The Electromagnetic Form Factor of the Pion at Large Q^2

The 43rd International Symposium on Lattice Field Theory

Ian van Schalkwyk on behalf of the QCDSF Collaboration

Adelaide University

Lattice 2026

Other QCDSF Talks

- Nabil Humphrey *Today 1710, Benjamin Banneker*
  - Tensor-polarised structure functions of the rho meson from lattice QCD using the Feynman-Hellmann method.
- Jordan Mckee *Wednesday at 1130, Benjamin Banneker*
  - Nachtmann moment of the parity-violating structure function from Feynman-Hellmann
- Mischa Batelaan, *Wednesday 1130, Margaret Brent B*
  - η and η' production in radiative decays of chromium

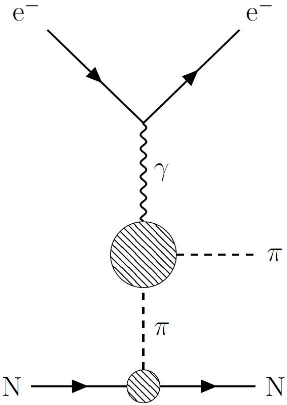
Motivation

- Pion is a pseudo-Goldstone boson
  - $m_\pi \approx \sqrt{m_q}$
- Investigate transition from non-perturbative to perturbative QCD
- Future experiments at JLab and EIC with momentum transfers greater than $Q^2 = 30\,\mathrm{GeV}^2$

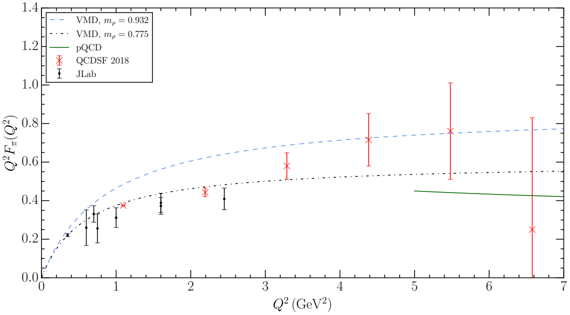
$$\langle \pi(\mathbf{p}') | \mathcal{V}(0) | \pi(\mathbf{p}) \rangle = -i(p' + p)_\mu F_\pi(Q^2)$$

Where,

$$\mathcal{V}_\mu(z) = \bar{q}(z)\gamma_\mu q(z)$$



Our Approach



1. Feynman-Hellmann (FH) technique to exact $F_\pi(Q^2)$
2. Modified momentum smearing
3. All-mode averaging
4. Variational techniques between interpolating operators

The Feynman-Hellmann Method

As in Quantum mechanics, the FH theorem relates small perturbation in the action to change in expectation value.

$$\frac{\partial E(\lambda)}{\partial \lambda} = \left\langle \psi \middle| \frac{\partial H(\lambda)}{\partial \lambda} \middle| \psi \right\rangle.$$

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For this work, we use the perturbation

$$S_{\mathrm{QCD}} \to S_{\mathrm{QCD}}(\lambda) = S_{\mathrm{QCD}} + \lambda \int d^4 z (e^{iq\cdot z} + e^{-iq\cdot z}) \mathcal{V}_4(z).$$

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This allows access to the following matrix element of interest

$$\left.\frac{\partial E(\lambda)}{\partial \lambda}\right|_{\lambda=0} = \langle \pi(\mathbf{p}') | \mathcal{V}_4(0) | \pi(\mathbf{p}) \rangle = -i \frac{[p' + p]_4}{2E_\pi(\mathbf{p})} F_\pi(Q^2)$$

The Feynman-Hellmann Method

- Only Breit frame pairs receive energy shift linear in[1] λ i.e. $\mathbf{p}' = -\mathbf{p}$

Example

Consider a momentum transfer of $\mathbf{q} = (\pm 4, 0, 0)\ (\frac{2\pi}{L})$.

