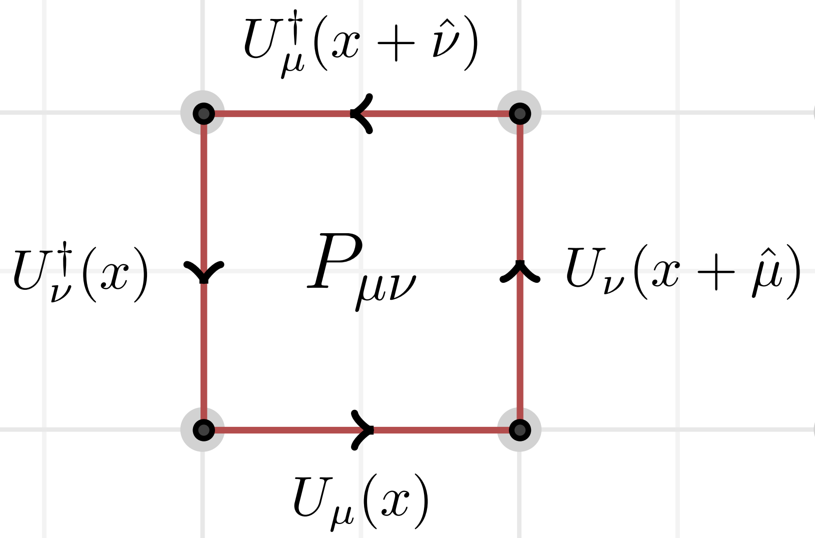
Sp(4) lattice gauge theory at finite temperature in the presence of fermions



Alexis Verney-Provatas, Chris Allton, Ed Bennett, Luigi Del Debbio, Ryan C. Hill, Deog Ki Hong, Jong-Wan Lee, C.-J. David Lin, Biagio Lucini, Maurizio Piai and Davide Vadacchino

(on behalf of the TELOS collaboration)

28 July 2026

Motivation

[1] See e.g. LISA collaboration [1702.00786], LIGO Scientific collaboration [1607.08697], M. Maggiore et al. [1912.02622]
[2] See e.g. V. Brdar et al. [1810.12306], E. Witten (Phys. Rev. D 30 (1984) 272)

Sp(2N) has not been studied at finite temperature with fermions.*

Next generation of gravitational wave detection experiments [1]:

- Window into primordial evolution of the universe
- First order phase transition → Relic Stochastic Gravitational Wave (SGW) Signal [2]

Strongly coupled, asymptotically free gauge theories:

- Many experience deconfinement phase transition at high T
- Studies of SU(N) Yang-Mills theories in the SGW context [3]
- Detectability of SGW signal depends on the strength of the phase transition [3]

Sp(4) with $N_f = 2$ fermions in fundamental representation: realisation of SIMP Dark Matter

* For $N \geq 2$

[3] See e.g. W.-C. Huang et al. [2012.11614], J. Correia [2505.17824]

Sp(4) gauge theory with $N_f = 2$ fermions in the fundamental representation

The Sp(2N) group family [1] is defined as:

$$\mathrm{Sp}(2N) \leq \mathrm{SU}(2N)$$

$$\forall\ U \in \mathrm{Sp}(2N) : U\Omega U^T = \Omega$$

$$\Omega = \begin{bmatrix} \mathbb{O}_{N\times N} & \mathbb{I}_{N\times N} \\ -\mathbb{I}_{N\times N} & \mathbb{O}_{N\times N} \end{bmatrix}$$

The theory is described by the Lagrangian density:

$$\mathscr{L} = -\frac{1}{2}\mathrm{Tr}G_{\mu\nu}G^{\mu\nu} + m_f \sum_j^{N_f} \overline{\psi}_\alpha^{(j)}\psi^{(j)\alpha} + \frac{1}{2}\sum_j^{N_f}\left(i\overline{\psi}_\alpha^{(j)}\gamma^\mu D_\mu \psi^{(j)\alpha} - i(D_\mu \overline{\psi}_\alpha^{(j)})\gamma^\mu\psi^{(j)\alpha}\right)$$

- Spectrum of the theory at $T = 0$ has been studied [2].

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

$$D_\mu = \partial_\mu + igA_\mu$$

[1] E. Bennett et al. [2304.01070]
[2] E. Bennett et al. [1909.12662]

Sp(4) gauge theory with $N_f = 2$ fermions in the fundamental representation

Programme of studying symplectic gauge theories on the lattice

Other Sp(4) talks at Lattice 2026:

- N. Brito: "Progress on scalar singlets and glueballs in lattice gauge theories coupled to matter fields in multiple representations (2 fundamental + 3 antisymmetric)"
- G. Simonetti: "Progress on Sp(4) Gauge Theory with $N_f = 2$ Fundamental Möbius Domain Wall Fermions"

Sp(4) Yang-Mills

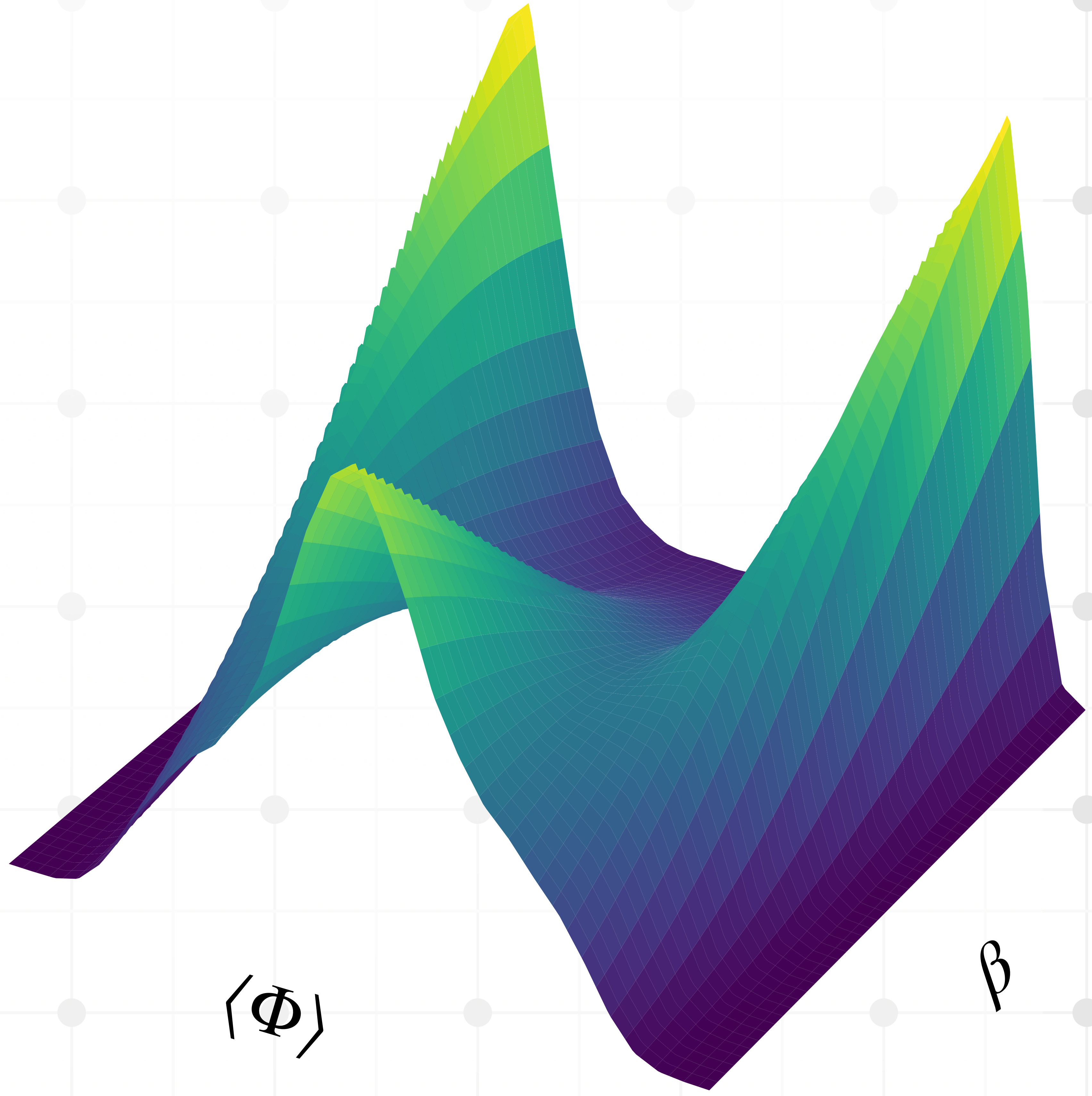
At finite temperature

The deconfinement phase transition:

- Polyakov loop is order parameter
- Is a first order phase transition [1]

Theory studied in the context of dark matter: [2]

- Symplectic glueball model of dark matter
- Computation of critical temperature, latent heat and interface tension



