Continuum and chiral extrapolations in the lattice calculation of inclusive D_s-meson semileptonic decays

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Introduction

Current status

Final goal: lattice calculation of inclusive decays of the $B_{(s)}$ meson

Starting point: inclusive decays of the D_s meson

Steps so far:

- Methodology of inclusive decays on the lattice [Lattice 2022, arXiv:2211.16830]
- Finite-volume effects in inclusive decays [Lattice 2023, arXiv:2312.16442]
- Systematic effects in the lattice calculation [Lattice 2024, arXiv:2411.18058]
    - Truncation errors → revisited today

⇒ Summarized in arXiv:2504.03358

Today:

- Continuum and chiral extrapolations [arXiv:2604.22201]

Inclusive Decays - Continuum

Total decay rate [2305.14092, 2504.03358]

$$\Gamma \sim \int_0^{\boldsymbol{q}^2_{max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2} \sum_{l=0}^{2} \bar{X}^{(l)}(\boldsymbol{q}^2)$$

$\bar{X}^{(l)}(\boldsymbol{q}^2)$ integral over energy of hadronic final states

$$\bar{X}^{(l)}(\boldsymbol{q}^2) = \int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu}(\boldsymbol{q},\omega) K_{\mu\nu}^{(l)}(\boldsymbol{q},\omega)$$

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$\bar{X}^{(l)}(\boldsymbol{q}^2)$ integral over energy of hadronic final states

Kernel function

$$\bar{X}^{(l)}(\boldsymbol{q}^2) = \int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu}(\boldsymbol{q},\omega) K_{\mu\nu}^{(l)}(\boldsymbol{q},\omega)$$

$$k_{\mu\nu}^{(l)}(\boldsymbol{q},\omega)\theta(\omega_{\max} - \omega)$$

Analytically known

Step function

l-th power of ω and $\boldsymbol{q}^2$

Inclusive Decays - Continuum

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Kernel function

$$\bar{X}^{(l)}(\boldsymbol{q}^2) = \int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu}(\boldsymbol{q},\omega) K^{(l)}_{\mu\nu}(\boldsymbol{q},\omega)$$

Free choice: $0 < \omega_0 \leq \omega_{\text{min}}$

No state below ω_{min}

Fix $\omega_0 = 0.9\omega_{\text{min}}$

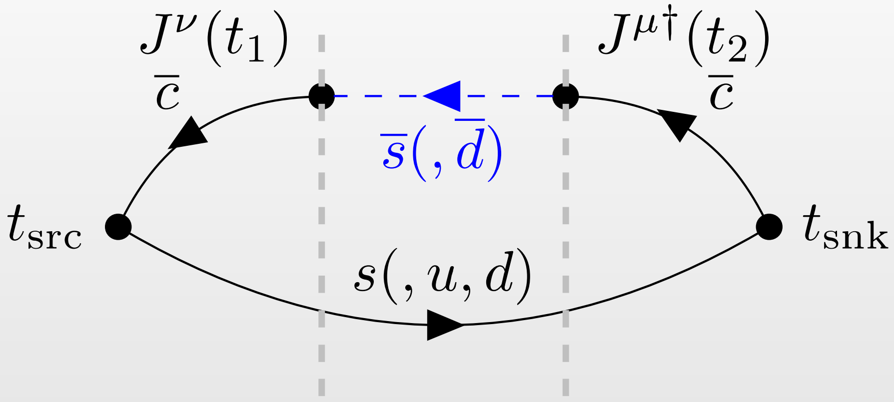
$$k^{(l)}_{\mu\nu}(\boldsymbol{q},\omega)\theta(\omega_{\text{max}} - \omega)$$

Analytically known

Step function

l-th power of ω and $\boldsymbol{q}^2$

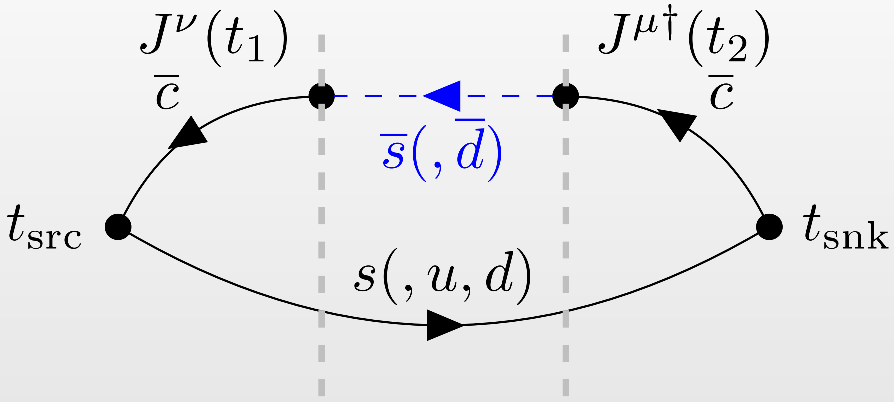
Inclusive decays – Lattice



- t_{src}, t_2, t_{snk} fixed
- $t_{src} \leq t_1 \leq t_2$
- $t = t_2 - t_1$

$$\boxed{C_{\mu\nu}(\boldsymbol{q}, t) = \int_0^\infty d\omega\, W_{\mu\nu}(\boldsymbol{q}, \omega)\, e^{-\omega t}}$$

Inclusive decays – Lattice



- t_{src}, t_2, t_{snk} fixed
- $t_{src} \leq t_1 \leq t_2$
- $t = t_2 - t_1$

$$C_{\mu\nu}(\boldsymbol{q}, t) = \int_0^{\infty} d\omega\, W_{\mu\nu}(\boldsymbol{q}, \omega)\, e^{-\omega t}$$

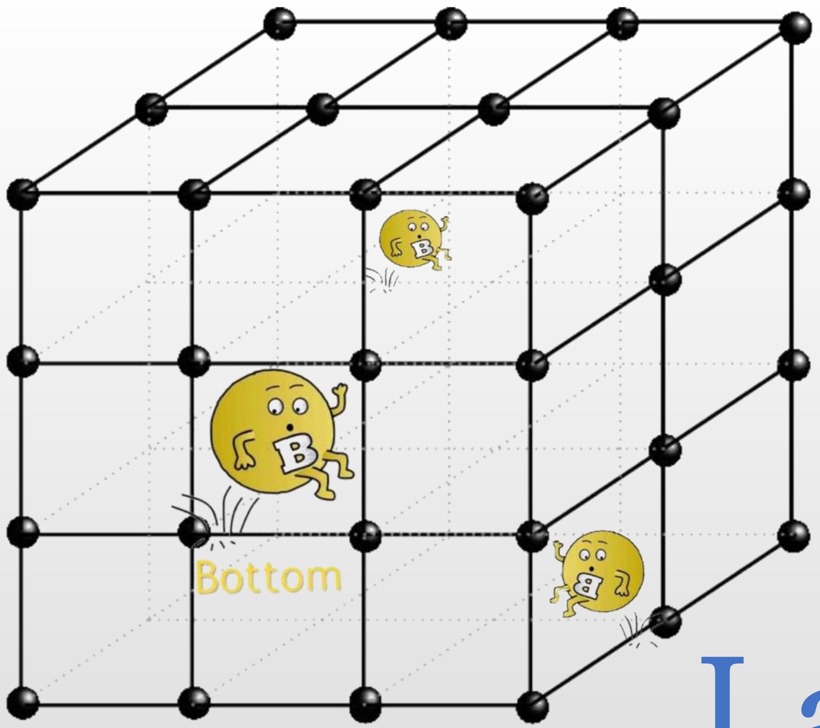
Continuum expression

$$\bar{X}^{(l)}(\boldsymbol{q}^2) = \int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu}(\boldsymbol{q}, \omega) K^{(l)}_{\mu\nu}(\boldsymbol{q}, \omega)$$

Approximate Kernel in polynomials of $e^{-\omega}$

$$K^{(l)}_{\sigma,\mu\nu}(\omega, \boldsymbol{q}) \simeq c^{(l)}_{\mu\nu,0}(\boldsymbol{q}; \sigma) + \\ + c^{(l)}_{\mu\nu,1}(\boldsymbol{q}; \sigma) e^{-\omega} + \ldots + c^{(l)}_{\mu\nu,N}(\boldsymbol{q}; \sigma) e^{-N\omega}$$

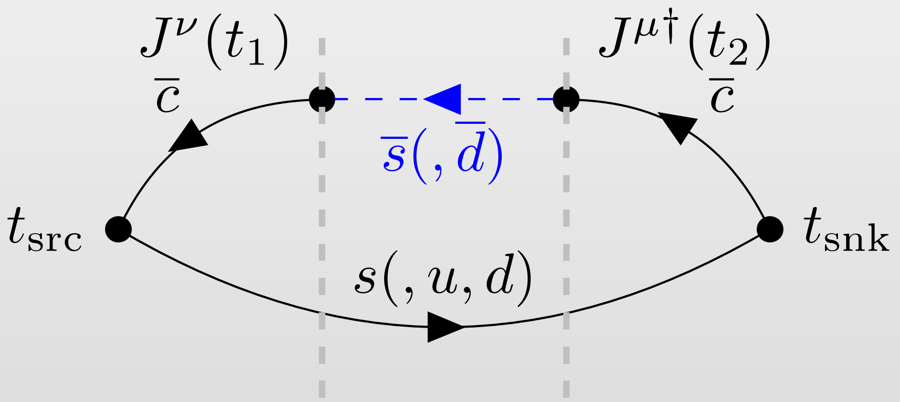
$$\bar{X}^{(l)}_{\sigma}(\boldsymbol{q}^2) \sim c^{(l)}_{\mu\nu,0} \underbrace{\int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu}(\boldsymbol{q}, \omega)}_{C(0)} + \cdots + c^{(l)}_{\mu\nu,N} \underbrace{\int_{\omega_0}^{\infty} d\omega\, W^{\mu\nu}(\boldsymbol{q}, \omega) e^{-N\omega}}_{C(N)}$$