- A sink momentum of $\mathbf{p}' = (2, 0, 0)\ (\frac{2\pi}{L})$ couples to source momenta:
  - $\mathbf{p} = (-2, 0, 0)(\frac{2\pi}{L})$, and,
  - $\mathbf{p} = (6, 0, 0)(\frac{2\pi}{L})$
- A sink momentum of $\mathbf{p}' = (-2, 0, 0)\ (\frac{2\pi}{L})$ couples to source momenta
  - $\mathbf{p} = (2, 0, 0)(\frac{2\pi}{L})$, and,
  - $\mathbf{p} = (-6, 0, 0)(\frac{2\pi}{L})$

[1] Phys. Rev. D 96, 114506

Momentum Smearing

- Method of quark smearing from the RQCD collaboration in 2016[2]
- Modifies existing quark smearing routines with modification to smearing kernel Φ for quark momentum $\mathbf{k}$

$$\Phi_{\mathbf{xy}} \to \Phi_{\mathbf{xy}}^{(\mathbf{k})} = e^{-i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\Phi_{\mathbf{xy}}$$

- Relate quark momentum $(\mathbf{k})$ to hadron momentum $(\mathbf{p})$ using $\mathbf{p}_{\text{meson}} = 2\mathbf{k}$, and $\mathbf{p}_{\text{baryon}} = 3\mathbf{k}$
- This significantly improves signal for hadrons with momentum $\mathbf{p}$

[2] Phys.Rev.D 93 (2016) 9, 094515

Momentum Smearing

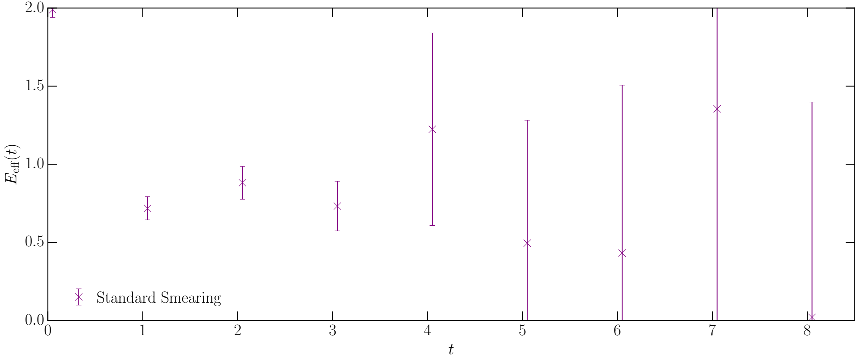
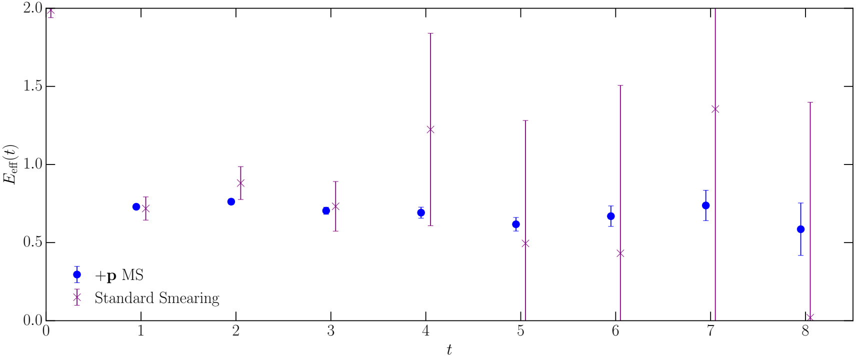