Reweighted Polyakov loop histograms

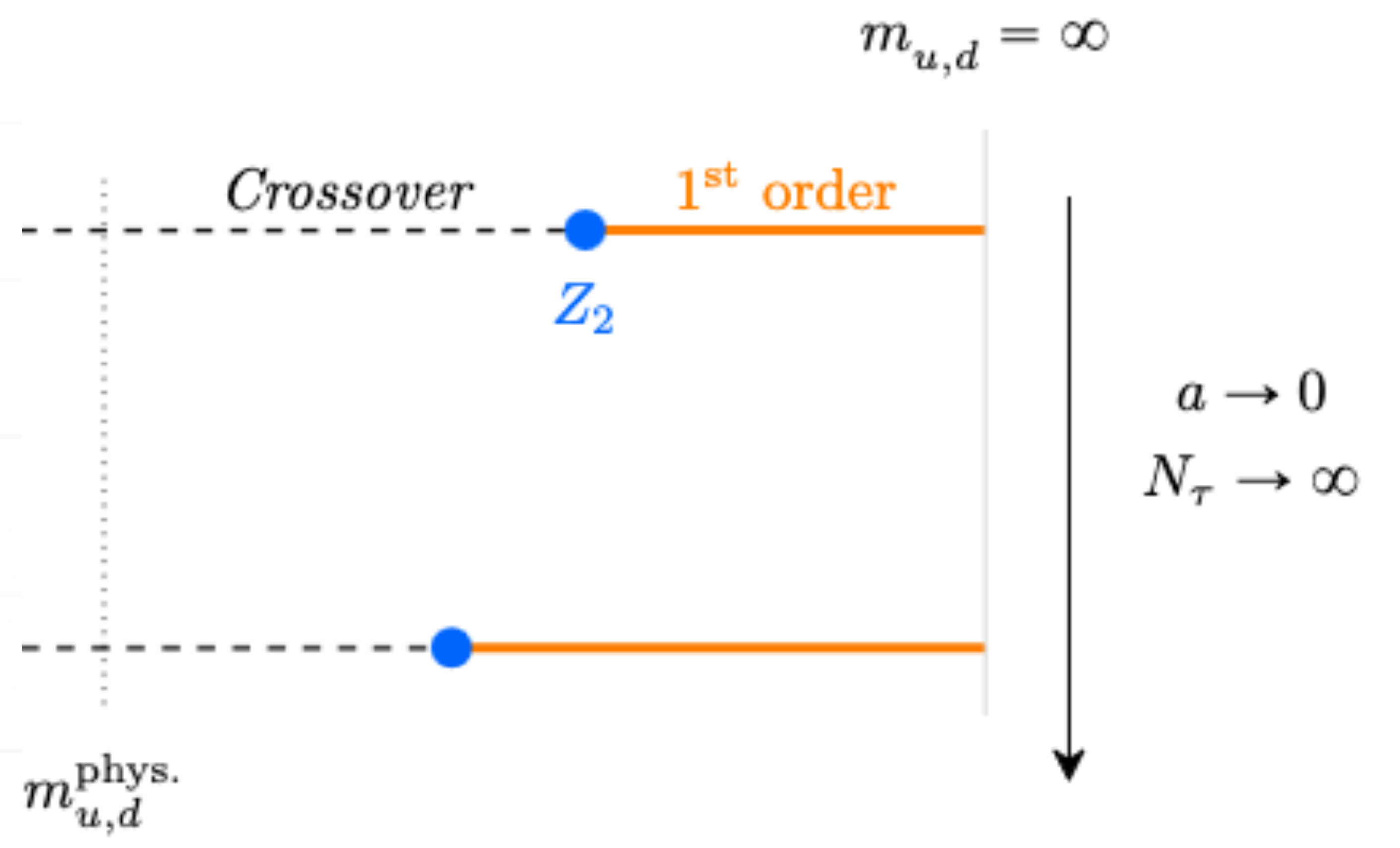
$$\Phi(\vec{x}) = \frac{1}{N_c}\mathrm{Tr}\left[\prod_{t=0}^{N_t-1} U_0(t,\vec{x})\right]$$

[1] K. Holland et al. [0312022], E. Bennett et al [2509.19009]

[2] M. Bruno et al. [2410.17122]

Diagram taken from [2]

Adding fermion fields



For the deconfinment phase transition in QCD:

- SU(3) Yang-Mills is first order
- At the physical point it is an analytic crossover [1]

The presence of heavy fermions weakens the phase transition [2]

- A first order region and a crossover are separated by a critical point, which is still a subject of investigation in SU(3)

Recent work has also investigated SU(4) [3]

What happens in the case of symplectic gauge group?

[1] Y. Aoki et al. [0611014]
[2] F. Cuteri et al. [2009.14033]
[3] V. Ayyar et al. [2602.23002]

Numerical approach

Lattice action:

- Wilson gauge action
- $N_f = 2$ dynamical Wilson fermions (fundamental representation)
- Sample the theory using the HMC via Grid on GPUs using `hmcdj`

Observables:

- Inspect Polyakov loop susceptibility scaling with spatial volume

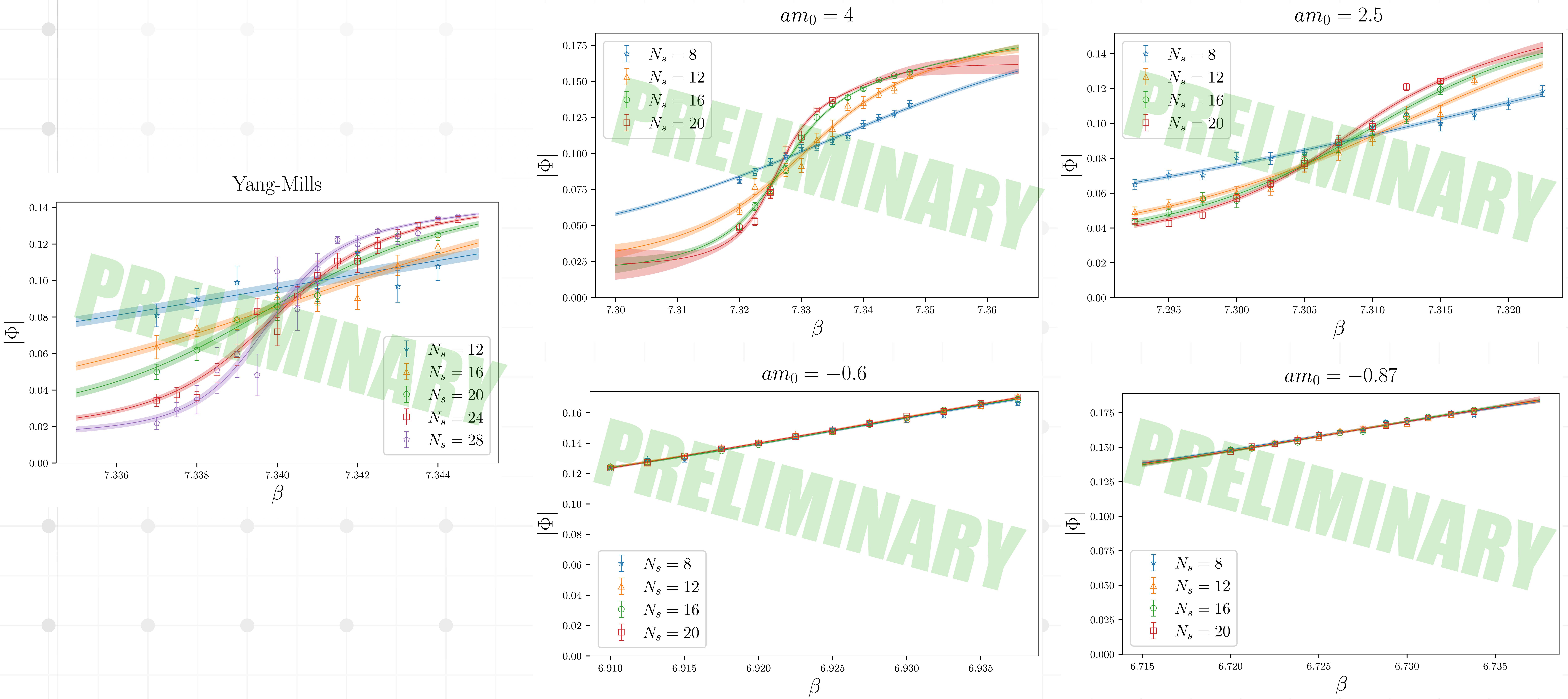
$$\chi_{|\Phi|} = \frac{\tilde{V}}{a}\left(\langle |\Phi|^2 \rangle - \langle |\Phi| \rangle^2\right)$$

- Use multi histogram reweighting (where possible)

$$\Phi(\vec{x}) = \frac{1}{N_c}\mathrm{Tr}\left[\prod_{t=0}^{N_t-1} U_0(t,\vec{x})\right]$$