Lattice Setup

Simulation details

Simulations conducted on Fugaku using Grid [P. Boyle et al., https://github.com/paboyle/Grid] and Hadrons [A. Portelli et al., https://github.com/aportelli/Hadrons] software packages



Lattice parameters/ ID	m_{ud}	m_s	M_π [MeV]	T	$[t_1, t_2]$	N_{Conf}	$N_{t_{\mathrm{src}}}$
$\beta = 4.17$, $a^{-1} = 2.453(1)$ GeV, $32^3 \times 64 \times 12$							
C-*ud*3-sa	0.0070	0.0400	309(1)	28	[11, 17]	100	2
C-*ud*4-sa	0.0120	0.0400	399(1)	28	[11, 17]	100	2
C-*ud*5-sa	0.0190	0.0400	499(1)	28	[11, 17]	100	2
$\beta = 4.17$, $a^{-1} = 2.453(1)$ GeV, $48^3 \times 96 \times 12$							
C-*ud*2-sa-L	0.0035	0.0400	226(1)	42	[16, 26]	100	4
$\beta = 4.35$, $a^{-1} = 3.61(1)$ GeV, $48^3 \times 96 \times 8$							
M-*ud*3-sa	0.0042	0.0250	300(1)	42	[16, 26]	50	8

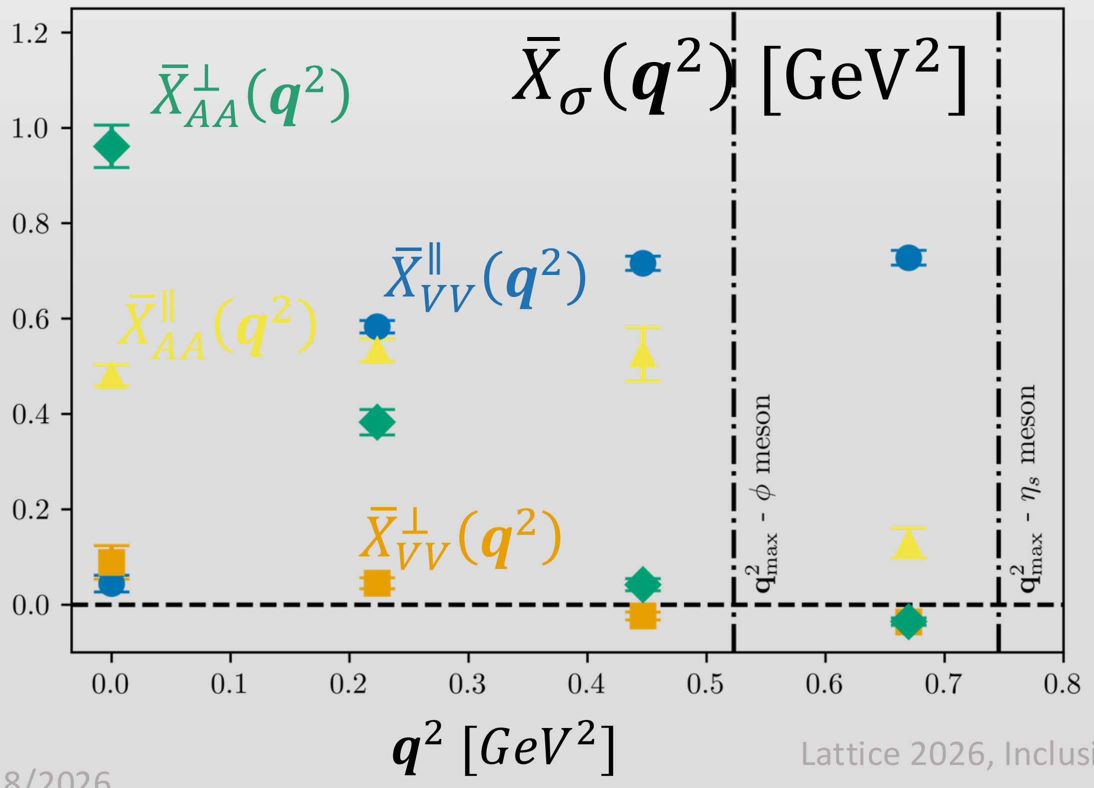
Simulation:
- 2+1 Möbius domain-wall fermions
- s, c quarks simulated at near-physical values
- Cover whole kinematical region $\boldsymbol{q} = (0,0,0) \rightarrow (1,1,1)$

Numerical Results

The differential rate $\bar{X} \sim \frac{d\Gamma}{d\boldsymbol{q}^2}$

$$\bar{X}_\sigma(\boldsymbol{q}^2) = \sum_{l=0}^{2} \left\langle D_s(\boldsymbol{0}) \middle| \tilde{J}_\mu^\dagger(-\boldsymbol{q}) K_{\sigma,\mu\nu}^{(l)}(\hat{H}, \boldsymbol{q}) \tilde{J}_\nu(\boldsymbol{q}) \middle| D_s(\boldsymbol{0}) \right\rangle$$

$N = 10, \sigma = 0.1$



Results for a single ensemble

Previously addressed systematics

1. Finite polynomial approximation
   - **Chebyshev polynomials** [arXiv:2305.14092, 2504.03358]
   - *HLT approach* [De Santis et al., arXiv:2504.06063]

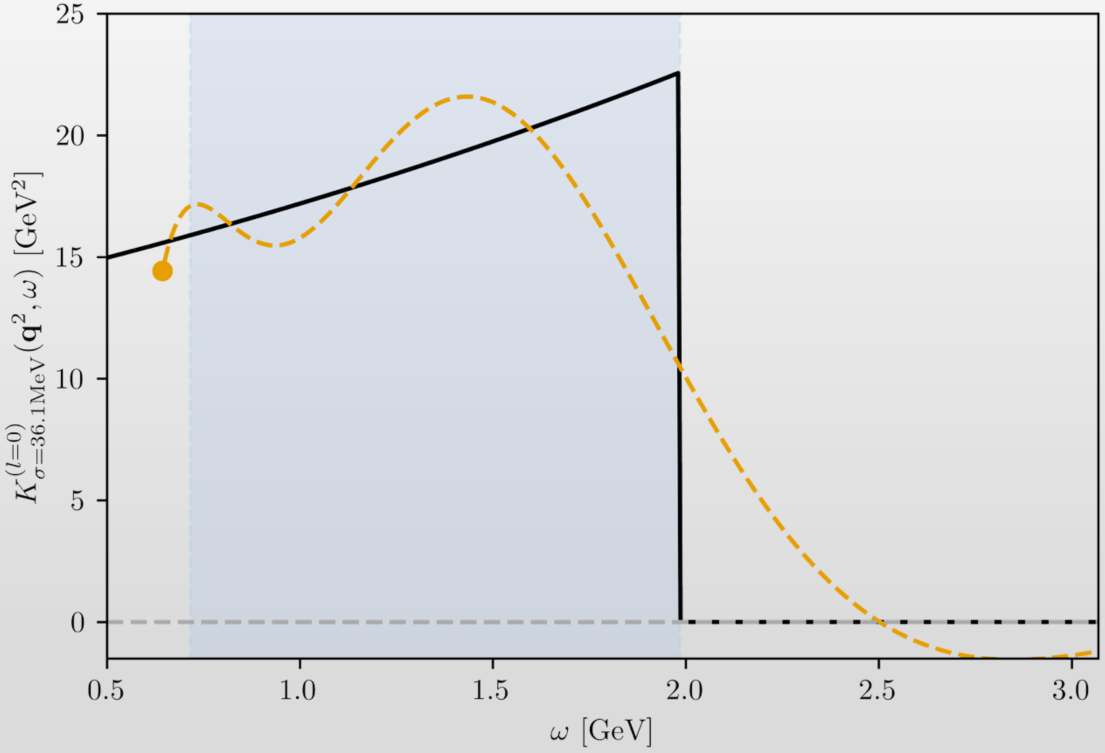
   \+ smearing
2. Finite-volume effects

Finite polynomial approximation

Consider

$$\bar{X}(\boldsymbol{q}^2) \sim \int_0^\infty d\omega\ \rho(\omega) \times K(\omega)$$

