Order-Order Interface in Pure $\mathrm{SU}(N)$ Gauge Theory

Aaron Haarti, Kari Rummukainen, Tobias Rindlisbacher

University of Helsinki

2025

Order-Order Interface in Pure $\mathrm{SU}(N)$ Gauge Theory

Content

- Introduction
- The twist and the formation of order-order interface
- Measuring interface tension of order-order interface with capillary wave theory and integration methods
- Result in $\mathrm{SU}(4)$ and $\mathrm{SU}(8)$

Introduction

Motivation and goal

Motivation:

- Pure $\mathrm{SU}(N)$ gauge theory is relevant to BSM physics for example dark matter candidates
- Understanding the nature of pure $\mathrm{SU}(N)$ gauge theory such as first order phase transitions and interfaces tensions

Goal:

Measure Casimir scaling hypothesis for gluon string tension [1]

$$\frac{k(N-k)}{s(N-s)}.$$

Evaluate perfect wetting hypothyses $\sigma_{oo}(T_c) = 2\sigma_{od}(T_c)$.

[1] Bursa and Teper 2005

Pure gauge theory on the lattice

Phase transition[2]

Low temperature color confining $Z(N)$-symmetric disordered phase:

$$\langle P(\vec{x}) \rangle = 0.$$

High temperature color non-confining $Z(N)$-breaking ordered phase:

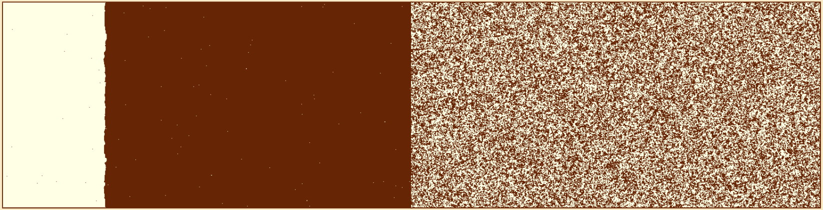
$$\langle P(\vec{x}) \rangle \sim z_n.$$

First order phase transition

[2] Polyakov 1978; Susskind 1979; Brown et al. 1988; Fukugita, Okawa, and Ukawa 1990

The twist and the formation of an o - o interface

The twist: ising



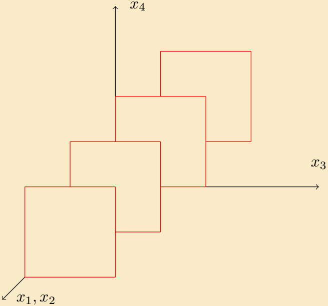
Low T: ordered phase High T: disordered phase

order-order interface:

1. two ordered phases
2. moves freely
3. location is independent of twist

The twist and the formation of an o - o interface

The twist: $\mathrm{SU}(N)$



$$T = \{U_{34}(x_1, x_2, 0, 0) \mid \forall x_1 \in \{1, \cdots, N_1\},\ x_2 \in \{1, \cdots, N_2\}\}$$

$$S_T[U] = \frac{\beta}{N} \sum_x \sum_{\mu<\nu} \Re\,\mathrm{Tr}[1 - z_n^{-1} U_{\mu\nu}(x)].^3$$

[3] Kajantie, Karkkainen, and K. Rummukainen 1991

Measuring interface tension of o - o interface

CWT method

The surface energy is given by[4]

$$F = \sigma \int_0^L \int_0^L dy\, dx \sqrt{1 + |\nabla z|^2} = \sigma \int_0^L \int_0^L dy\, dx \left[1 + \tfrac{1}{2} |\nabla z|^2 - \tfrac{1}{8} |\nabla z|^4 \ldots \right].$$

Far below the UV cutoff we can approximate

$$\beta F \approx \frac{\sigma}{T} \int_0^L \int_0^L dy\, dx \left[1 + \tfrac{1}{2} |\nabla z|^2 \right].$$