Results

$N_t = 4$ Polyakov loop

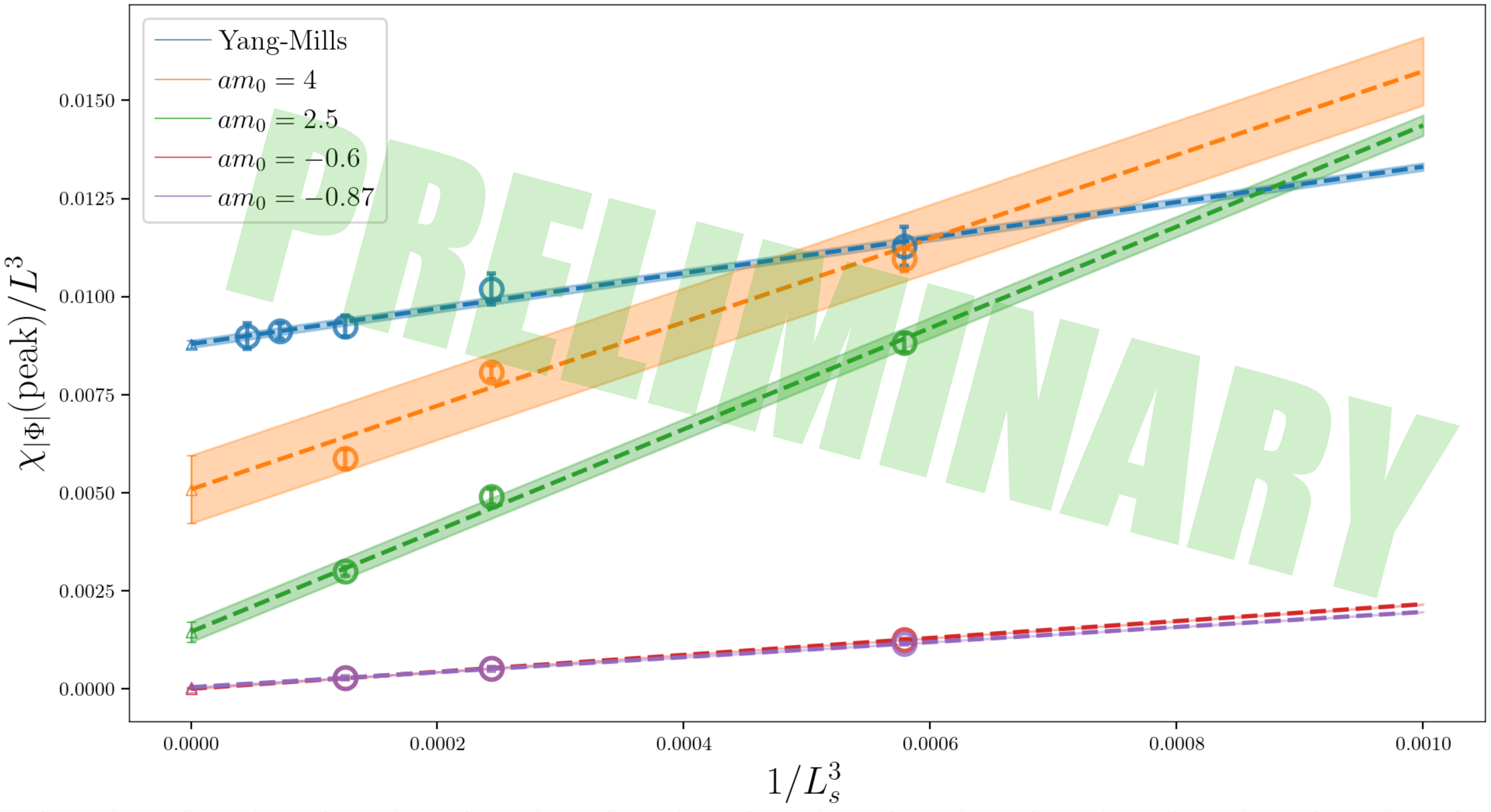


$N_t = 4$ Polyakov loop susceptibility

$N_t = 4$ Polyakov loop susceptibility peak volume scaling

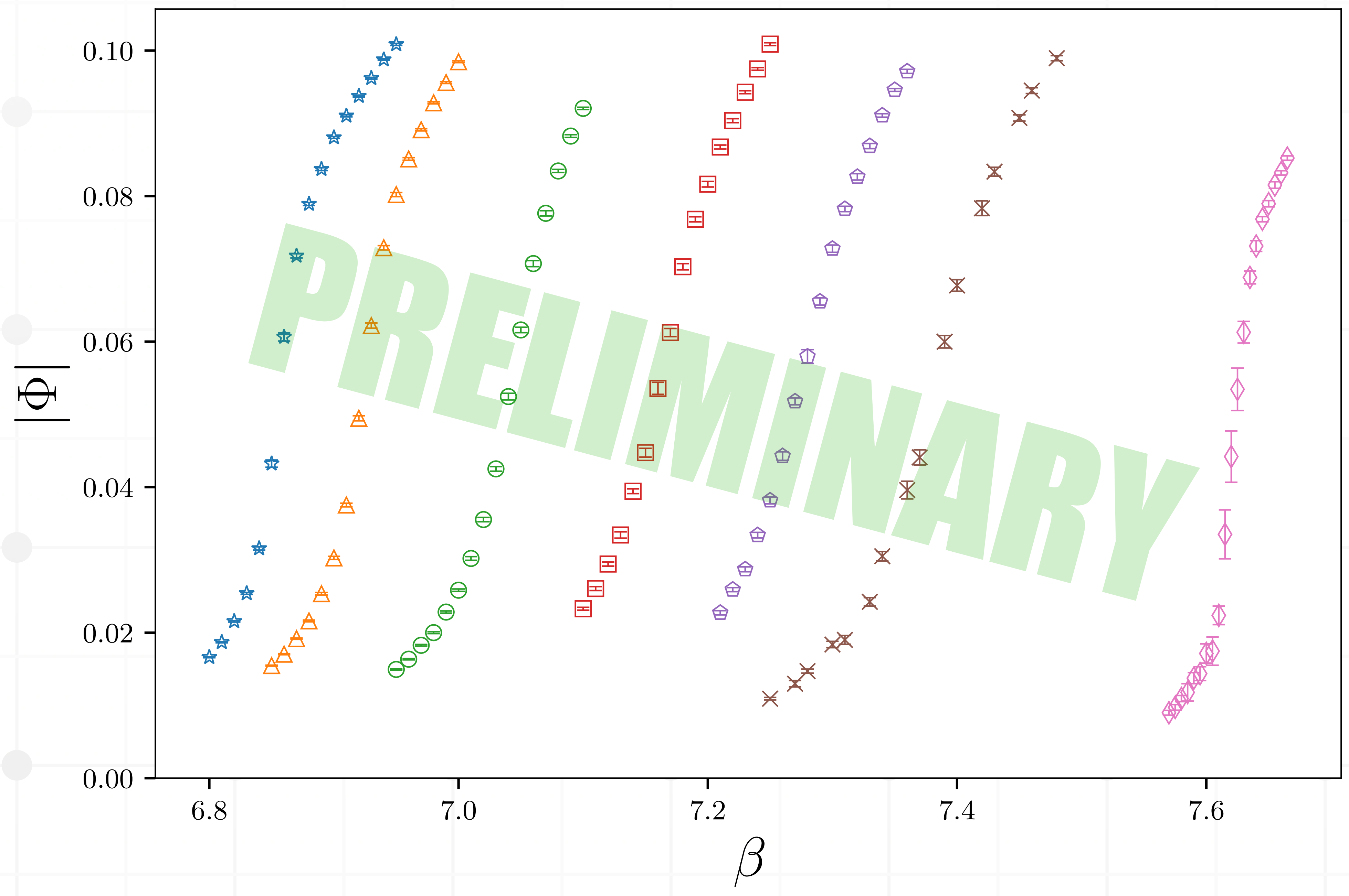
- Inspect the infinite volume limit of $\chi_{|\Phi|} / L_s^3$
- For a crossover this should tend to 0
- For a first order phase transition this should tend to a finite, non-zero constant

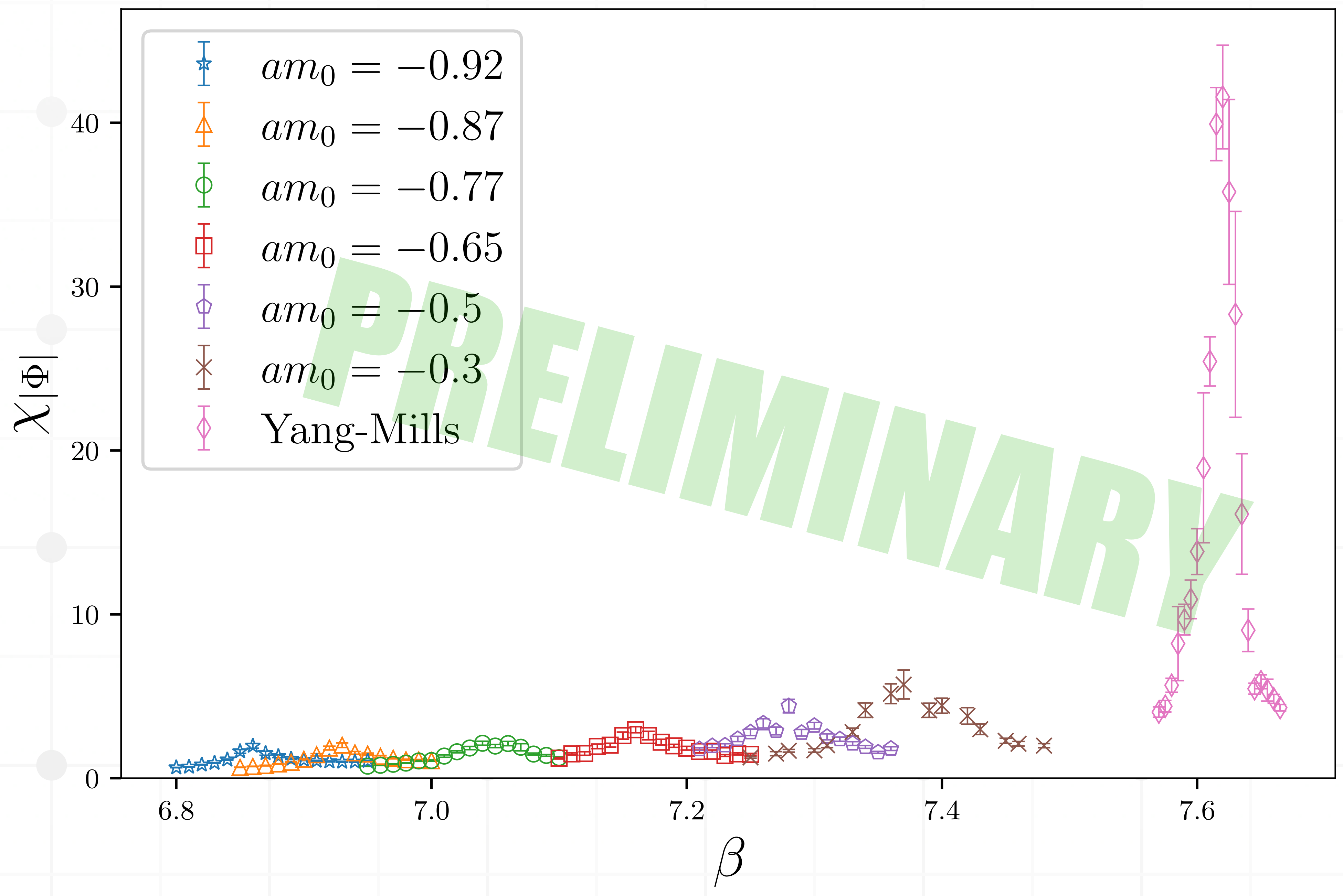
- Yang-Mills is first order
- Heavier masses compatible with first order
- Lighter masses compatible with crossover