Depending on $\rho(\omega)$ errors behave differently



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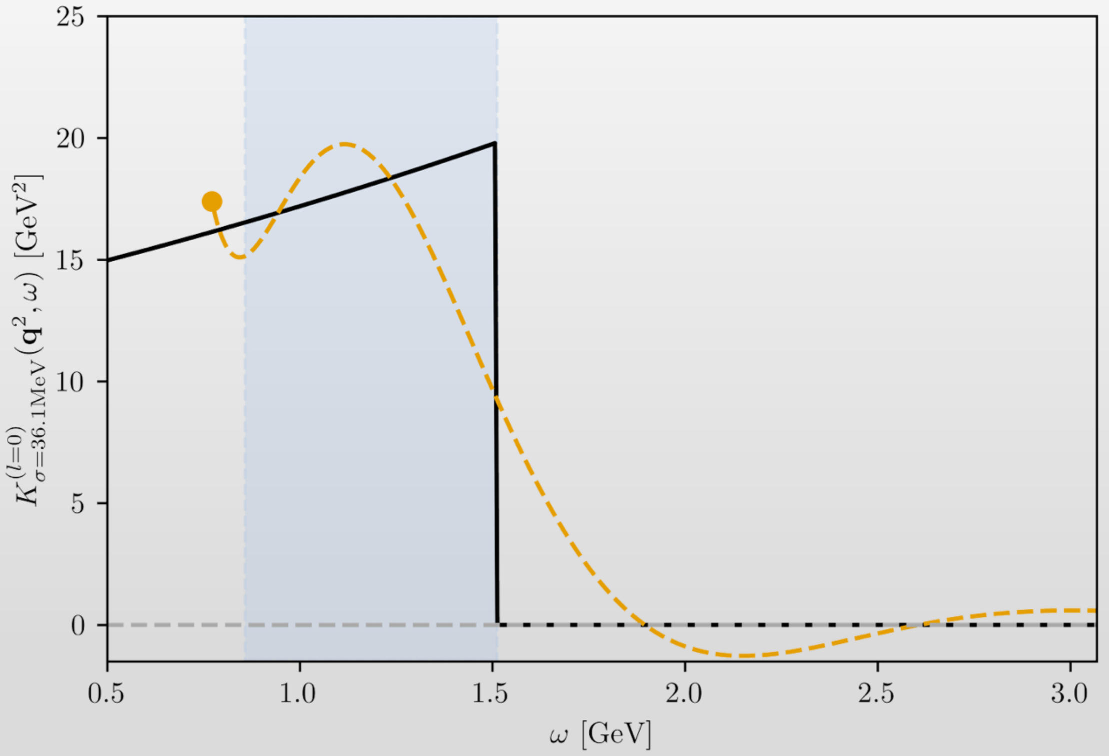
As momentum increases → Phase-space becomes narrower



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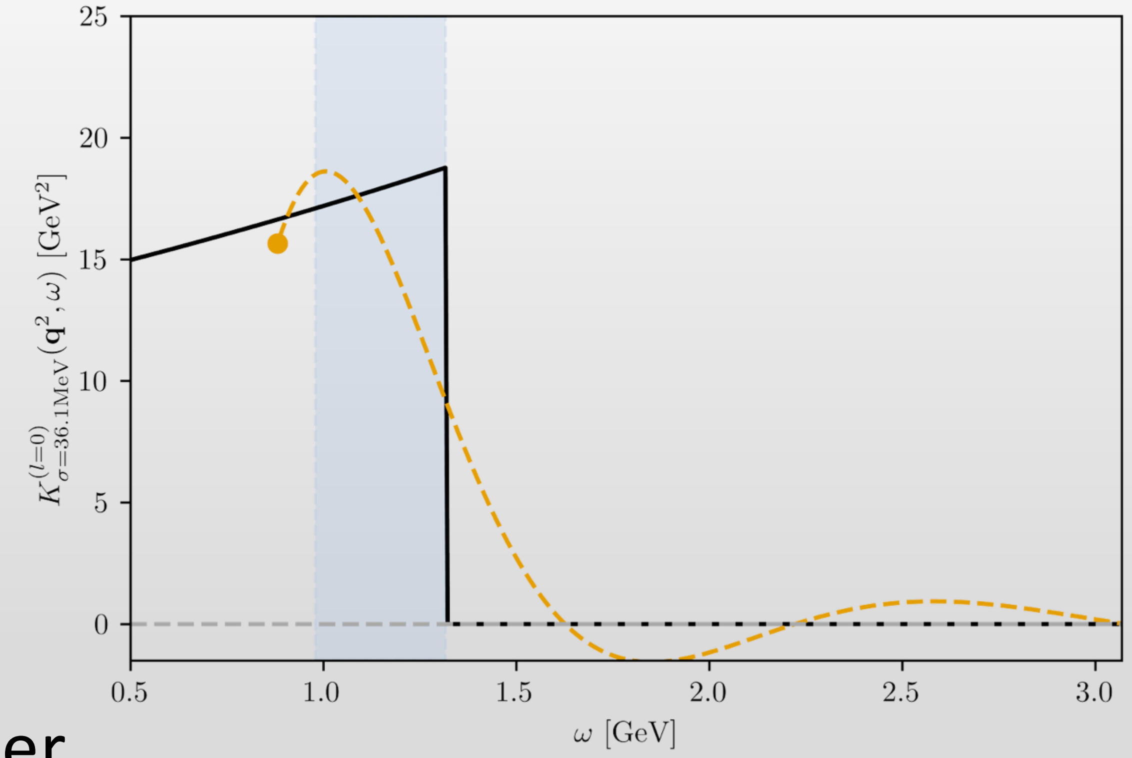
Window for excited states becomes smaller



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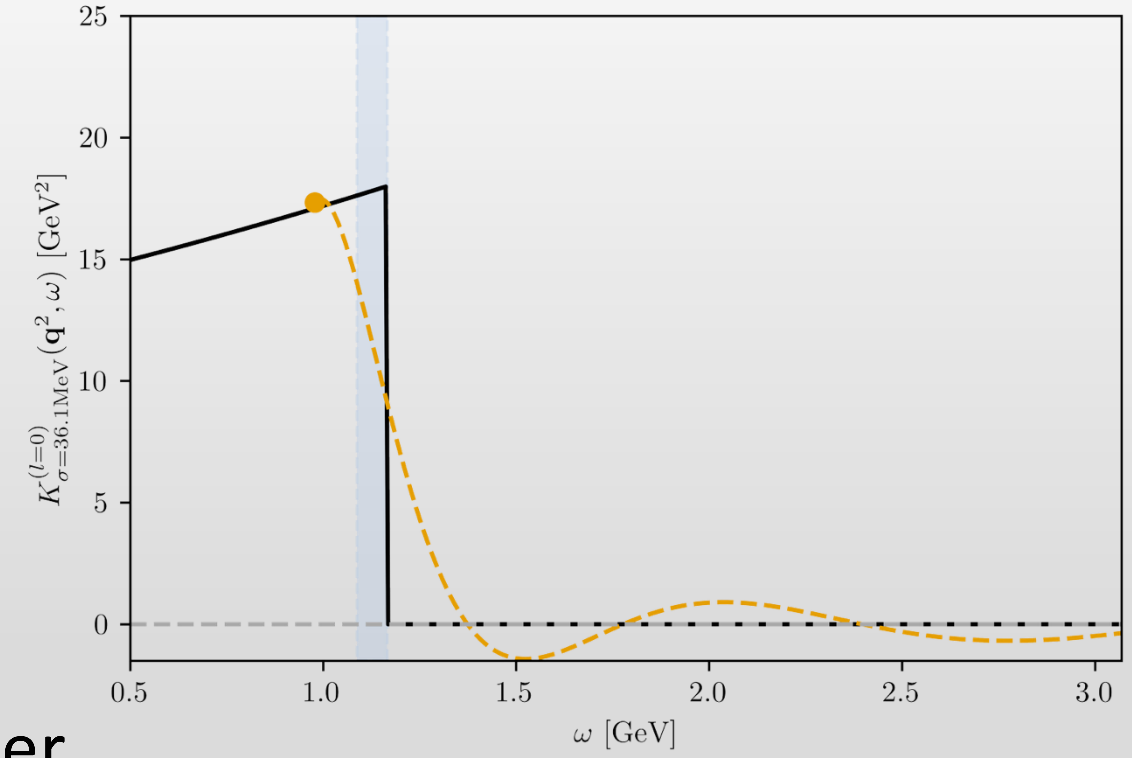