Figure: Effective mass of pion with momentum $p = (2, 2, 0)\frac{2\pi}{L}$

Momentum Smearing



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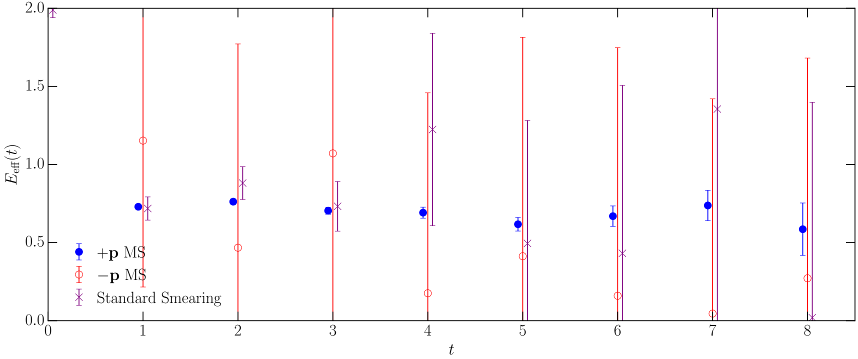


Figure: Effective mass of pion with momentum $p = (2,2,0)\frac{2\pi}{L}$

Momentum-averaged Smearing

- Problematic because of the superposition $\mathbf{p} \pm \mathbf{q}$
- Instead, at source, superimpose two momentum smeared sources smeared with $\mathbf{p}$ and $-\mathbf{p}$ respectively.

Example

Consider again a momentum transfer of $\mathbf{q} = (\pm 4, 0, 0)\left(\frac{2\pi}{L}\right)$.

1. Construct two sources at same location, one smeared for $\mathbf{p} = (2, 0, 0)\left(\frac{2\pi}{L}\right)$ and one for $\mathbf{p} = (-2, 0, 0)\left(\frac{2\pi}{L}\right)$
2. Average these two sources and perform propagator inversion as usual
3. At sink then smear as required
    - In our case, smear propagator for $\mathbf{p}$ and $-\mathbf{p}$ at sink and again average