- Spatial volumes still quite small

$N_t = 6$ Sparse scan

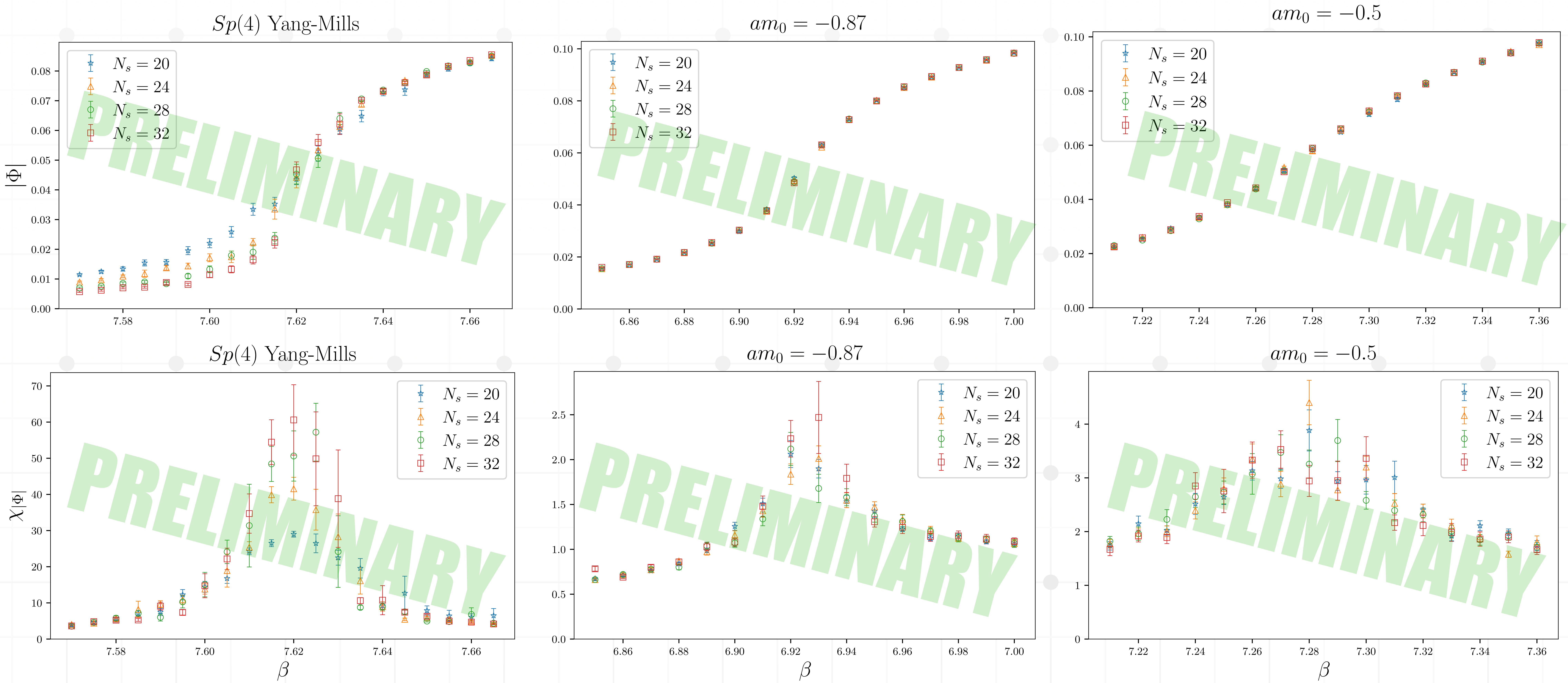




- Preliminary scan of the phase space
- The bare masses checked so far have all been crossovers
- Reweighting can only be used in denser scans in β

Figures from lattices of volume 6×24^3

$N_t = 6$

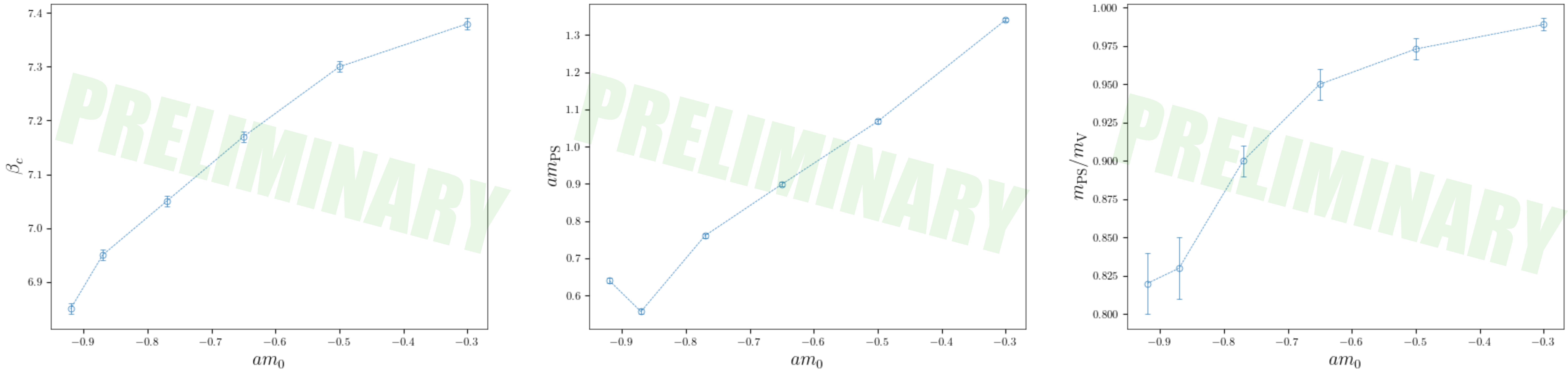


Sp(4) Yang-Mills
am0 = −0.87
am0 = −0.5
Ns = 20
Ns = 24
Ns = 28
Ns = 32
|Φ|
χ|Φ|
β
PRELIMINARY

Scale setting

Performed at β_c for $N_t = 6$

- Estimate value of β_c at each value of am_0
- Perform zero temperature measurements at parameter values



- $am_{\rm PS} > 1$ and $m_{\rm PS} / m_{\rm V} \sim 1$ while still in the crossover phase region
- Require values of (am_0, β_c) calculated at larger N_t
  - For finite volume effects
  - For smaller ratios of $m_{\rm PS} / m_{\rm V}$

Next steps

$N_t = 4$

- Estimation of critical bare mass value requires more data

$N_t = 6$

- Search for first order region in larger bare mass
- Obtaining an estimate on the critical values allows approach to continuum limit

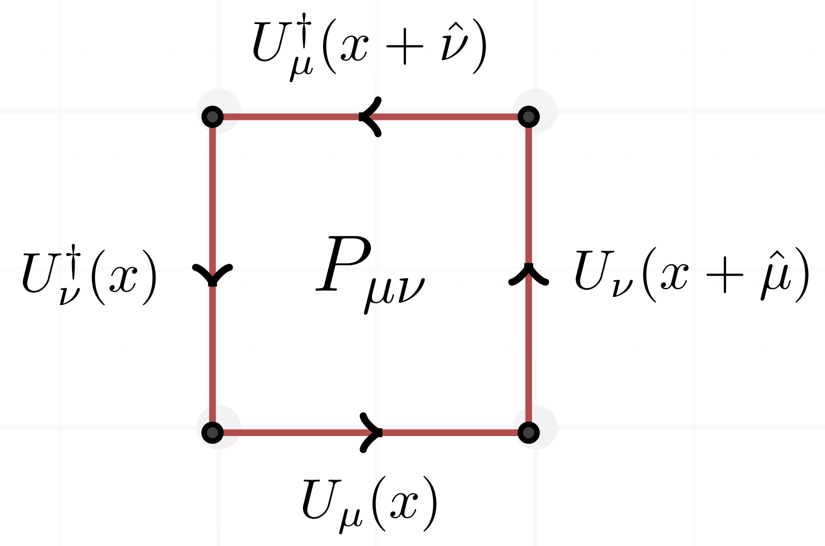
Larger N_t

- Should provide pairs of (am_0, β_c) with more interesting zero temperature physics near the confinement critical point
- Following lines of constant physics would allow estimation of continuum limit values of the latent heat.