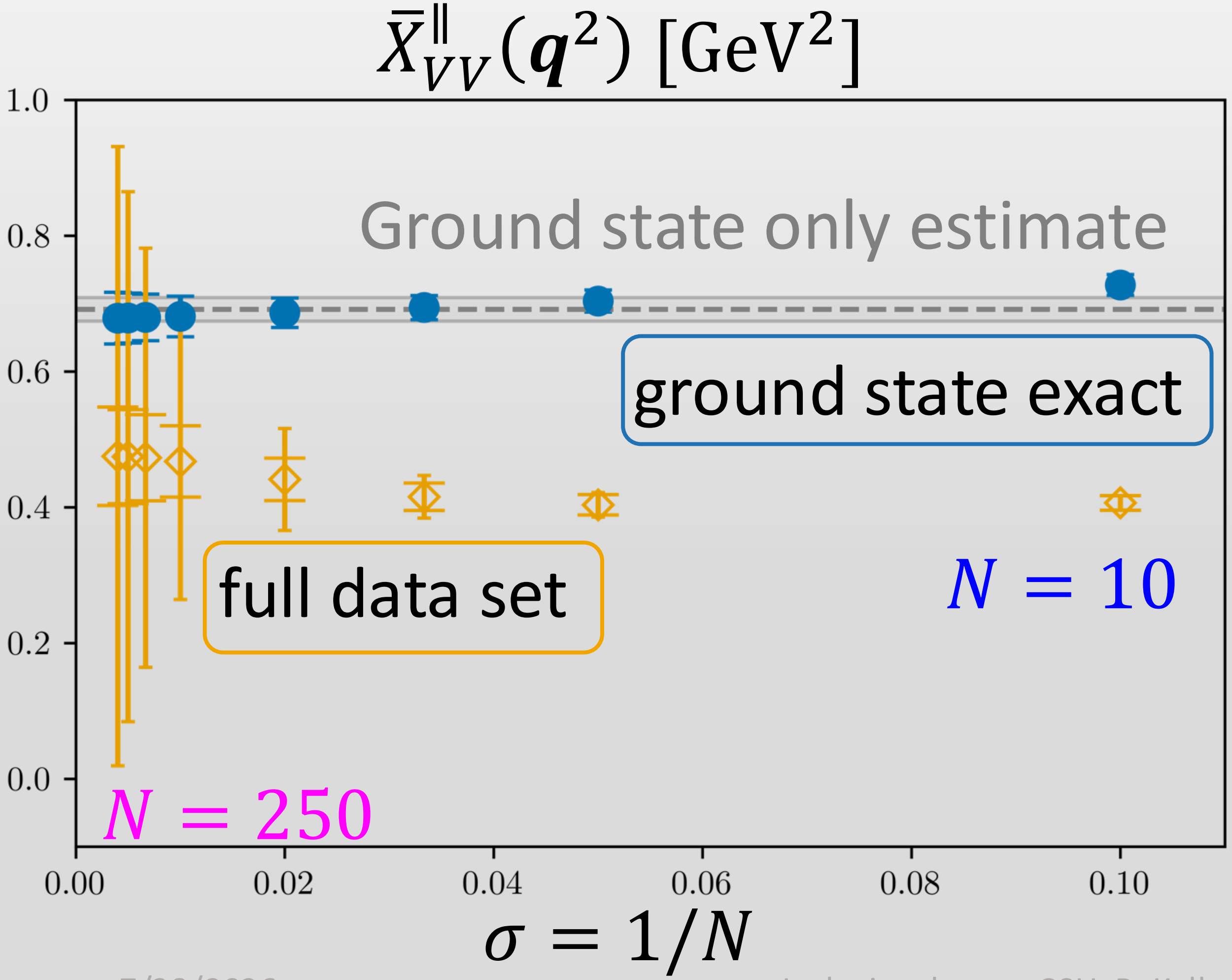
Near maximal recoil momentum

Only ground state → large dependence on σ

Ground-state well-known ⇒ remove from analysis

Systematic error - Approximation

Application for $\bar{X}^{\parallel}_{VV}(\boldsymbol{q}^2)$ for $\boldsymbol{q} = (1,1,1)$



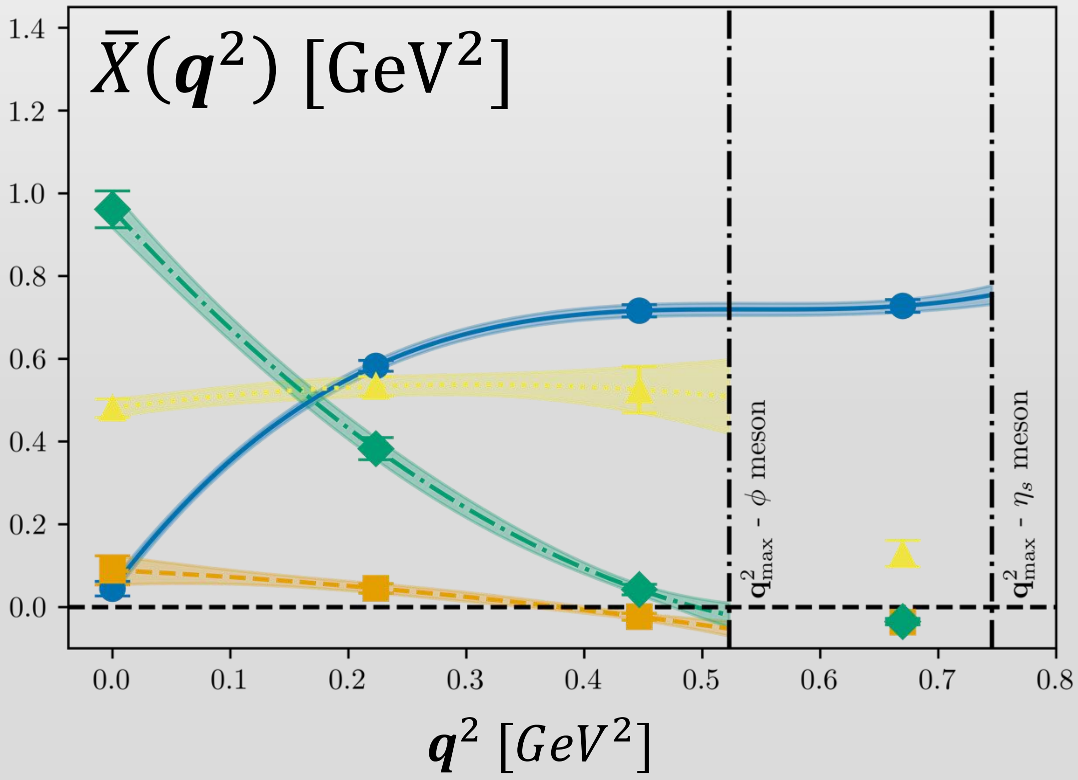
In the $\sigma \to 0$ limit:

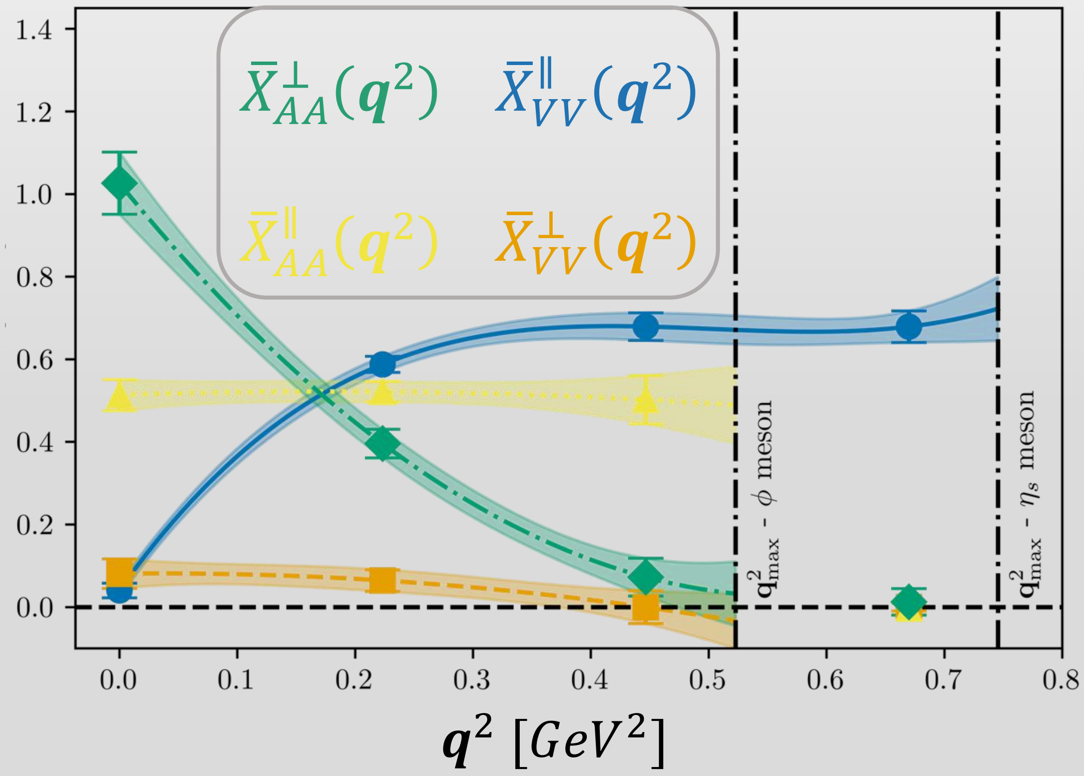
1. Full data set:
    - shifts in central values + below ground state estimate
    - substantial increase in errors
    - Error bounds: no determination relevant for phenomenology
2. Ground state exact:
    - stable central values + agrees with groud state estimate
    - small increase in errors

Estimating the systematic corrections

Channels:

1. AA: infinite-volume limit + $\sigma \to 0$ limit
2. VV: finite-volume corrections expected small; only $\sigma \to 0$ limit





Continuum & chiral extrapolations

Define global fitting strategy using all ensembles as input

$$\bar{X}(\boldsymbol{q}^2, m_\pi^2, a^2)$$
$$= \left(\sum_{i=0}^{N_P} c_i (\boldsymbol{q}^2)^i\right) \times \left(1 + \tilde{c}_{m_\pi} \left(\frac{m_\pi}{\Lambda_C}\right)^2\right) \times (1 + \tilde{c}_a (a\Lambda_C)^2) \times \left(1 + \tilde{c}_{a\boldsymbol{q}} (a\boldsymbol{q})^2\right)$$

Continuum & chiral extrapolations

Define global fitting strategy using all ensembles as input

Mass scale to make coefficients dimensionless

$\Lambda_C = 0.7$ GeV

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Estimating systematics:

I. Include $\mathcal{O}\left((m_\pi/\Lambda_C)^4\right)$

II. Remove either $(1 + \tilde{c}_a (a\Lambda_C)^2)$ or $\left(1 + \tilde{c}_{a\boldsymbol{q}}(a\boldsymbol{q})^2\right)$