[4] Moore and Turok 1997

Measuring interface tension of o - o interface

CWT method

Taking the Fourier transform we can evaluate the surface integral,

$$\beta F \approx \frac{\sigma}{T}\int_0^L\int_0^L dy\,dx\left[1+\tfrac{1}{2}|\nabla z|^2\right] \approx \sum_n \frac{4\pi^2\sigma n^2 \hat{z}_n^2}{2T}.$$

By equipartition, with $n^2 \equiv n_x^2 + n_y^2 \neq 0$,

$$\langle F_n\rangle = \frac{T}{2} \iff n^2\langle \hat{z}_n^2\rangle = \frac{T}{4\pi^2\sigma} \quad \overset{\text{lattice}}{\Longrightarrow} \quad n^2\langle \tilde{z}_n^2\rangle = \frac{T}{a^2 4\pi^2\sigma}.$$

Inverting gives the surface tension

$$\frac{\sigma}{T^3} = \frac{N_t^2}{4\pi^2 n^2\langle \hat{z}_n^2\rangle},$$

UV lattice effects scale as $\mathcal{O}(n^2/L^2)$, thus taking $n \to 0$ yields σ/T^3.

Measuring interface tension of o - o interface

Integration method[5]

$$\frac{\partial}{\partial\beta}\frac{F}{T} = \frac{1}{\beta}\langle S_G[U]\rangle.$$

$$\alpha_{\text{o-o}} A = F_{\text{twist}} - F_{\text{no-twist}} = T\int d\beta \frac{1}{\beta}(\langle S_{\text{twist}}[U] - S_G[U]\rangle).$$

$$A = N_1 N_2 a^2 \text{ and } a = \tfrac{1}{N_t T}:$$

$$\Rightarrow \frac{\sigma_{o-o}}{T^3} = \frac{N_t^2}{N_1 N_2}\int_{\beta_0}^{\beta_1} d\beta \frac{1}{\beta}\langle S_{\text{twist}}[U] - S_G[U]\rangle.$$

[5] Kajantie, Karkkainen, and K. Rummukainen 1991

Measuring interface tension of o - o interface

Critical freezing

The transition barrier grows $\sigma_{c-d} \propto N^2$, so standard Monte Carlo does not suffice in $\mathrm{SU}(N > 3)$.

We use **Multicanonical Monte Carlo** to sample the full distribution of the first order phase transition. Given an order parameter $\mathcal{O}(s)$ and weight function $W(\mathcal{O}(s))$,

$$\langle A \rangle = \frac{\langle A(s) e^{W(\mathcal{O}(s))} \rangle}{\langle e^{W(\mathcal{O}(s))} \rangle}.$$

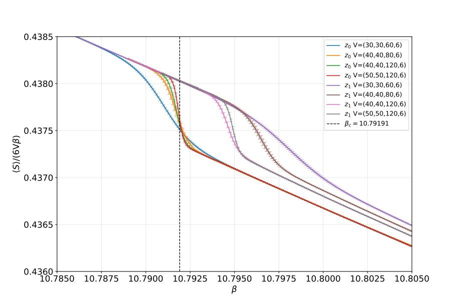
The order parameter is the Polyakov line $|\langle P^s(x) \rangle|$ smeared perpendicular to the surface.

Multihistogram reweighting, modified for the multicanonical weights, and jackknife with blocking give a smooth function.[6]

[6] Rindlisbacher, Kari Rummukainen, and Salami 2025; Berg and Neuhaus 1992; Kari Rummukainen, Seppä, and Weir 2026; Hällfors and Kari Rummukainen 2025; Ferrenberg and Swendsen 1988; Gattringer and Lang 2009

Measuring interface tension of o - o interface

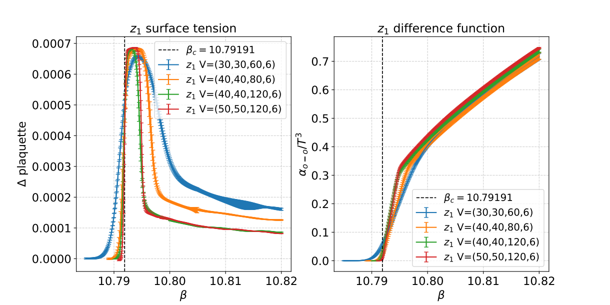
Finite volume



- Multicanonical Monte Carlo and multihistogram reweighting produces very smooth functions
- Since $\lim_{N_z \to \infty} \Delta\beta_c = \beta_{c,t} - \beta_c = 0$, we need an ansatz for extrapolating to β_c