Summary

First investigation of Sp(4) at finite temperature with fermions

- $N_f = 2$ Wilson fermions in fundamental representation
- Theory is a minimal realisation of SIMP
- Obtained first picture of finite temperature phase space in the presence of fermions



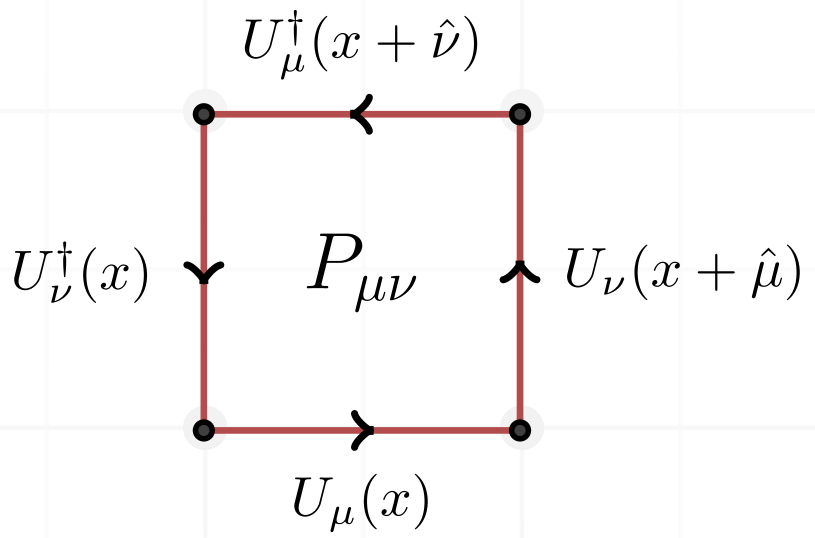
Phase structure at $N_t = 4$

- Crossover at intermediate masses
- Signal of first order phase transition at heavy masses

Next steps

- Extend study to larger N_t aiming at continuum limit
- Larger N_t will also allow scale setting

Backup slides



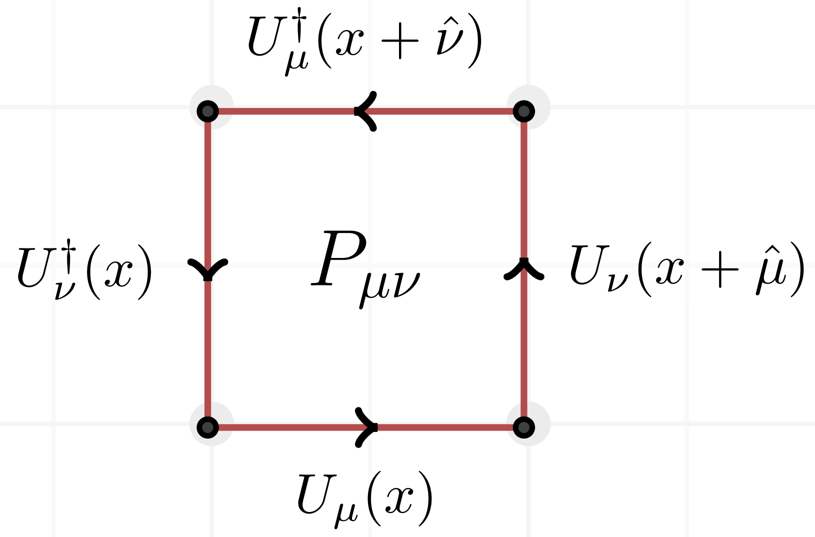
Lattice action

Wilson gauge action coupled to $N_f = 2$ dynamical Wilson fermions in the fundamental representation:

$$S := S_g + S_f$$

Gauge action

$$S_g = \beta \sum_x \sum_{\mu<\nu} \left(1 - \frac{1}{2N}\mathrm{ReTr}P_{\mu\nu}\right)$$



Fermion action

$$S_f = a^4 \sum_{j=1}^{N_f} \overline{\psi}^{(j)} D_m \psi^{(j)}$$

$$D_m\psi(x) = (\frac{4}{a} + m_0)\psi(x) - \frac{1}{2a}\sum_\mu \Big[(1-\gamma_\mu)U_\mu(x)\psi(x+\hat{\mu}) + (1+\gamma_\mu)U_\mu^\dagger(x-\hat{\mu})\psi(x-\hat{\mu})\Big]$$

Multihistogram Reweighting

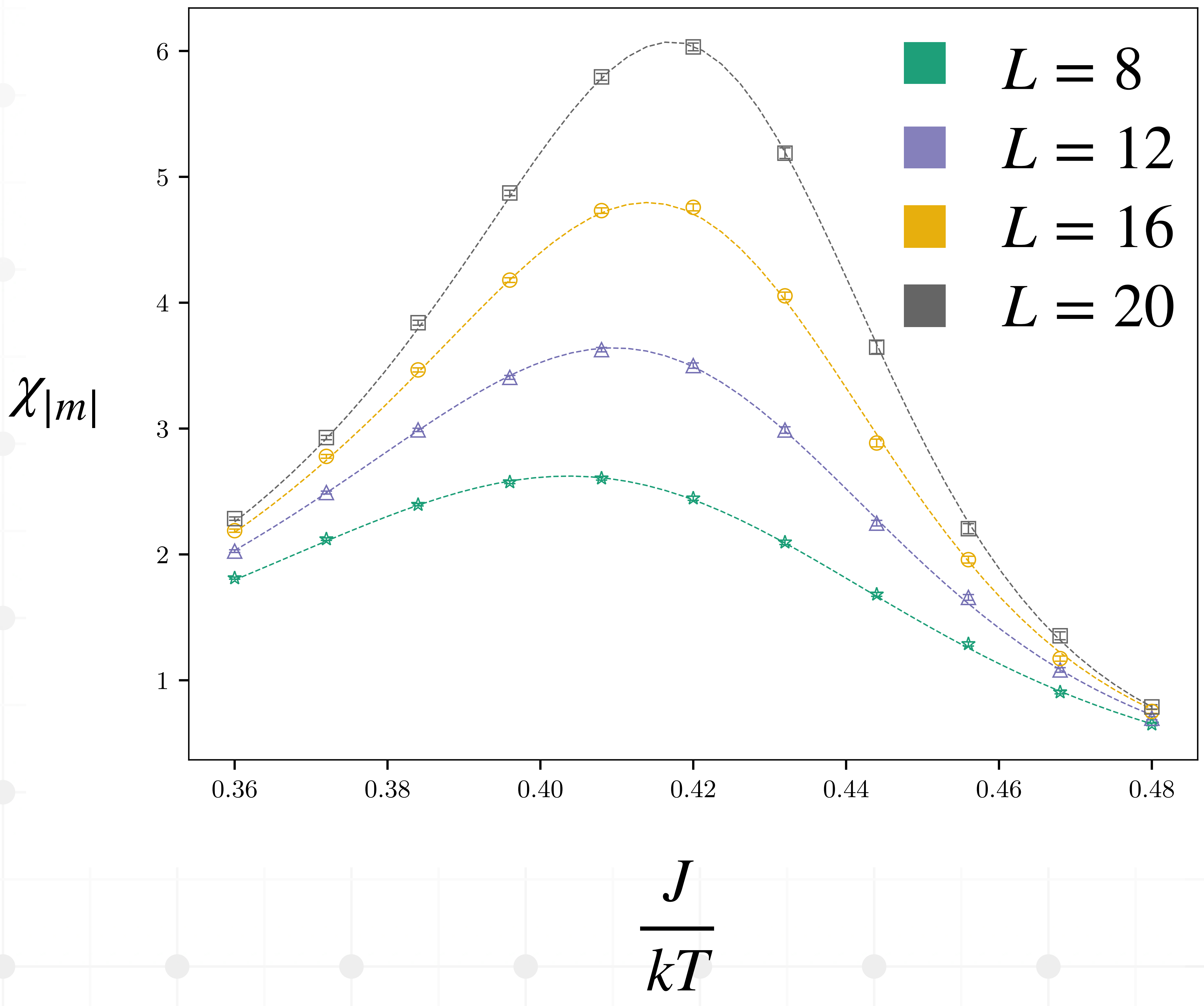
Solve:

$$Z(\beta) = \sum_{i,s} \frac{1}{\sum_j n_j Z_j \exp[(\beta - \beta_j)E_{is}]}$$

- Iteratively for source values β_k
- Fix against drift
- Use logsumexp trick

Interpolate observable O at target β:

$$\langle O(\beta) \rangle = \frac{1}{Z(\beta)} \sum_{i,s} \frac{O_{i,s}}{\sum_j n_j Z_j \exp[(\beta - \beta_j)E_{is}]}$$