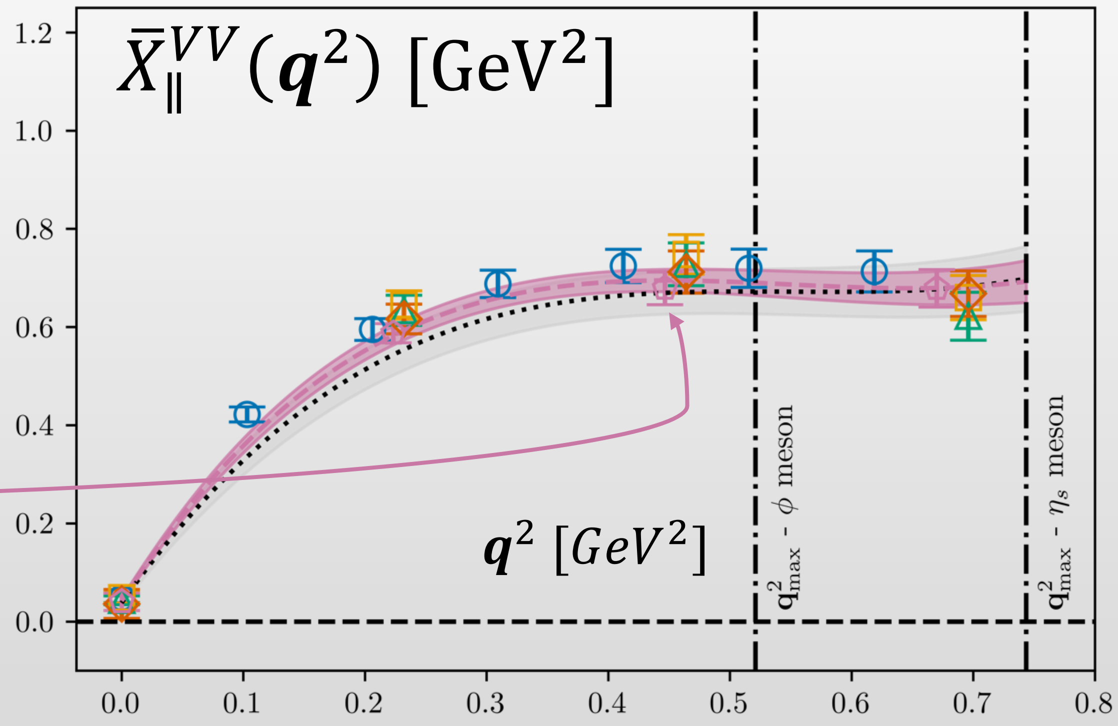
III. Remove higher order terms, e.g. $\mathcal{O}\left((\boldsymbol{q}^2)^3\,(m_\pi/\Lambda_C)^2 (a\Lambda_C)^2\right)$

Continuum & chiral extrapolations

Apply for $\bar{X}_{\parallel}^{VV}(\boldsymbol{q}^2)$

Ensembles:

- $\beta = 4.17$; $m_{ud} =$ 0.0035, 0.0070, 0.012, 0.0190
- $\beta = 4.35$; $m_{ud} =$ 0.0042



Dashed: fit w/ $\beta = 4.35$ & $m_{ud} = 0.0042$

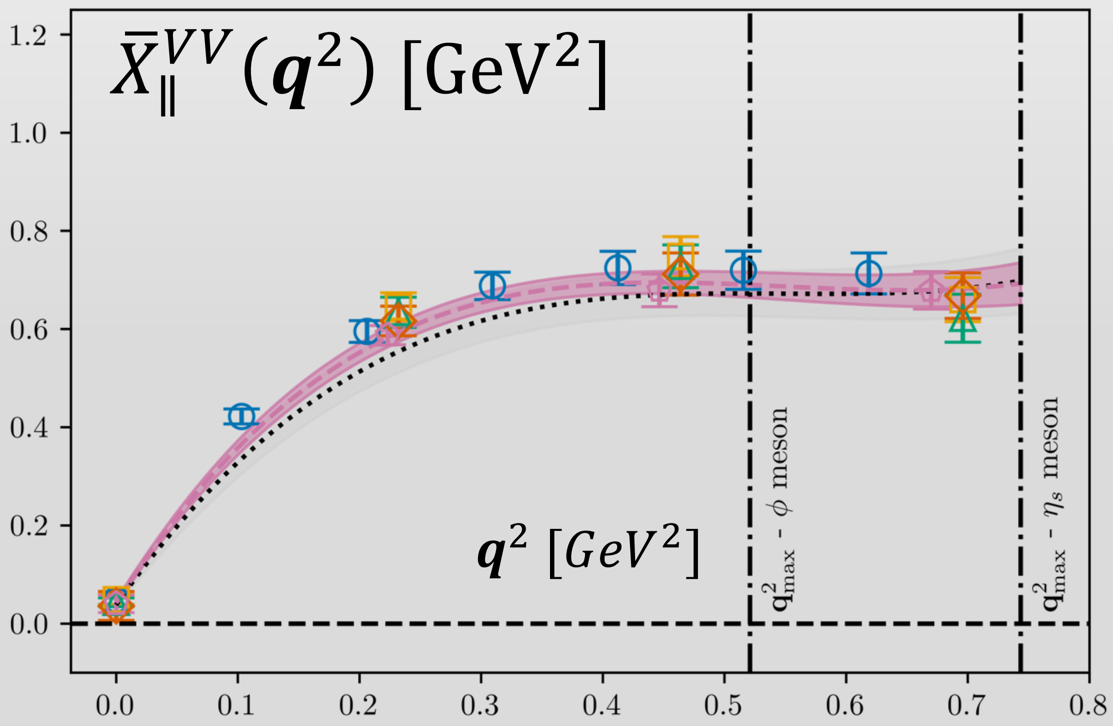
Black-dotted: fit w/ $a^2 = 0$ & $m_\pi^2 = \left(m_\pi^{\text{Phys}}\right)^2$

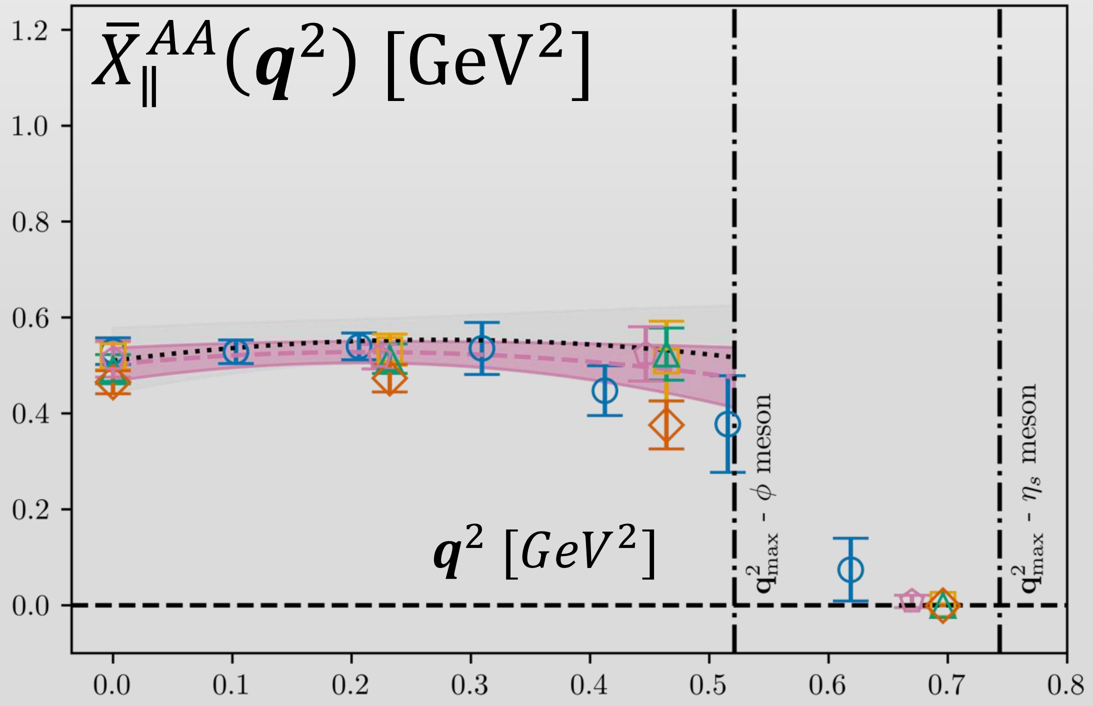
Most severe shift towards the physical point

Continuum & chiral extrapolations

Ensembles:

- $\beta = 4.17; m_{ud} = 0.0035, 0.0070, 0.012, 0.0190$
- $\beta = 4.35; m_{ud} = 0.0042$