Momentum-Averaged Smearing

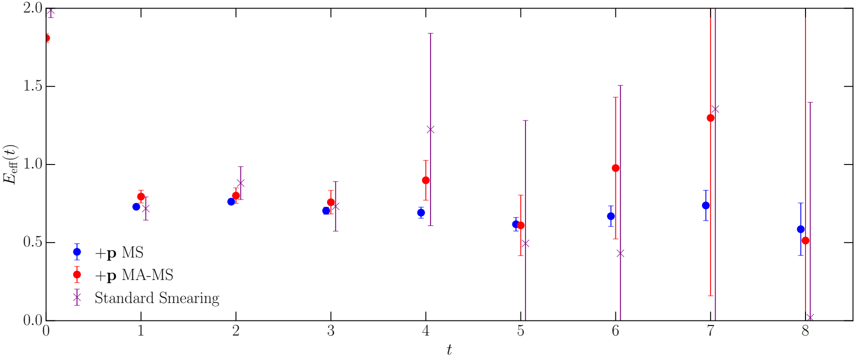


Figure: Effective mass of pion with momentum $p = (2,2,0)\frac{2\pi}{L}$

Momentum-Averaged Smearing

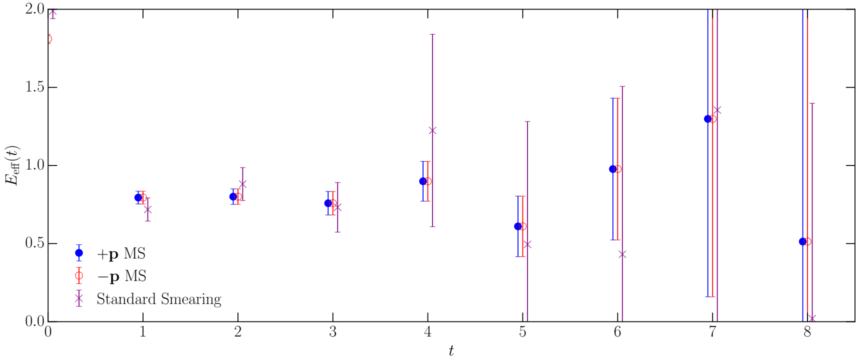


Figure: Effective mass of pion with momentum $p = (2, 2, 0)\frac{2\pi}{L}$

Momentum-Averaged Smearing

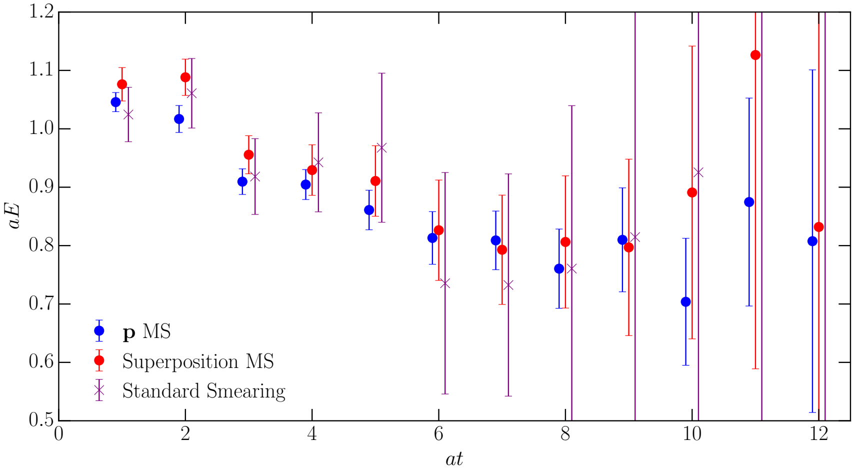


Figure: Effective mass of baryon with momentum $p = (3, 0, 0)\frac{2\pi}{L}$

All-Mode Averaging

Correction factor:

$$C_{\text{corr}} = \frac{1}{N_{\text{strict}}} \sum_{i=1}^{N_{\text{strict}}} (C^i_{\text{strict}} - C^i_{\text{approximate}})$$

Improved Approximate Correlator:

$$C^j_{\text{improved}} = C^j_{\text{approximate}} + C_{\text{corr}}$$

The AMA[3] correlators then are:

$$C_{\text{AMA}} = \frac{1}{N_{\text{approximate}}} \left(\sum_{i=1}^{N_{\text{strict}}} C^i_{\text{strict}} + \sum_{j=N_{\text{strict}+1}}^{N_{\text{approximate}}} C^j_{\text{improved}} \right)$$

$C_{\text{approximate}}$ are ≈ 4 times faster to calculate than C_{strict}

[3] Phys. Rev. D 91, 114511

Variational Techniques

We construct a GEVP using basis of interpolating operators

$$\begin{pmatrix} C_{(\gamma 5,\ \gamma 5)} & C_{(\gamma_5,\ \gamma_4\gamma_5)} \\ C_{(\gamma_4\gamma_5,\ \gamma_5)} & C_{(\gamma_4\gamma_5,\ \gamma_4\gamma_5)} \end{pmatrix}$$

After diagonalising time dependence of C an improved may be extracted.
See also [a]