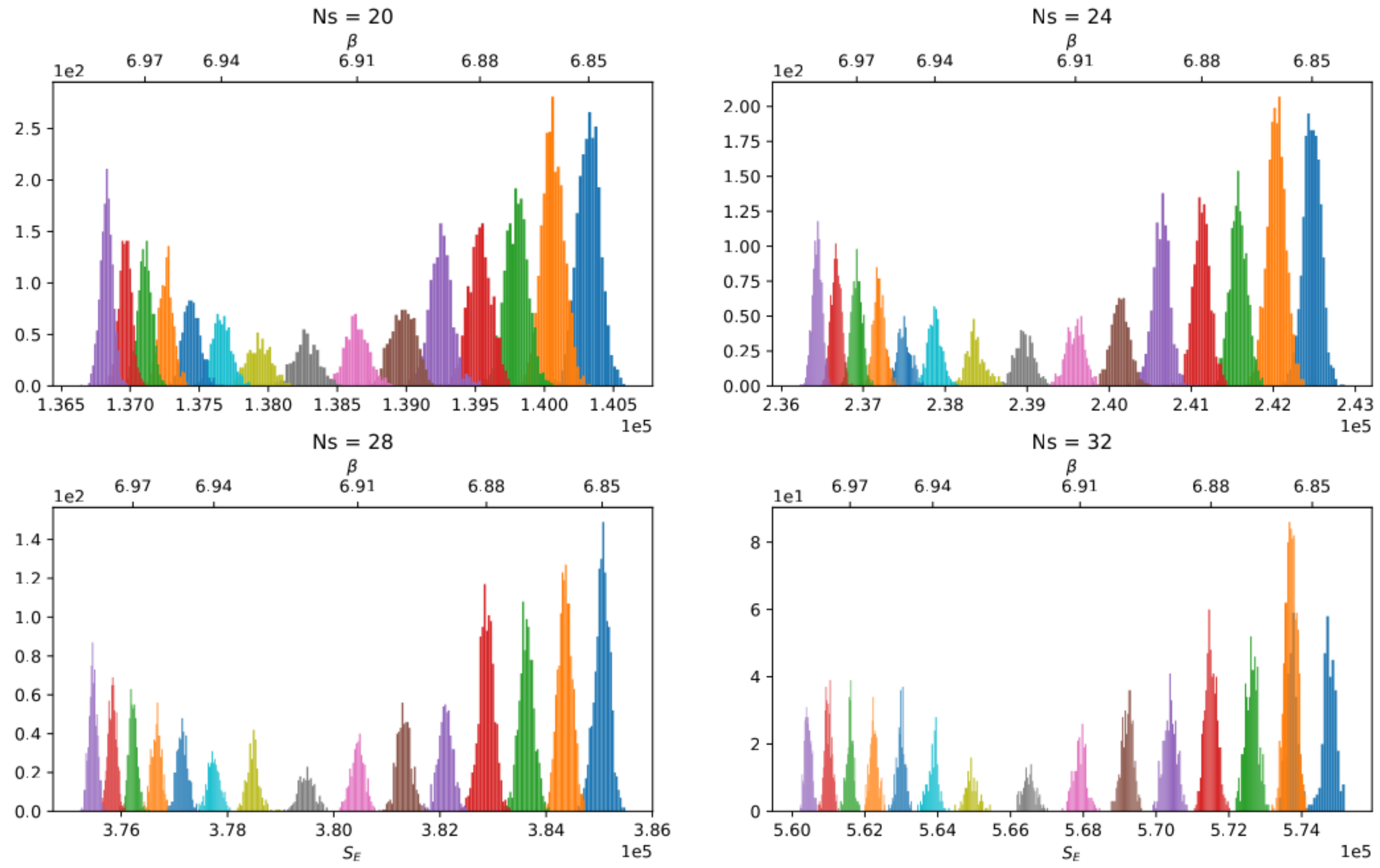
Reweighted 2D Ising model magnetic susceptibility at $H = 0$

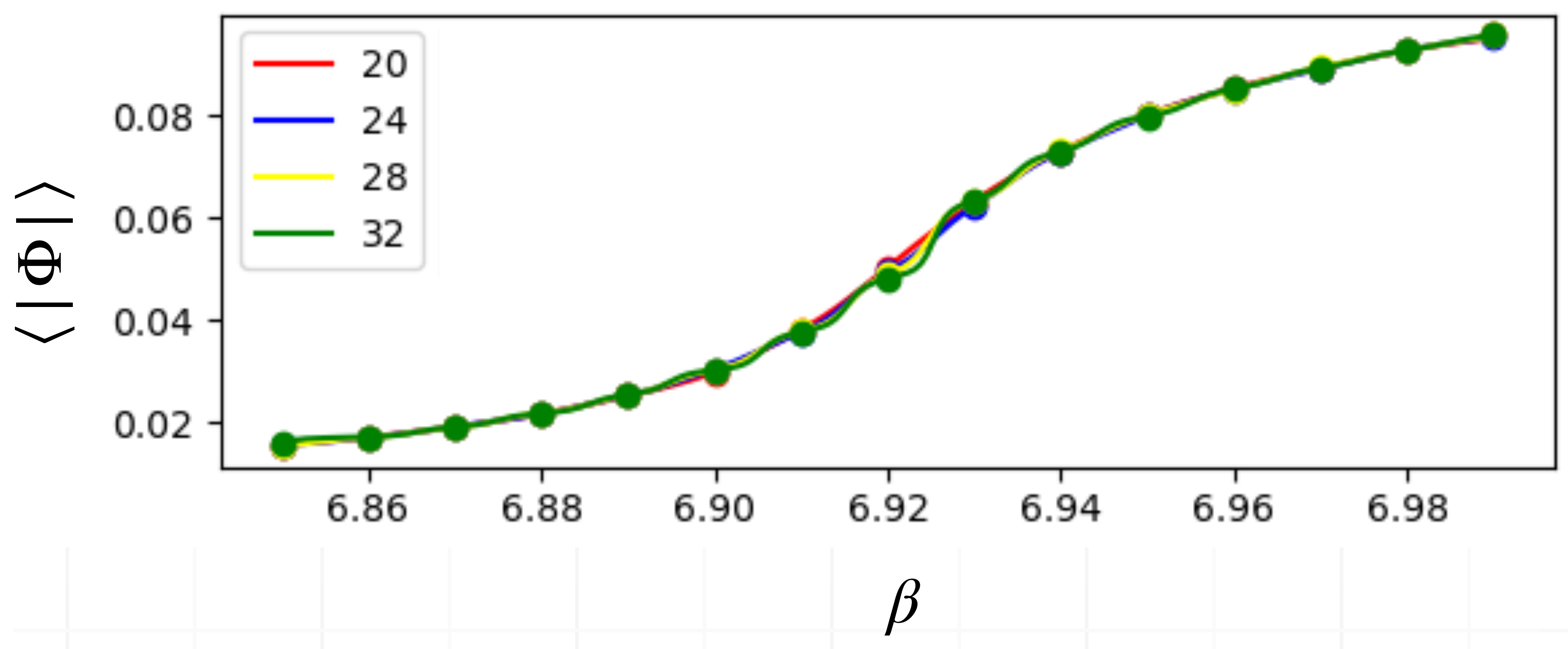
[1] M.E.J. Newman and G.T. Barkema, Monte Carlo methods in statistical physics, Clarendon Press
[2] AVP, "pyRw: Fast multiple histogram reweighting in Python using numba", https://github.com/vataspro/pyRw

Multihistogram Reweighting

Fundamental limitation: $|E(\beta_0) - E(\beta)| \leq \sigma_E$

- Energy (Euclidean action) histograms from source β must overlap with target β





[1] M.E.J. Newman and G.T. Barkema, Monte Carlo methods in statistical physics, Clarendon Press
[2] AVP, "pyRw: Fast multiple histogram reweighting in Python using numba", https://github.com/vataspro/pyRw

Autocorrelation

The MCMC chains sampled have large autocorrelation times

$$C_O(t) = \sum_{i=1}^{N-\tau} \frac{(O_t - \langle O \rangle)(O_{t+\tau} - \langle O \rangle)}{\mathrm{var}(O)}$$

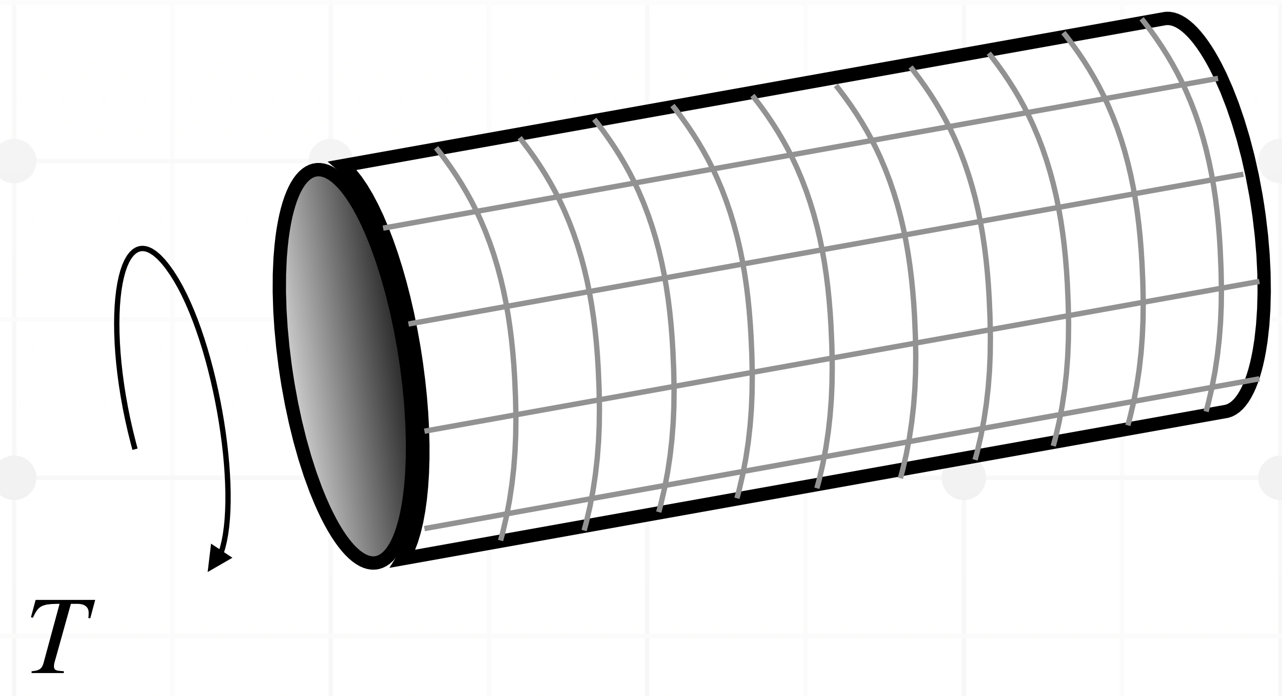
$$\tau_{\mathrm{int}}^{O} = \frac{1}{2} + \sum_{i=1}^{\infty} C_O(i)$$