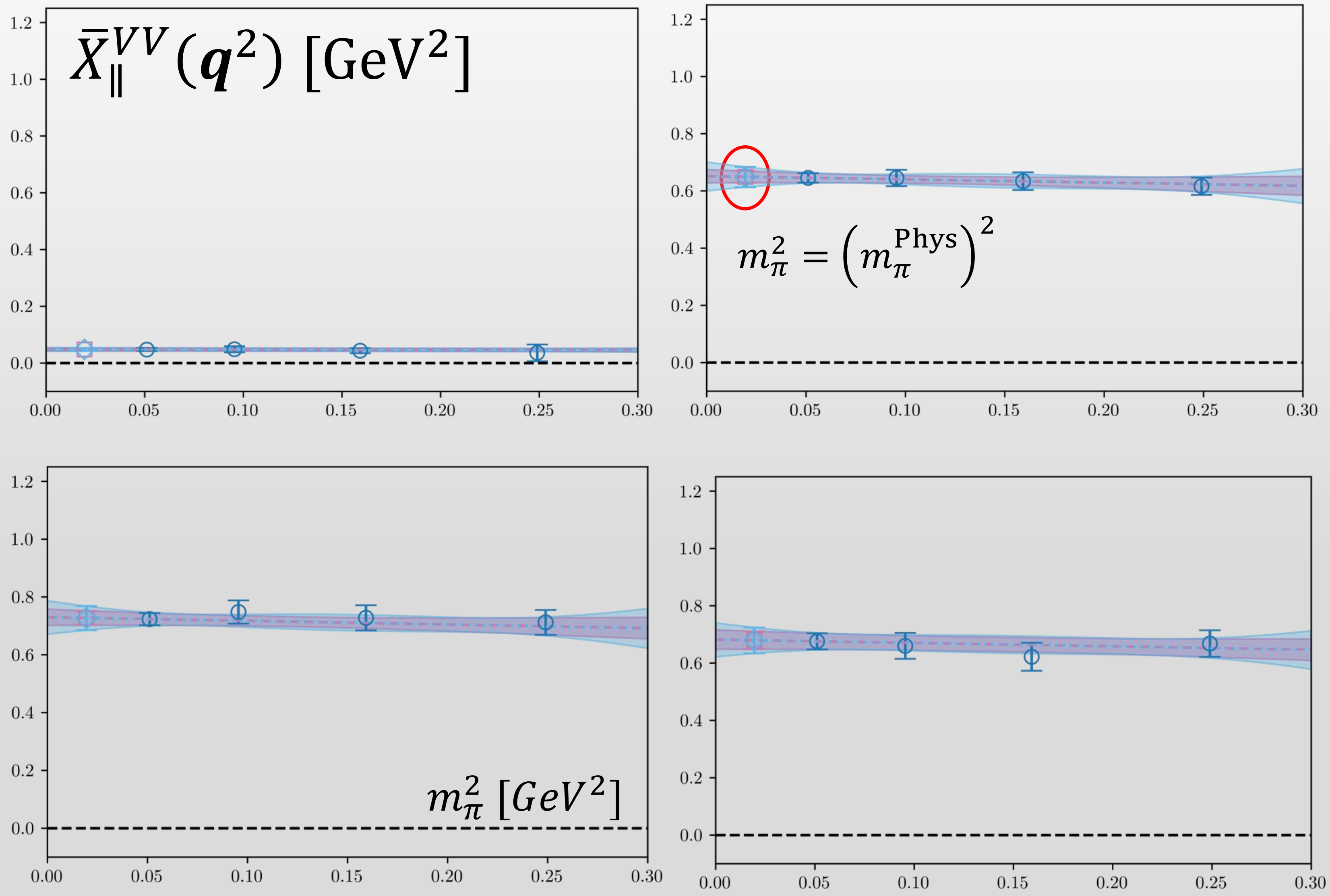
Chiral extrapolation

Focus on $\bar{X}_{\parallel}^{VV}(\boldsymbol{q}^2)$

Dashed:
$\mathcal{O}\left((m_\pi/\Lambda_C)^2\right)$

Dotted:
$\mathcal{O}\left((m_\pi/\Lambda_C)^4\right)$

Overall:
- Weak dependence on m_π^2
- Larger errors w/ m_π^4 term



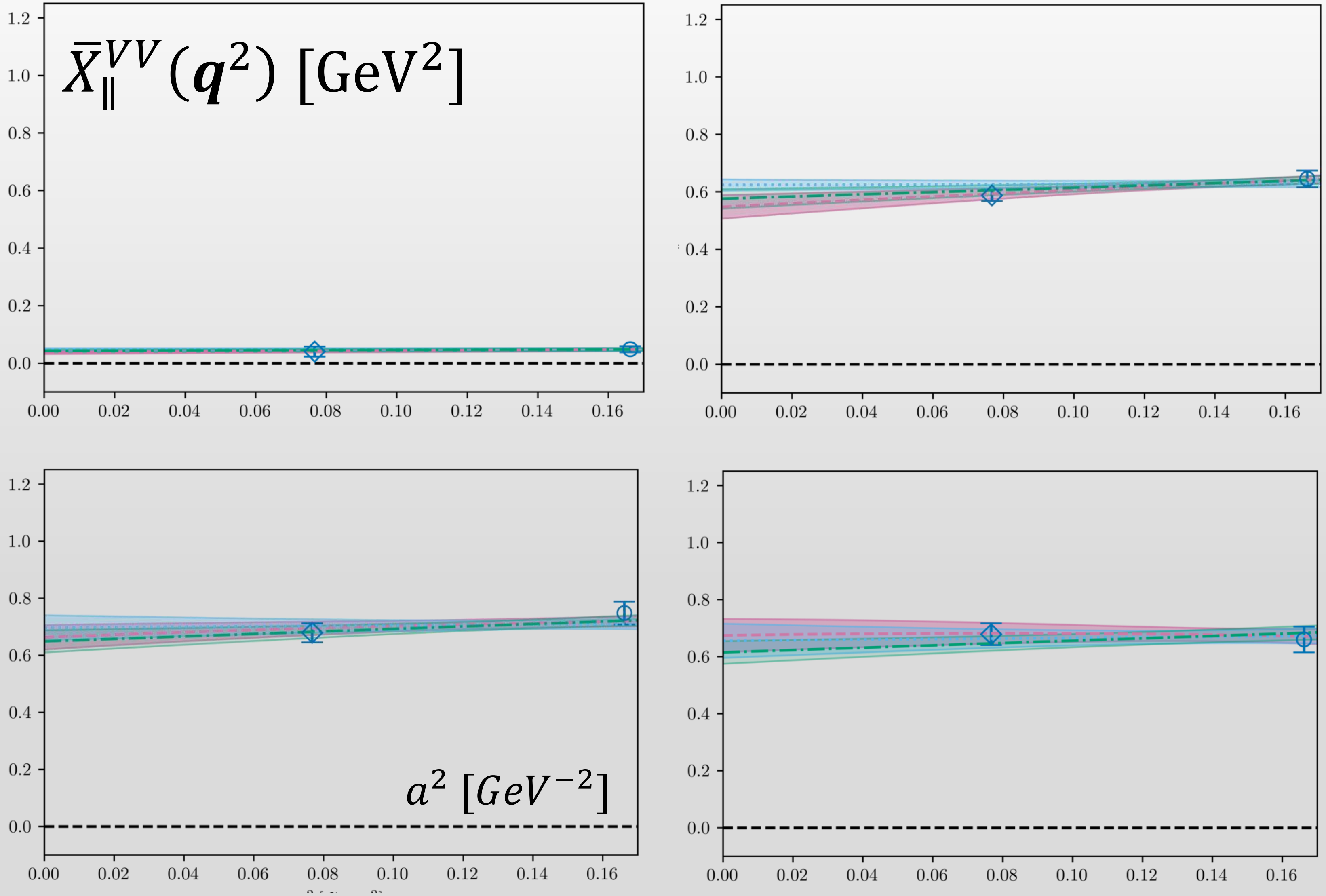
Continuum extrapolation

Focus on $\bar{X}_{\parallel}^{VV}(\boldsymbol{q}^2)$

Dashed:
$\mathcal{O}\big((1+(a\Lambda_C)^2)$
$\times\ (1+(a\boldsymbol{q})^2)\big)$

Dotted:
$\mathcal{O}\big((1+(a\boldsymbol{q})^2)\big)$

Dashdotted:
$\mathcal{O}\big((1+(a\Lambda_C)^2)\big)$



Continuum extrapolation

Focus on $\bar{X}_{\parallel}^{VV}(\boldsymbol{q}^2)$

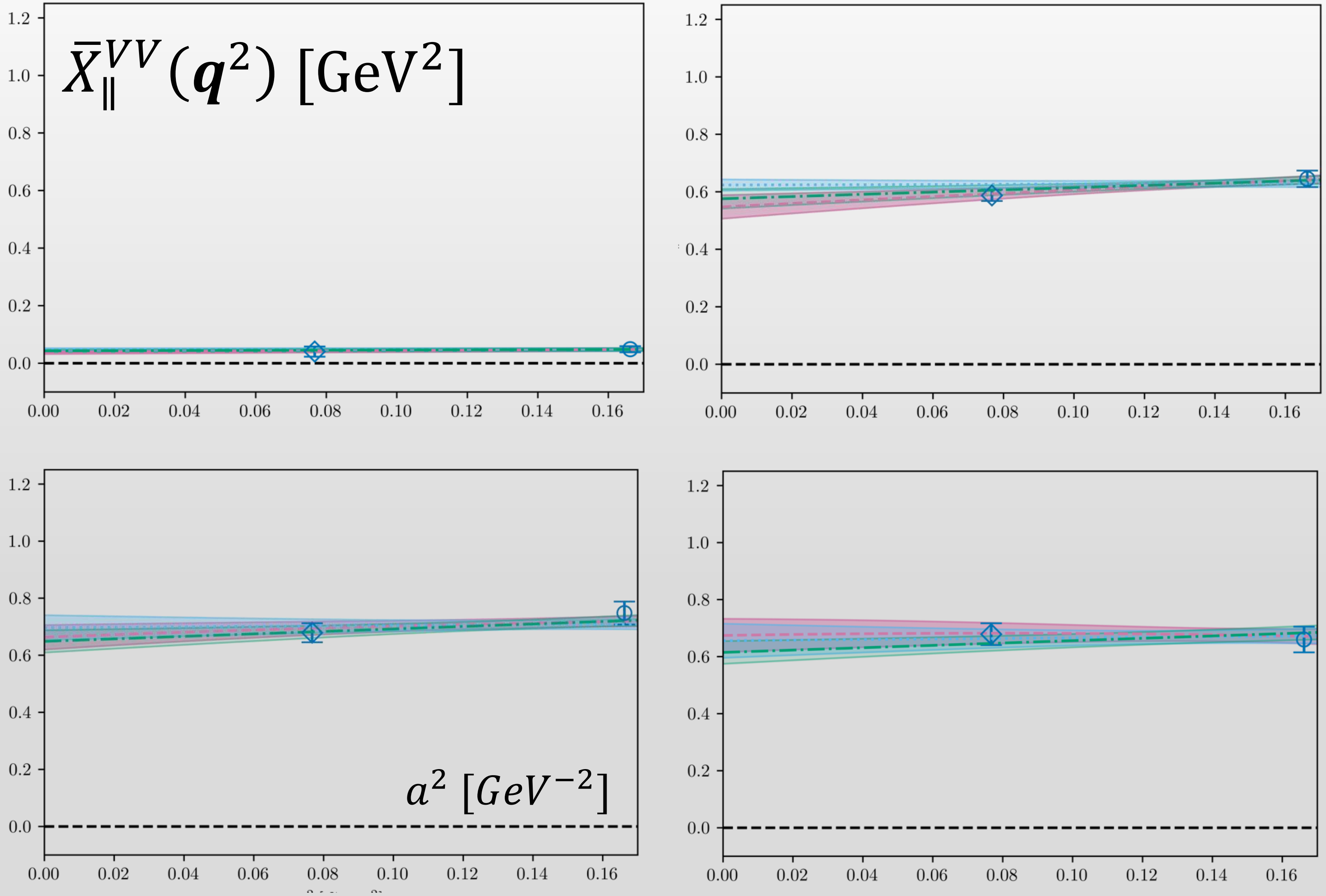
Dashed: $\mathcal{O}\big((1+(a\Lambda_C)^2) \times (1+(a\boldsymbol{q})^2)\big)$