[a] Phys.Rev.D 112 (2025) 5

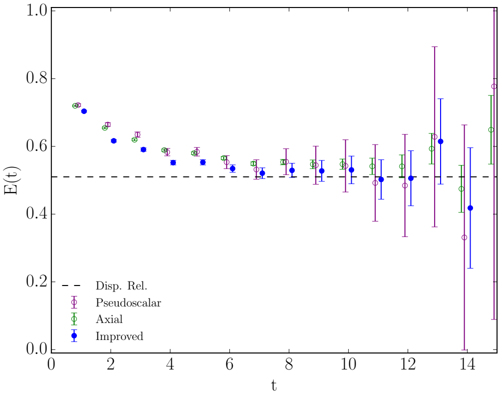


Figure: $\mathbf{p} = (3, 2, 1)\left(\frac{2\pi}{L}\right)$

Simulation Details: Pion Form Factor

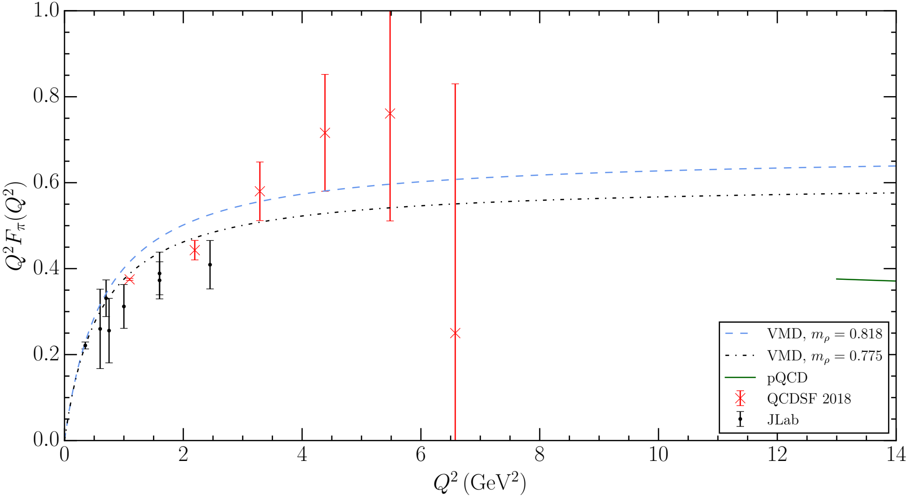
β	$\kappa_l,\ \kappa_s$	$N_L \times N_T$	m_π (MeV)	N_{cfg}
5.65	$0.122005, 0.122005$	48×96	413	550

Table: Configuration Details

Q^2 (GeV^2)	$\mathbf{q}\ \left(\frac{2\pi}{L}\right)$	$\mathbf{p}\ \left(\frac{2\pi}{L}\right)$	N_{strict}	$N_{\text{approximate}}$
4.62	$(4, 4, 0)$	$(2, 2, 0)$	1	10
8.08	$(6, 4, 2)$	$(3, 2, 1)$	1	25
10.39	$(6, 6, 0)$	$(3, 3, 0)$	1	25

Table: Simulation Parameters, for $\text{tol}_{\text{strict}} = 10^{-12}, \text{tol}_{\text{approximate}} = 10^{-5}$

The Electromagnetic Form Factor of the Pion at the $\mathrm{SU}(3)$ Symmetric Point ($m_\pi = 413\,\mathrm{MeV},\ a = 0.068\,\mathrm{fm}$)



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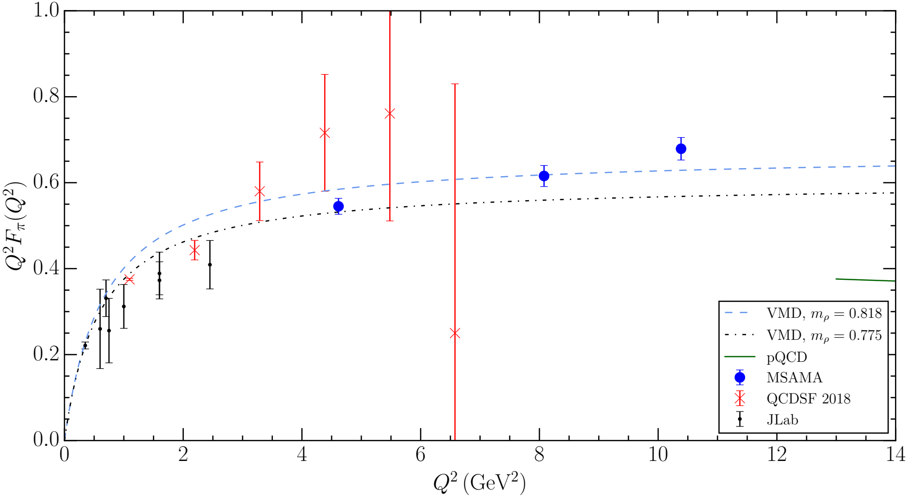
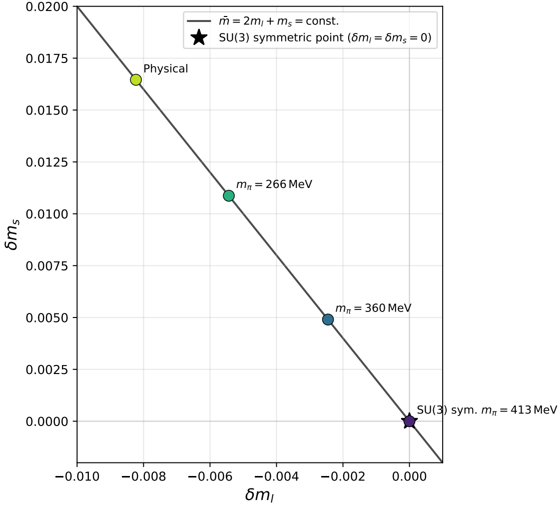