Measuring interface tension of o - o interface

Finite volume



- Multicanonical Monte Carlo and multihistogram reweighting produces very smooth functions
- Since $\lim_{N_z \to \infty} \Delta\beta_c = \beta_{c,t} - \beta_c = 0$, we need an ansatz for extrapolating to β_c

Measuring interface tension of o - o interface

Finite volume effects

Comparing the Z_i and Z_0 systems, with $\lim_{V\to\infty} \hat{\sigma}_V = \sigma/T^3$,

$$Z_i/Z_0 = e^{-\hat{\sigma}_V \frac{L^2}{N_t^2}} = Ae^{-\hat{\sigma}\frac{L^2}{N_t^2}}.$$

The prefactor $A = Z_{bulk}Z_{zm}Z_{cw}$ collects bulk Gaussian fluctuations, zero mode IR divergences $\propto N_z$, and capillary waves K_{cw} [7]:

$$\sigma_V(\beta, N_t, L, N_z) = \sigma(\beta, N_t) - \frac{N_t^2}{L^2}\left(\log N_z + \log\left(K_{cw}\tilde{c}(\beta, N_t)\right)\right).$$

The infinite volume $\sigma(\beta, N_t)$ then extrapolates to T_c with an ansatz that assumes perfect wetting:

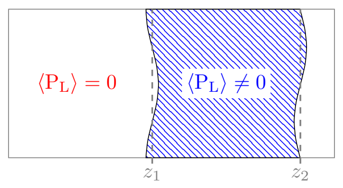
$$\frac{\sigma_{dd}}{T^3} = \frac{2\sigma_{cd}(1-3t)}{T_c^3} + \frac{\delta r_0 t}{T_c^2}\left(1 - \log(\delta r_0 T_c t/\gamma)\right)$$
$$\Rightarrow y = a(1-3x) + bx(1 - \log(cx)).$$

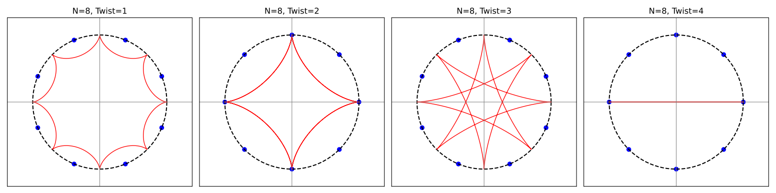
[7] Iwasaki et al. 1994; Holland and Wiese 2000

Measuring interface tension of o - o interface

Perfect wetting

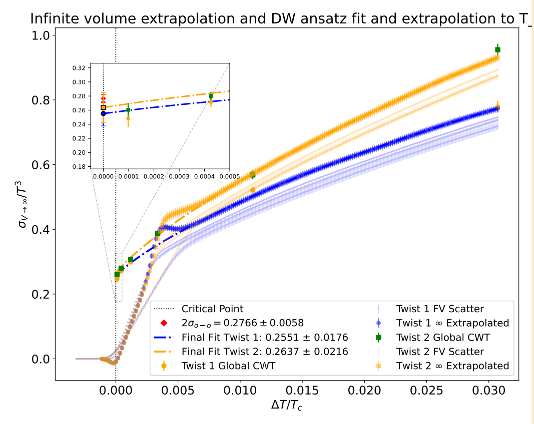
Perfect wetting deconfined-confined surfaces at β_c join "perfectly" into a deconfined-deconfined surface with $\sigma_{oo} = 2\sigma_{od}$