Dotted: $\mathcal{O}\big((1+(a\boldsymbol{q})^2)\big)$

Dashdotted: $\mathcal{O}\big((1+(a\Lambda_C)^2)\big)$

Overall:

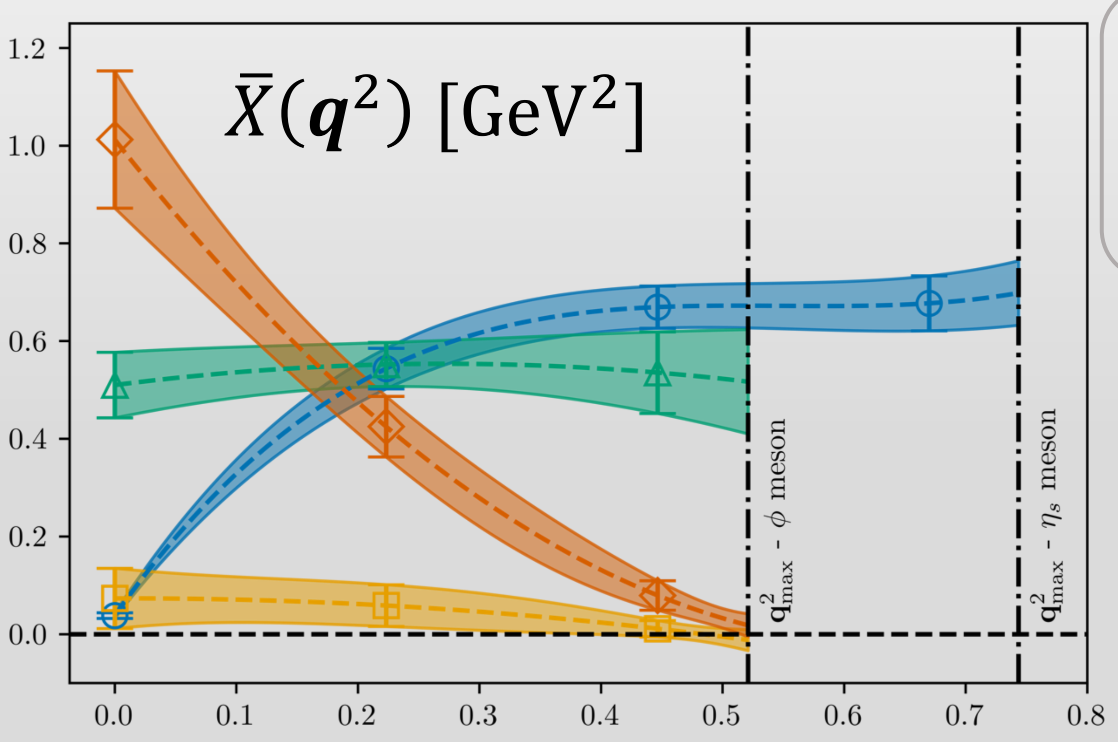
- Again, weak dependence on a^2
- **BUT**: small downward trend
- **AND**: extrapolation is based on only two points



Determination of inclusive rate

Channels making up $\bar{X}(\boldsymbol{q}^2)$ at the physical point

''Data points'' are based on fit



$\bar{X}^{\parallel}_{AA}(\boldsymbol{q}^2)$ $\bar{X}^{\parallel}_{VV}(\boldsymbol{q}^2)$

$\bar{X}^{\perp}_{AA}(\boldsymbol{q}^2)$ $\bar{X}^{\perp}_{VV}(\boldsymbol{q}^2)$

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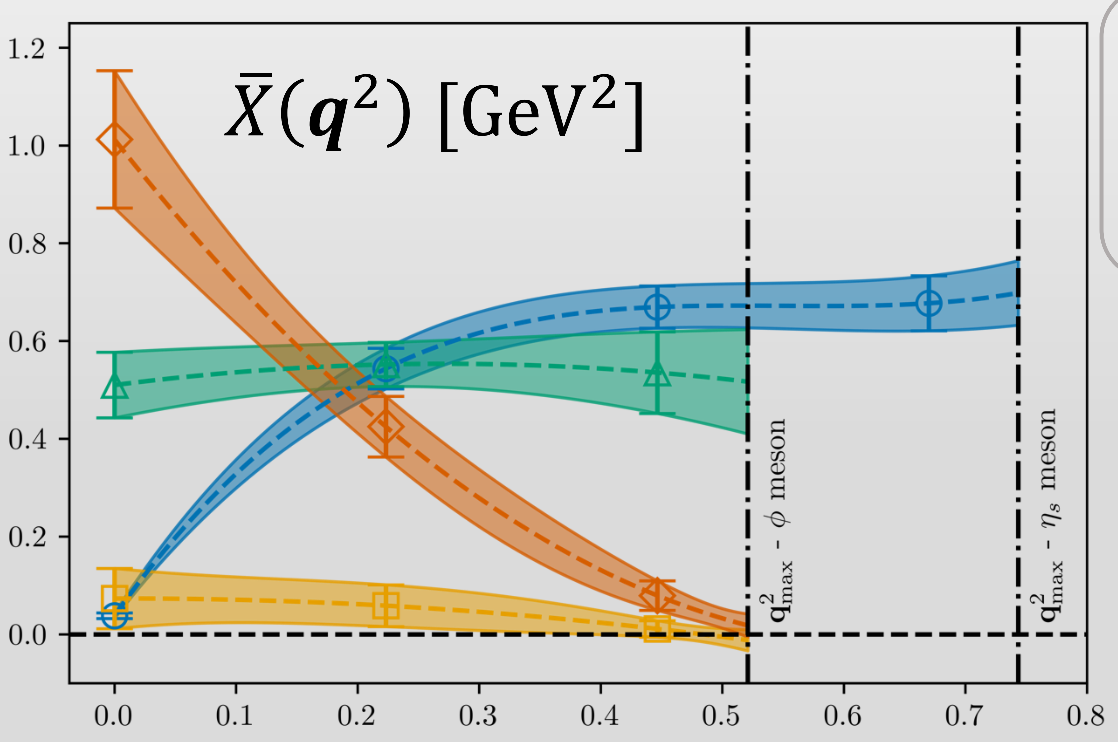
"Data points" are based on fit



$\bar{X}^{\parallel}_{AA}(\boldsymbol{q}^2)$ $\bar{X}^{\parallel}_{VV}(\boldsymbol{q}^2)$

$\bar{X}^{\perp}_{AA}(\boldsymbol{q}^2)$ $\bar{X}^{\perp}_{VV}(\boldsymbol{q}^2)$

Total decay rate

$$\frac{\Gamma}{|V_{cs}|^2} = \frac{G_F^2}{24\pi^3} \int_0^{q^2_{max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\, \bar{X}(\boldsymbol{q}^2)$$

Determination of inclusive rate

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$\bar{X}^{\parallel}_{AA}(\boldsymbol{q}^2)$ $\bar{X}^{\parallel}_{VV}(\boldsymbol{q}^2)$