- Use the integrated autocorrelation time
- Estimate it using MS self consistent windowing
- Skip $2\lceil \tau_{\mathrm{int}} \rceil$ samples
- Follow by bootstrap for statistical error estimation

Finite temperature lattice gauge theory

By compressing the temporal dimension we can identify:

$$T = \frac{1}{aN_T}$$

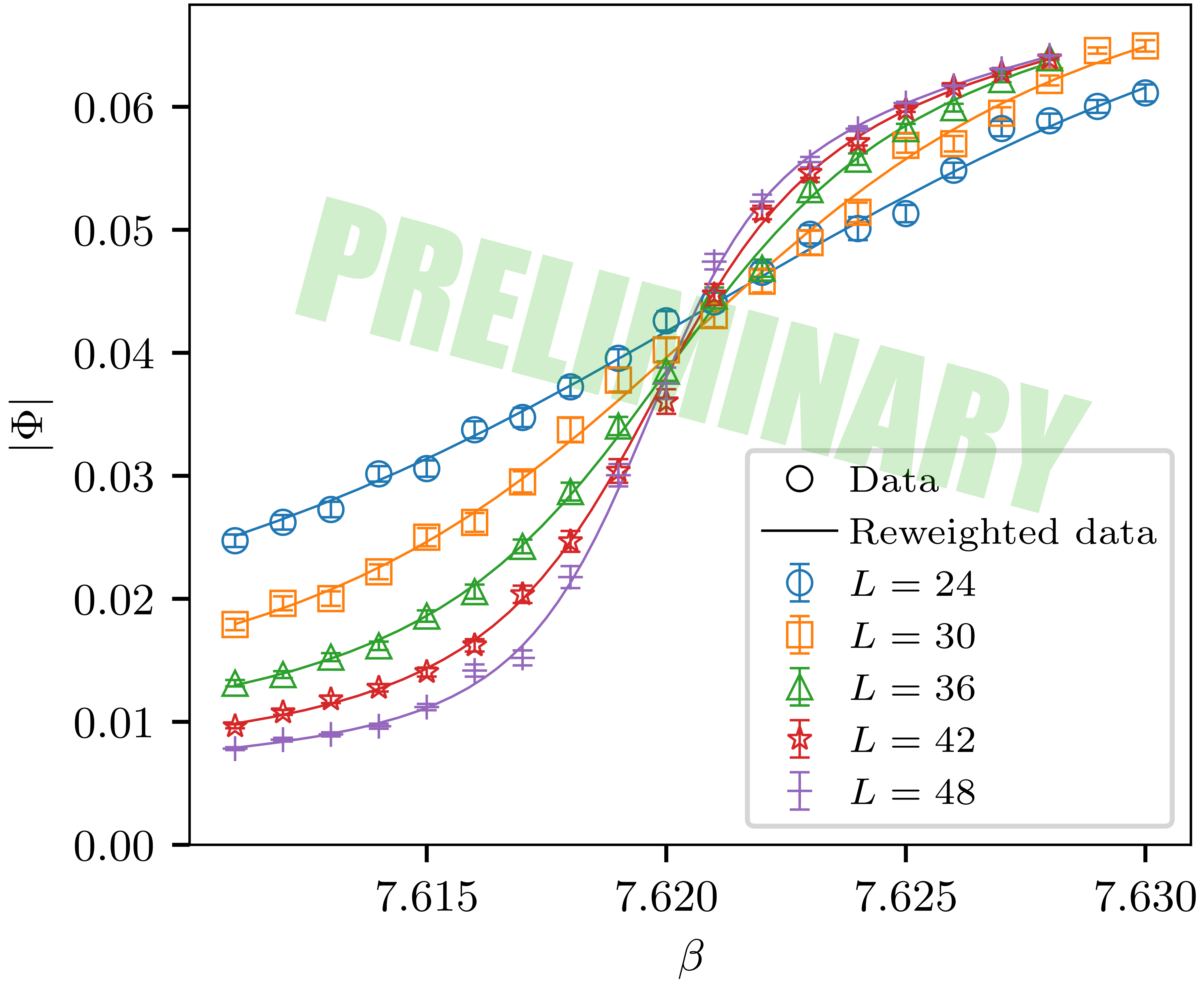


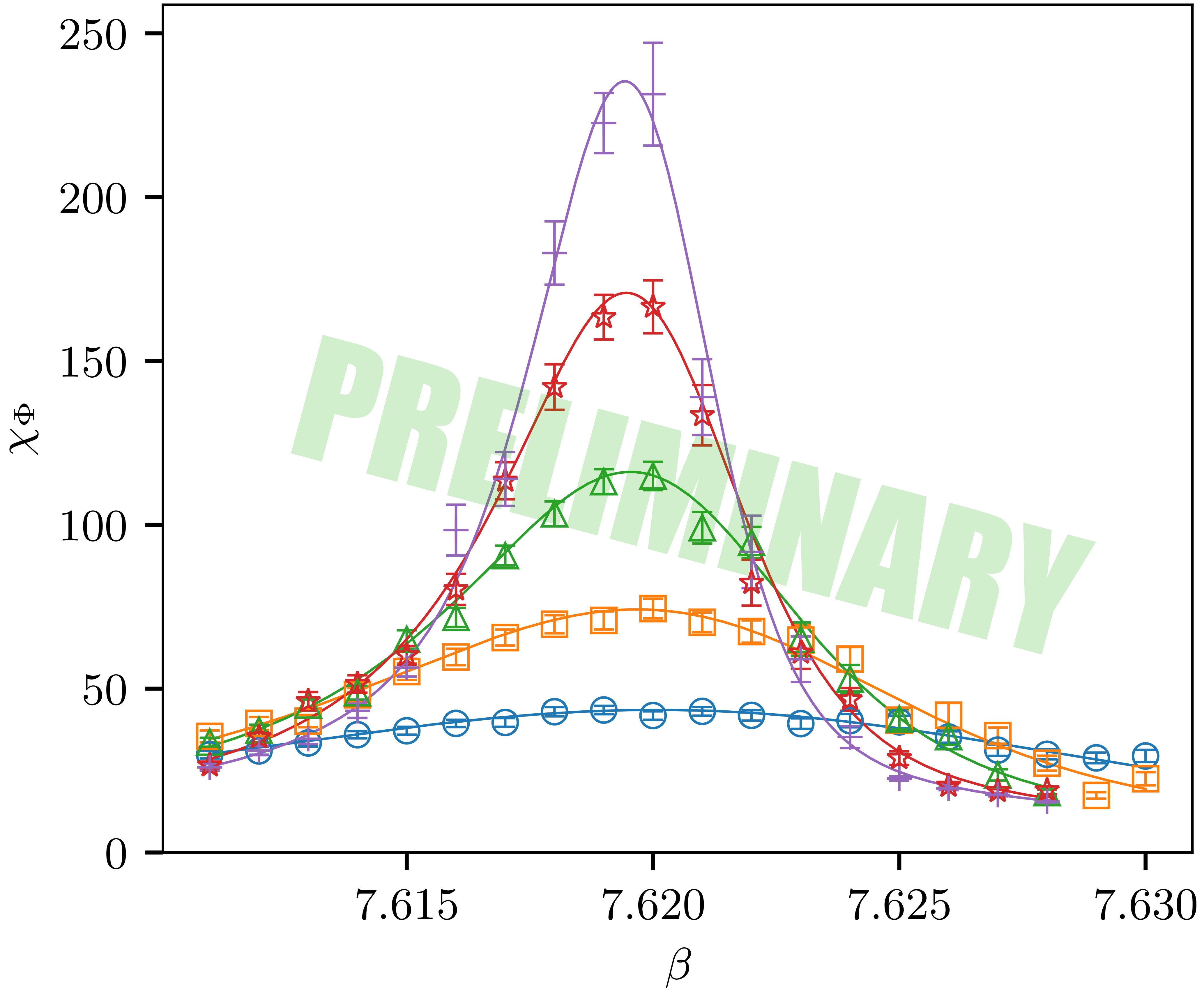
The expectation value of the Polyakov loop can be identified as:

