\nu}(x,y) = \langle\Omega|\mathcal{T}\{J_A^\mu(x)J_A^\nu(y)\mathcal{O}_\pi^\dagger(0)\}|\Omega\rangle$$

$$J_A^\mu = Z_A^{(0)}(1+\tilde{b}_A a\tilde{m}_{ij})\left[\overline{\psi}\gamma_\mu\gamma_5\Psi + ac_A\partial_\mu\overline{\psi}\gamma_5\Psi - a\frac{c'_A}{4}\left(\overline{\psi}\gamma_\mu\gamma_5(\overrightarrow{D}+m_\Psi)\Psi - \overline{\psi}(\overleftarrow{D}+m_\psi)\gamma_\mu\gamma_5\Psi\right) + (\psi\leftrightarrow\Psi)\right]$$

$$\begin{aligned} G_3^{\mu\nu}(x,y) = Z^2(a)Z_\pi\langle\Omega|T\Big\{ & \Big(\overline{\Psi}^{(0)}(x)\gamma^\mu\gamma^5\psi^{(0)}(x) + \overline{\psi}^{(0)}(x)\gamma^\mu\gamma^5\Psi^{(0)}(x) \\ & + ac_A\partial^\mu\Big\{\overline{\Psi}^{(0)}(x)\gamma^5\psi^{(0)}(x)\Big\} + ac_A\partial^\mu\Big\{\overline{\psi}^{(0)}(x)\gamma^5\Psi^{(0)}(x)\Big\}\Big) \\ & \times\Big(\overline{\Psi}^{(0)}(y)\gamma^\nu\gamma^5\psi^{(0)}(y) + \overline{\psi}^{(0)}(y)\gamma^\nu\gamma^5\Psi^{(0)}(y) \\ & + ac_A\partial^\nu\Big\{\overline{\Psi}^{(0)}(y)\gamma^5\psi^{(0)}(y)\Big\} + ac_A\partial^\nu\Big\{\overline{\psi}^{(0)}(y)\gamma^5\Psi^{(0)}(y)\Big\}\Big) \\ & \times[\overline{\psi}^{(0)}(0)\gamma^5\psi^{(0)}(0)]^\dagger\Big\}|\Omega\rangle + \mathcal{O}(a^2), \end{aligned}$$

$$\frac{ac'_A}{4}\sum_{\mathbf{x}_e,\mathbf{x}_m,x}\left[D_u^{-1}(0|x_m)_{da}^{\eta\alpha}\right]\left[D_d^{-1}(x_e|0)_{cd}^{\varepsilon\zeta}\right]\left[D_\Psi^{-1}(x|x_e)_{bc}^{\gamma\delta}\right]\left[D_\Psi(x_m|x)_{ab}^{\beta\gamma}\right]$$
$$\left(\gamma^\nu\gamma^5\right)^{\alpha\beta}\left(\gamma^\mu\gamma^5\right)^{\delta\varepsilon}\left(\gamma^5\right)^{\zeta\eta}e^{i\mathbf{p}_e\cdot\mathbf{x}_e}e^{i\mathbf{p}_m\cdot\mathbf{x}_m}.$$

$$\sum_x D_\Psi(x_m|x)_{ab}^{\beta\gamma}D_\Psi^{-1}(x|x_e)_{bc}^{\gamma\delta} = \delta^{(4)}(x_m-x_e)\delta^{\beta\delta}\delta_{ac}$$

- The $ac_A\partial^\mu P$ insertion does **not** generate this antisymmetric $\epsilon^{\mu\nu\alpha\beta}$ structure,

- The c'_A terms reduce to contact-type contributions and do not affect the nonzero-τ matrix element used for the moment extraction.