Measuring interface tension of o - o interface

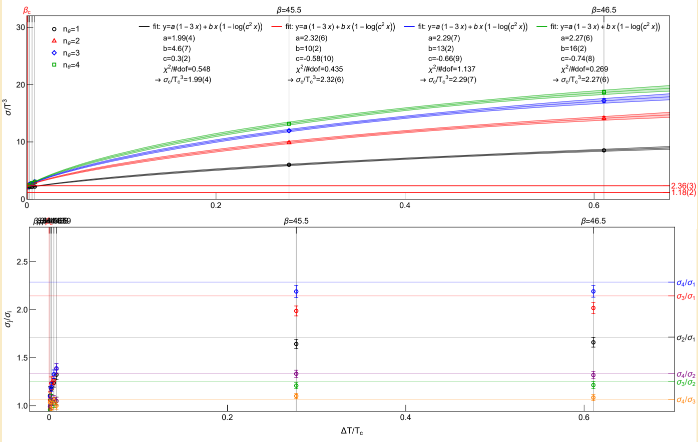
Perfect wetting result in $\mathrm{SU}(4)$



- Perfect wetting aligns with z_2 not necessarally with z_1
- Both methods are in agreement
- CWT easier than integration but can extrapolate to β_c more accurately

Measuring interface tension of o - o interface

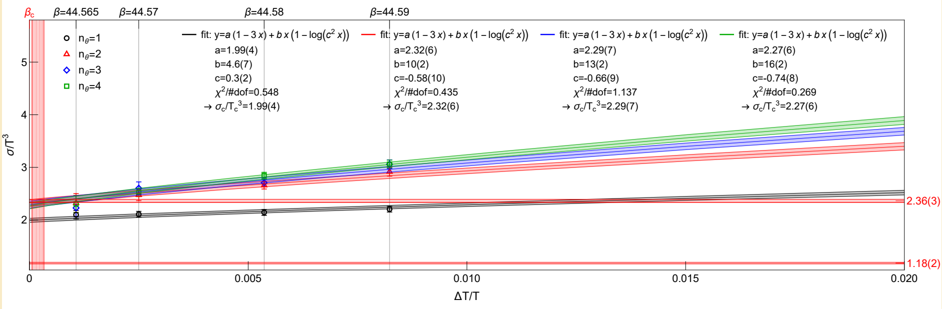
Casimir scaling in $\mathrm{SU}(8)$, $N_t = 6$



Tension ratios are in good agreement with Casimir scaling.

Measuring interface tension of o - o interface

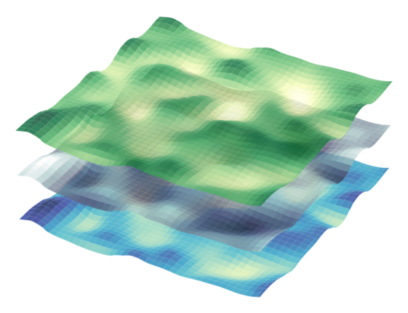
Result in $\mathrm{SU}(8)$, $N_t = 6$



- The z_1 interface tension does not have perfect wetting
- The remaining twists are consistent with perfect wetting, and we cannot resolve whether they differ
- Closer you go to β_c the harder it is to detect the surface

Measuring interface tension of o - o interface

Conclusions



- Both methods are in agreement
- Casimir scaling is in agreement with measurement as $\beta \to \infty$
- Not all twists exhibit perfect wetting

References I

Berg, B. A. and T. Neuhaus (1992). “Multicanonical ensemble: A New approach to simulate first order phase transitions”. In: *Phys. Rev. Lett.* 68, pp. 9–12. DOI: 10.1103/PhysRevLett.68.9. arXiv: hep-lat/9202004.

Brown, Frank R. et al. (Oct. 1988). “Nature of the Deconfining Phase Transition in SU(3) Lattice Gauge Theory”. In: *Phys. Rev. Lett.* 61 (18), pp. 2058–2061. DOI: 10.1103/PhysRevLett.61.2058. URL: https://link.aps.org/doi/10.1103/PhysRevLett.61.2058.