Flavour Breaking Effects



$$\bar{m} = 2m_l + m_s = \text{constant}$$

Where δ_l are defined as

$$\delta m_l = \frac{1}{2}\left(\frac{1}{\kappa_l} - \frac{1}{\kappa_0}\right) = \frac{1}{2}\delta m_s$$

Where κ_0 is κ at the $\mathrm{SU}(3)$ symmetric point.
Relate δm_l and δm_s with

$$\delta m_s = -2\delta m_l$$

Simulation Details: Flavour Breaking Effects

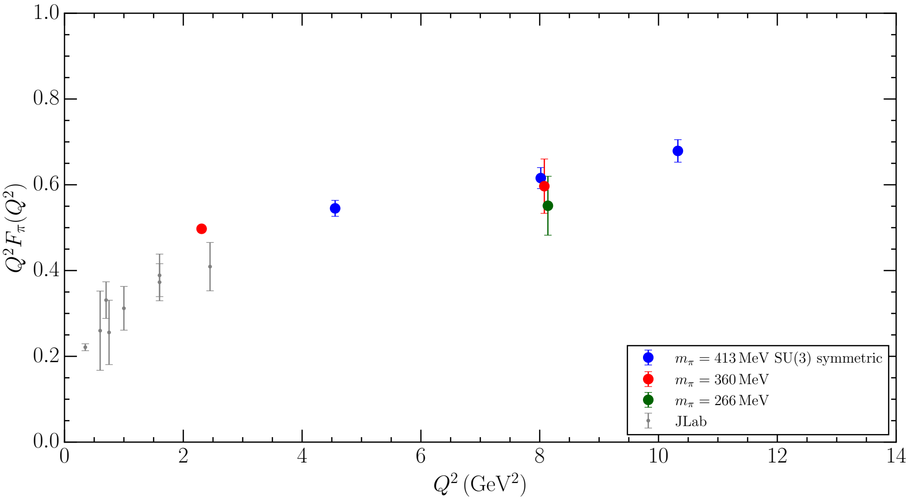
β	$\kappa_l,\ \kappa_s$	$N_L \times N_T$	m_π (MeV)	m_K (MeV)	N_{cfg}
5.65	0.122005, 0.122005	48×96	413	413	550
5.65	0.122078, 0.121859	48×96	360	445	550
5.65	0.122167, 0.121682	48×96	266	475	550

Table: Configuration Details

Q^2 (GeV^2)	$\mathbf{q}$ $(\frac{2\pi}{L})$	$\mathbf{p}$ $(\frac{2\pi}{L})$	N_{strict}	$N_{\text{approximate}}$
8.08	$(6,4,2)$	$(3,2,1)$	1	25

Table: Simulation Parameters for $\text{tol}_{\text{strict}} = 10^{-12}, \text{tol}_{\text{approximate}} = 10^{-5}$