$$V_q^{\mu\nu(0)}(q,p) = \bar{v}(\frac{1}{2}(1-x_0)p,\downarrow)\left[\gamma^\mu\gamma_5\frac{i}{\not{q}+\frac{x_0\not{p}}{2}-m_\Psi}\gamma^\nu\gamma_5+\gamma^\nu\gamma_5\frac{i}{-\not{q}+\frac{x_0\not{p}}{2}-m_\Psi}\gamma^\mu\gamma_5\right]u(\frac{1}{2}(1+x_0)p,\uparrow)$$

$$\begin{aligned} V_q^{[\mu\nu](0)}(q,p,m_\Psi,x_0) &= \epsilon^{\mu\nu\rho\sigma}q_\rho\bar{v}\gamma_\sigma\gamma_5 u\left[\frac{1}{q^2+x_0 p\cdot q-m_\Psi^2}+\frac{1}{q^2-x_0 p\cdot q-m_\Psi^2}\right] \\ &= -\frac{2\epsilon^{\mu\nu\rho\sigma}q_\rho}{\tilde{Q}^2}\bar{v}\gamma_\sigma\gamma_5 u\sum_{\substack{n=0,\\ \text{even}}}^{\infty}\mathcal{F}_n^{(0)}(\tilde{\omega})\phi_n^{(0)}(x_0)\,, \end{aligned}$$

$$\mathcal{F}_n^{(0)}(\tilde{\omega}) = \frac{3}{2}\frac{\sqrt{\pi}(n+1)(n+2)n!}{2^{n+2}\Gamma\left(n+\frac{5}{2}\right)}\left(\frac{\tilde{\omega}}{2}\right)^n {}_2F_1\left(\frac{n+1}{2},\frac{n+2}{2};n+\frac{5}{2};\frac{\tilde{\omega}^2}{4}\right).$$

$$\mathcal{F}_n(\tilde{Q}^2,\mu^2,\tilde{\omega},\tau) = \mathcal{F}_n^{(0)}(\tilde{\omega})\left[1+\frac{\alpha_s C_F}{4\pi}\gamma_n^{(0)}\ln\frac{\mu^2}{\tilde{Q}^2}\right]+\frac{\alpha_s C_F}{4\pi}\mathcal{R}_n^{(1)}(\tilde{Q}^2,\mu^2,\tau,\tilde{\omega})+\mathcal{O}(\alpha_s^2)$$

$$\phi_n(x_0,\epsilon) = \phi_n^{(0)}(x_0)\left[1+\frac{\alpha_s C_F}{4\pi}\frac{\gamma_n^{(0)}}{\epsilon'}+\mathcal{O}(\alpha_s^2)\right]$$

Beyond leading logarithm, the conformal symmetry is broken, so the Gegenbauer moments start to mix.

1 High-dimensional operator

$$O_H \equiv O_{\{111\}} = \bar{\psi}\,\gamma_{\{1}\,D_1 D_{1\}}\,\psi - \text{traces}$$

$$d_H = [\bar{\psi}] + [\psi] + [D] + [D] = 3/2 + 3/2 + 1 + 1 = 5$$

second-moment twist-two operator

2 Lower-dimensional operator

$$O_L = \bar{\psi}\,\gamma_1\,\psi$$

$$d_L = [\bar{\psi}] + [\psi] = 3/2 + 3/2 = 3$$

$$O_{\{111\}} \in \mathbf{4}_1, \qquad O_L \in \mathbf{4}_1$$

same hypercubic irrep $\Rightarrow$ mixing allowed by W_4

3 Dimensional argument

$$O_H^{\text{ren}} = Z_{HH}\,O_H^{\text{lat}} + Z_{HL}\,O_L^{\text{lat}} + \cdots$$

$$[Z_{HL}] = d_H - d_L = 5 - 3 = 2$$

$$\Lambda_{\text{UV}} \sim 1/a$$

$$\boxed{Z_{HL} \propto \Lambda_{\text{UV}}^2 \sim 1/a^2}$$

$$O_{\{111\}}^{\text{ren}} = Z_{HH}\,O_{\{111\}}^{\text{lat}} + \frac{c(g_0)}{a^2}\,\bar{\psi}\,\gamma_1\,\psi + \cdots$$

$$\Lambda_H^{\text{lat}}(p,a) = \gamma_1\,p_1^2 + \frac{g_0^2}{16\pi^2}\left[A\,\log(a^2p^2)\,\gamma_1\,p_1^2 + \frac{B}{a^2}\gamma_1 + \cdots\right]$$

Because a dimension-5 operator mixes with a dimension-3 operator allowed by the hypercubic symmetry, the coefficient is power divergent rather than logarithmic.

In contrast:

$$O_{\{123\}} \in \mathbf{4}_2,$$

so it does not mix with the vector current ($\mathbf{4}_1$).