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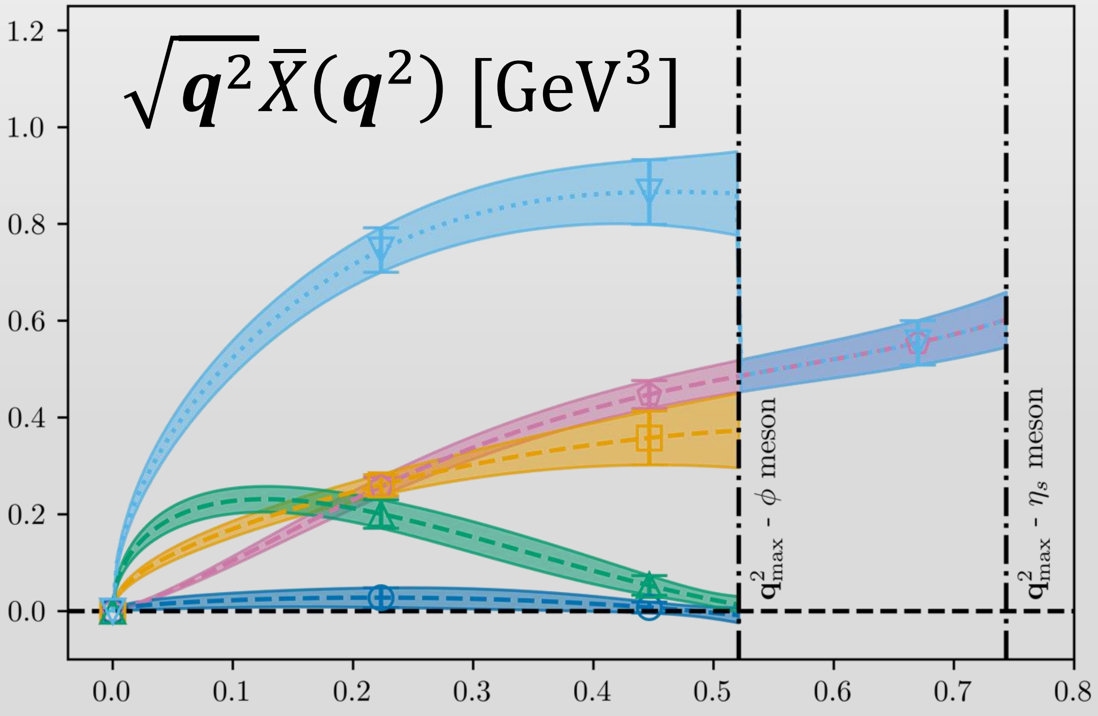

Determination of inclusive rate

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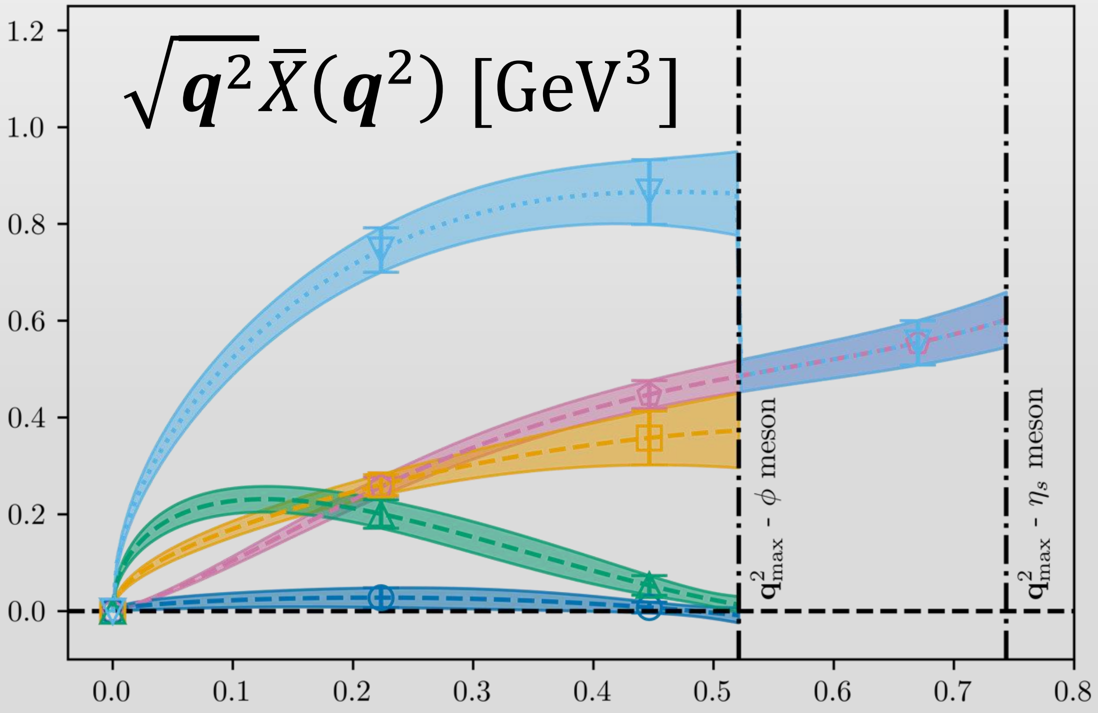

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$$\frac{\Gamma}{|V_{cs}|^2} = 0.884(47)^{+0.044}_{-0.055} \times 10^{-13}\,\text{GeV}$$

Determination of inclusive rate

Channels making up $\bar{X}(\boldsymbol{q}^2)$ at the physical point

''Data points'' are based on fit

$\bar{X}_{AA}^{\parallel}(\boldsymbol{q}^2)$ $\bar{X}_{VV}^{\parallel}(\boldsymbol{q}^2)$

$\bar{X}_{AA}^{\perp}(\boldsymbol{q}^2)$ $\bar{X}_{VV}^{\perp}(\boldsymbol{q}^2)$

Total decay rate

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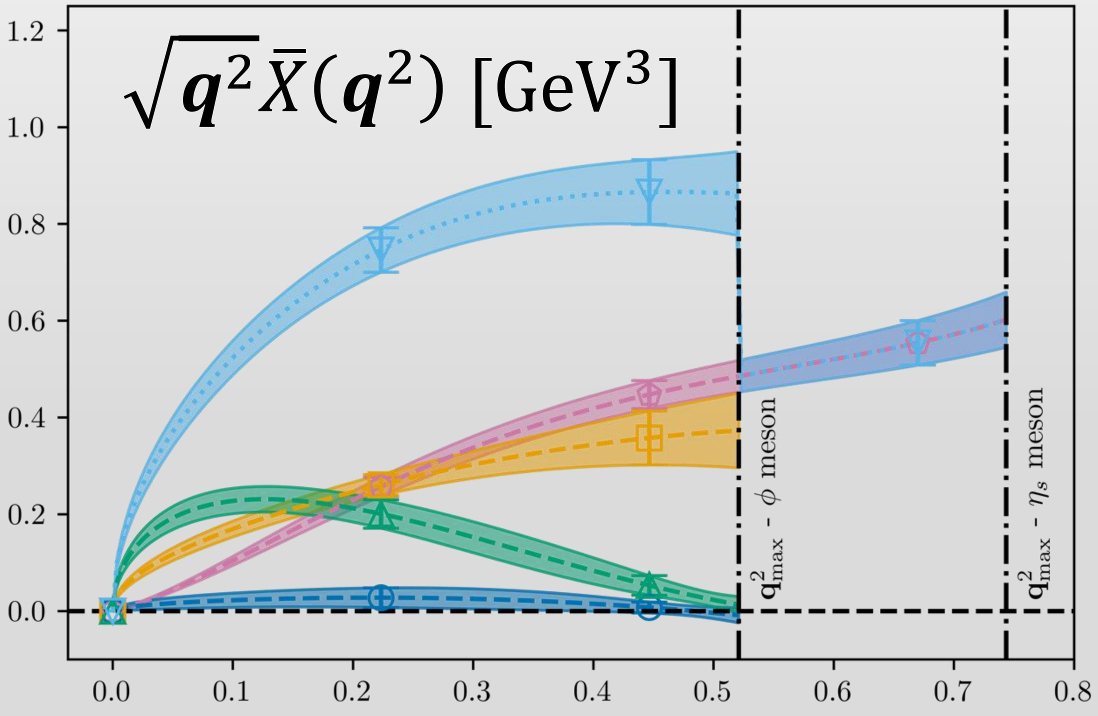

Determination from A. De Santis et al.

[arXiv:2504.06063]

- obtained using HLT approach

$$\frac{\Gamma}{|V_{cs}|^2} = 0.853(46)(30) \times 10^{-13}\,\mathrm{GeV}$$

Determination of inclusive rate

Total decay rate

$$\frac{\Gamma}{|V_{cs}|^2} = 0.884(47)^{+0.044}_{-0.055} \times 10^{-13}\,\mathrm{GeV}$$

Experimental data:

BESIII [2104.07311]: $\Gamma = 0.827(22) \times 10^{-13}$ GeV $\Longrightarrow$ $|V_{cs}|_{BESIII} = 0.967(28)^{+0.024}_{-0.029}$

CLEO [0912.4232]: $\Gamma = 0.856(55) \times 10^{-13}$ GeV $\Longrightarrow$ $|V_{cs}|_{CLEO} = 0.984(41)^{+0.024}_{-0.029}$

Determination of inclusive rate

Total decay rate

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Promising when compared to PDG

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The lattice is now able to study inclusive semileptonic decay at a level relevant for phenomenology!

Summary & Outlook

Summary

- Systematic effects in the inclusive analysis of semileptonic decays on the lattice
  - Error from Chebyshev polynomial approximation
    - Estimate based on mathematical properties
  - Finite volume corrections
    - Model to estimate corrections; supplement with data

  arXiv:2504.03358

  - Chiral & continnum extrapolations
    - Under control; supplement continuum extrapolation with one more data point

  arXiv:2604.22201

Outlook

- Extend towards a full analysis in the bottom sector (in collab. with RBC/UKQCD) (A. Elgaziari, Lattice 2025 2604.26381, De Santis et al. 2607.01116) De Santis, Fri., July 31st, 5:10 PM
- Disconnected $s\bar{s}$ diagramms (η/η')?
- Improve statistics with new techniques? (De Santis et al arXiv:2607.01116)
- Extend to different observables, e.g. moments (De Santis et al arXiv:2504.06063)
- Excl. form factors from inclusive lattice simulation (Zhi Hu, HQL2025, arXiv:2601.09480)