Pion Form Factor



Kaon Form Factor

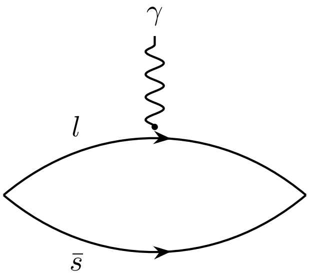


Figure: $F_K^l(Q^2)$

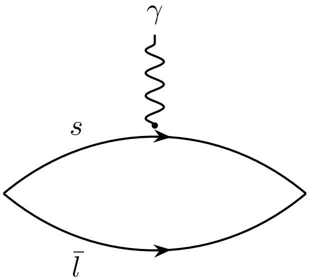
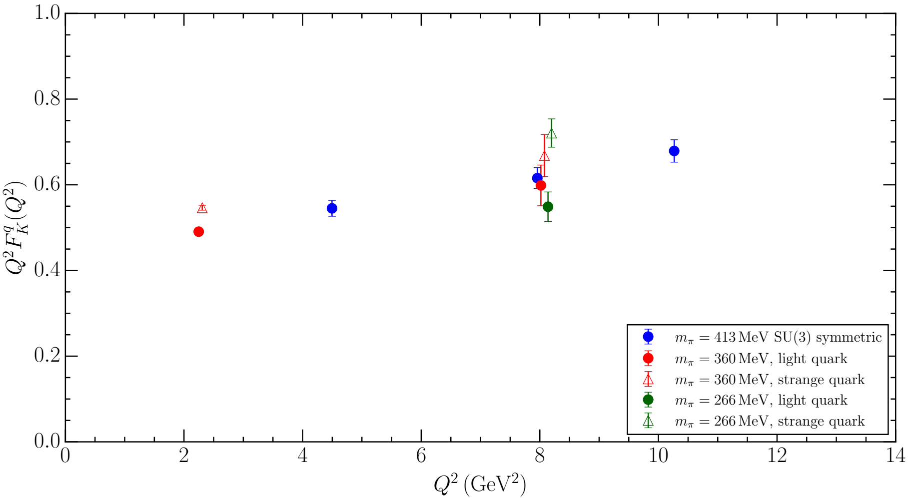


Figure: $F_K^s(Q^2)$

Kaon Form Factor



Flavour Breaking Expansion

For multiple m_l and m_s mass splittings, δm_l, along the $\bar{m}$-constant line, we can describe the flavour breaking effects using fan plots[4].

Note $A_{K\eta K} = \langle K|J^\eta|K\rangle$
$J^\pi = \frac{1}{\sqrt{2}}(\bar{u}\gamma u - \bar{d}\gamma d)$ and $J^\eta = \frac{1}{\sqrt{6}}(\bar{u}\gamma u + \bar{d}\gamma d - 2\bar{s}\gamma s)$

$$F_1 \equiv \frac{1}{\sqrt{3}}(A_{K\eta K} - A_{\bar{K}\eta\bar{K}}) = \frac{2}{\sqrt{3}}A_{K\eta K} = 2f - \frac{2}{\sqrt{3}}s_2\delta m_l,$$
$$F_2 \equiv (A_{K\pi K} + A_{\bar{K}\pi\bar{K}}) = 2A_{K\pi K} = 2f + 4s_1\delta m_l,$$
$$F_3 \equiv A_{\pi\pi\pi} = 2f + (-2s_1 + \sqrt{3}s_2)\delta m_l,$$
$$F_4 \equiv \frac{1}{\sqrt{2}}(A_{\pi K\bar{K}} - A_{KK\pi}) = 2f - 2s_1\delta m_l,$$
$$F_5 \equiv \frac{1}{\sqrt{3}}(A_{\eta K\bar{K}} - A_{KK\eta}) = 2f - \frac{2}{\sqrt{3}}(\sqrt{3}s_1 - s2)\delta m_l.$$