Bursa, Francis and Michael Teper (2005). “Casimir scaling of domain wall tensions in the deconfined phase of D=3+1 SU(N) gauge theories”. In: *JHEP* 08, p. 060. DOI: 10.1088/1126-6708/2005/08/060. arXiv: hep-lat/0505025.

Ferrenberg, Alan M. and Robert H. Swendsen (Dec. 1988). “New Monte Carlo technique for studying phase transitions”. In: *Phys. Rev. Lett.* 61 (23), pp. 2635–2638. DOI: 10.1103/PhysRevLett.61.2635. URL: https://link.aps.org/doi/10.1103/PhysRevLett.61.2635.

Fukugita, Masataka, Masanori Okawa, and Akira Ukawa (1990). “Finite-size scaling study of the deconfining phase transition in pure SU(3) lattice gauge theory”. In: *Nuclear Physics B* 337.1, pp. 181–232. ISSN: 0550-3213. DOI: https://doi.org/10.1016/0550-3213(90)90256-D. URL: https://www.sciencedirect.com/science/article/pii/055032139090256D.

Gattringer, C. and C. Lang (2009). *Quantum Chromodynamics on the Lattice: An Introductory Presentation*. Lecture Notes in Physics. Springer Berlin Heidelberg. ISBN: 9783642018497. URL: https://books.google.fi/books?id=l2hZKnlYDxoC.

Hällfors, Jaakko and Kari Rummukainen (2025). *Expanded ensemble method for bubble nucleation*. arXiv: 2502.14610 [hep-lat]. URL: https://arxiv.org/abs/2502.14610.

References II

Holland, Kieran and Uwe-Jens Wiese (Nov. 2000). "The Center symmetry and its spontaneous breakdown at high temperatures". In: ed. by M. Shifman and Boris Ioffe, pp. 1909–1944. DOI: 10.1142/9789812810458˙0040. arXiv: hep-ph/0011193.

Iwasaki, Y. et al. (Apr. 1994). "Interface tension in quenched QCD". In: *Physical Review D* 49.7, pp. 3540–3545. ISSN: 0556-2821. DOI: 10.1103/physrevd.49.3540. URL: http://dx.doi.org/10.1103/PhysRevD.49.3540.

Kajantie, K., Leo Karkkainen, and K. Rummukainen (1991). "Tension of the interface between two ordered phases in lattice SU(3) gauge theory". In: *Nucl. Phys. B* 357, pp. 693–712. DOI: 10.1016/0550-3213(91)90486-H.

Moore, Guy D. and Neil Turok (May 1997). "Classical field dynamics of the electroweak phase transition". In: *Physical Review D* 55.10, pp. 6538–6560. ISSN: 1089-4918. DOI: 10.1103/physrevd.55.6538. URL: http://dx.doi.org/10.1103/PhysRevD.55.6538.

Polyakov, A.M. (1978). "Thermal properties of gauge fields and quark liberation". In: *Physics Letters B* 72.4, pp. 477–480. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(78)90737-2. URL: https://www.sciencedirect.com/science/article/pii/0370269378907372.

Rindlisbacher, Tobias, Kari Rummukainen, and Ahmed Salami (Dec. 2025). "Confined-deconfined interface tension and latent heat in $\mathrm{SU}(N)$ gauge theory". In: *Phys. Rev. D* 112 (11), p. 114507. DOI: 10.1103/hxfx-8jqb. URL: https://link.aps.org/doi/10.1103/hxfx-8jqb.

Rummukainen, Kari, Riikka Seppä, and David J. Weir (Mar. 2026). "Lattice study of the critical bubble in $\mathrm{SU}(8)$ deconfinement transition". arXiv: 2603.22088 [hep-lat].

References III

Susskind, Leonard (Nov. 1979). "Lattice models of quark confinement at high temperature". In: *Phys. Rev. D* 20 (10), pp. 2610–2618. DOI: 10.1103/PhysRevD.20.2610. URL: https://link.aps.org/doi/10.1103/PhysRevD.20.2610.