[4] Phys.Rev.D 84 (2011) 054509, Phys.Rev.D 100 (2019) 11, 114516

Flavour Breaking Expansion

Only have access to F_1 to F_3:

$$\begin{aligned} F_1 &\equiv \frac{1}{\sqrt{3}}(A_{K\eta K} - A_{\bar{K}\eta\bar{K}}) = \frac{2}{\sqrt{3}}A_{K\eta K} = 2f - \frac{2}{\sqrt{3}}s_2\delta m_l, \\ F_2 &\equiv (A_{K\pi K} + A_{\bar{K}\pi\bar{K}}) = 2A_{K\pi K} = 2f + 4s_1\delta m_l, \\ F_3 &\equiv A_{\pi\pi\pi} = 2f + (-2s_1 + \sqrt{3}s_2)\delta m_l, \end{aligned}$$

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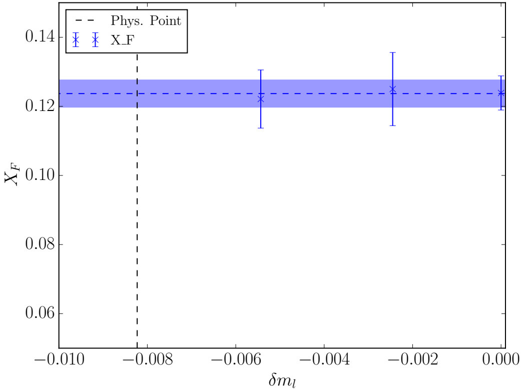
As well as a flavour singlet quantity

$$X_F \equiv \frac{1}{6}(3F_1 + F_2 + 2F_3) = 2f + \mathcal{O}(\delta m_l^2)$$

Flavour Singlet X_F



$$X_F = \frac{1}{6}(3F_1 + F_2 + 2F_3) = 2f + \mathcal{O}(\delta m_l^2)$$

$$Q^2 = 8.08\,\mathrm{GeV}^2,\ m_\pi = 266, 360.413\,\mathrm{MeV}$$

Fan-Plot

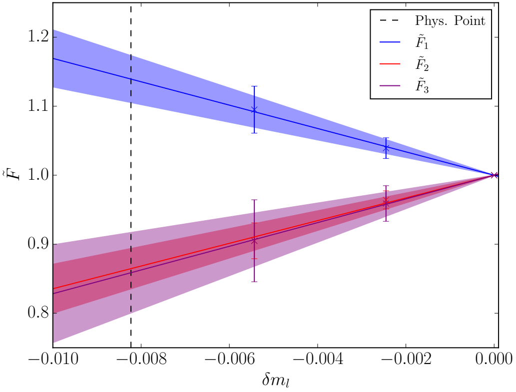
$$\tilde{F}_1 \equiv F_1/X_F = 1 - \frac{2}{\sqrt{3}} a_2 \delta m_l,$$

$$\tilde{F}_2 \equiv F_2/X_F = 1 + 4a_1 \delta m_l,$$

$$\tilde{F}_3 \equiv F_3/X_F = 1 + (-2a_1 + \sqrt{3}a_2)\delta m_l.$$

Where $a_i = s_i/2f$

$$Q^2 = 8.08\,\mathrm{GeV}^2,\ m_\pi = 266, 360.413\,\mathrm{MeV}$$



Summary

- We have tested and implemented new momentum-averaged smearing technique to our existing FH method
- Increase precision of the $F_\pi(Q^2)$ calculation using AMA and variation techniques
- Shown preliminary analysis of $F_\pi(Q^2)$ behaviour at different quark mass splittings

Future work:

- Investigate more $\mathrm{SU}(3)$ broken configurations and fit f-fan at greater than linear order
- Investigate f-fan at variety of Q^2
- Extrapolate Form Factors to physical point

Thanks and Acknowledgements

Pheonix, Pawsey, Gottigen, GSF, Grant?

Backup Slides

Environmental Effect