$$\langle \Phi \rangle = e^{-\beta F}$$

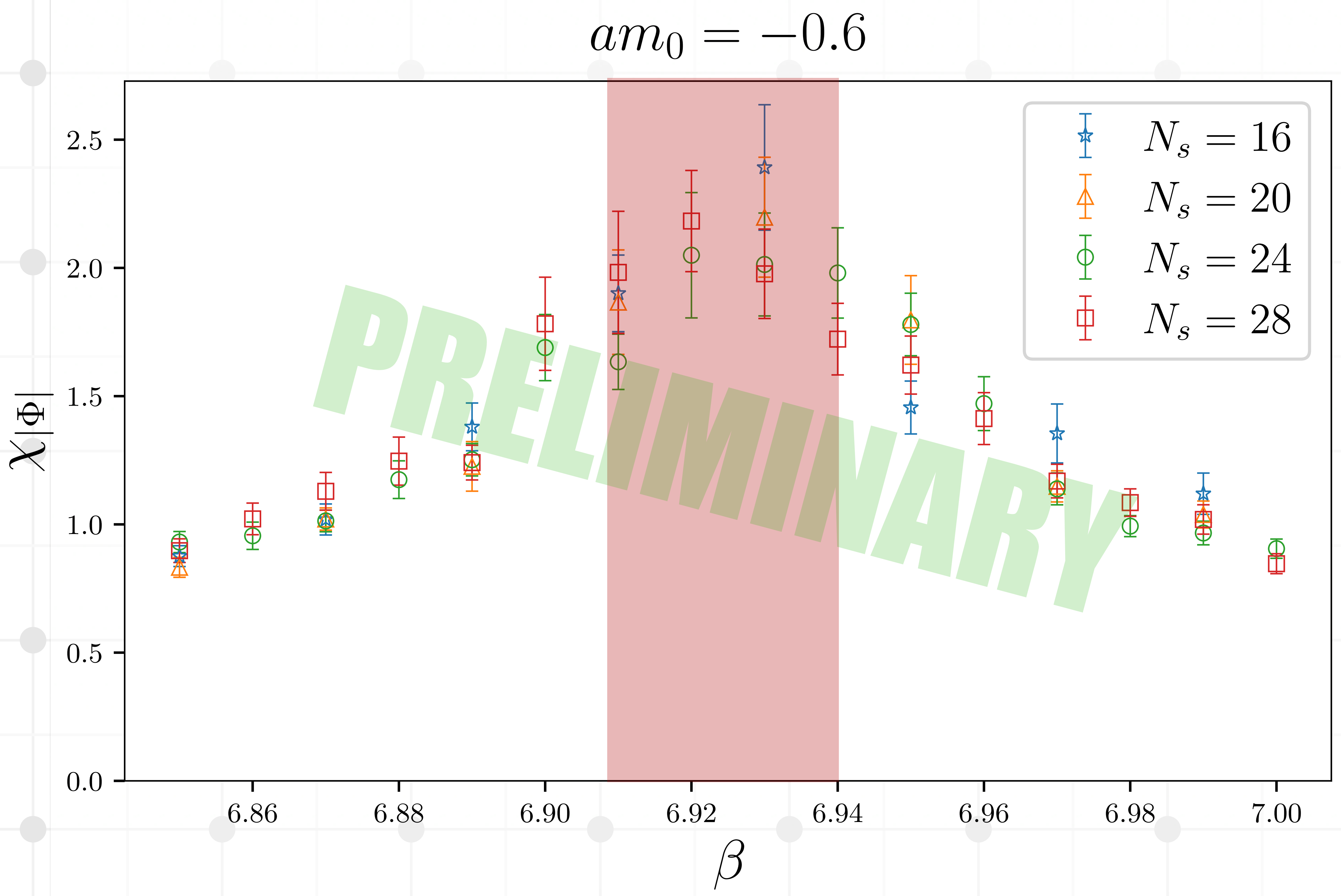
Yang-Mills heat bath

Theory sampled at $N_t = 6$

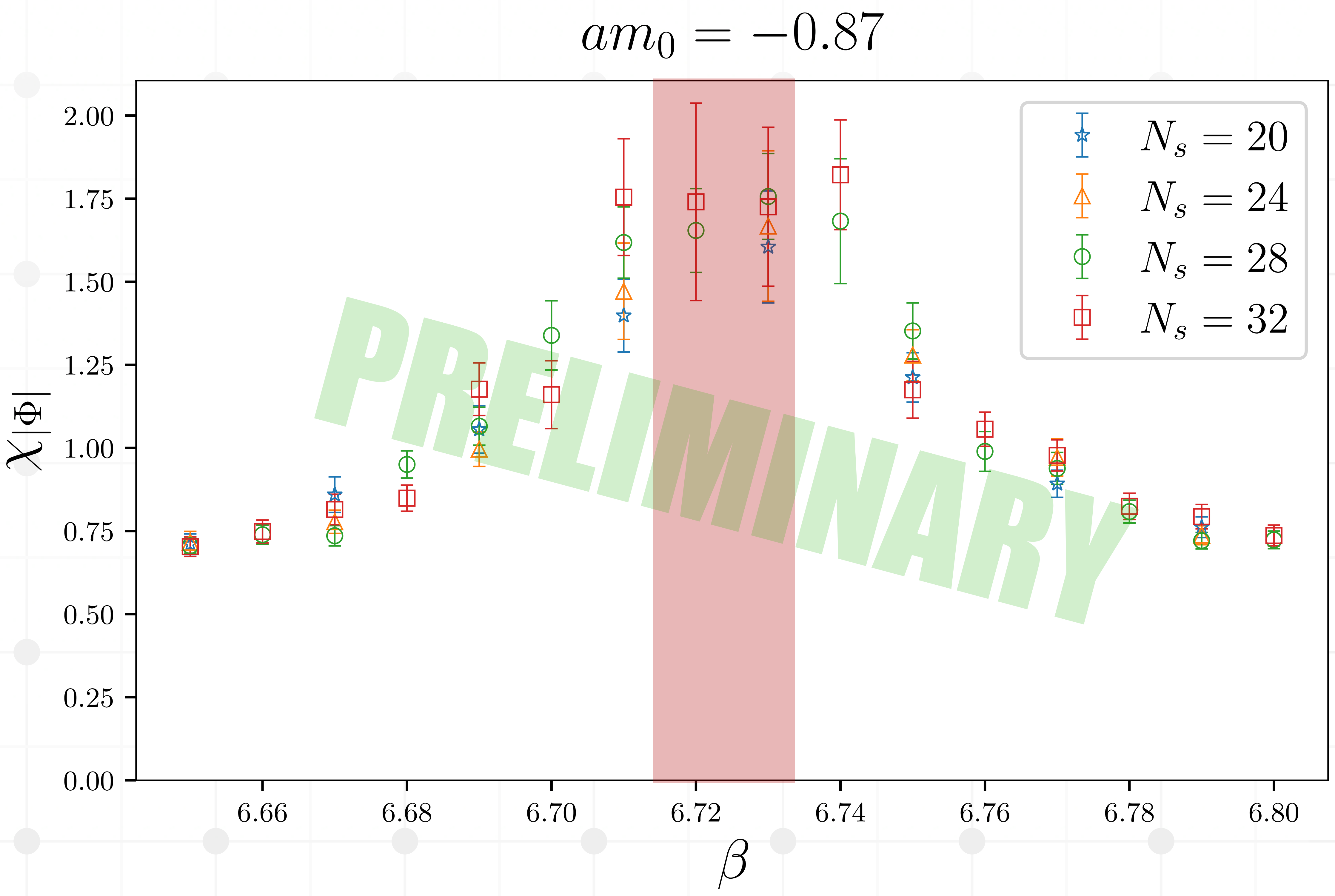




Sparse $N_t = 4$



$am_0 = -0.6$
$\chi_{|\Phi|}$
β
$N_s = 16$
$N_s = 20$
$N_s = 24$
$N_s = 28$
PRELIMINARY



$am_0 = -0.87$
$\chi_{|\Phi|}$
β
$N_s = 20$
$N_s = 24$
$N_s = 28$
$N_s = 32$
PRELIMINARY

Scale setting

am_0	β	am_{PS}	am_{V}	$w_0 m_{\mathrm{PS}}$	$w_0 m_{\mathrm{V}}$	$m_{\mathrm{PS}}/m_{\mathrm{V}}$
−0.3	7.38	1.341(4)	1.357(4)	1.708(5)	1.728(6)	0.989(4)
−0.5	7.3	1.068(5)	1.098(6)	1.315(6)	1.351(7)	0.973(7)
−0.65	7.17	0.899(6)	0.950(8)	0.914(6)	0.966(9)	0.95(1)
−0.77	7.05	0.761(6)	0.85(1)	0.669(6)	0.75(1)	0.90(1)
−0.87	6.95	0.557(7)	0.67(2)	0.494(7)	0.60(1)	0.83(2)
−0.92	6.85	0.641(8)	0.78(2)	0.394(5)	0.48(1)	0.82(2)

PRELIMINARY

Spectrum measurements

$$C_{\mathcal{O}_M}(t) = \sum_{\vec{x}} \langle 0 | \mathcal{O}_M(\vec{x}, t)\mathcal{O}^\dagger_M(\vec{0},0) | 0 \rangle$$

$$\mathcal{O}_{\mathrm{PS}} = \bar{\psi}^i \gamma_5 \psi^j$$

$$\mathcal{O}_{\mathrm{V}} = \bar{\psi}^i \gamma_\mu \psi^j$$

$$\lim_{t\to\infty} C_{\mathcal{O}_M}(t) \to \langle 0 | \mathcal{O}_M | M \rangle \langle 0 | \mathcal{O}_M | M \rangle^* \frac{1}{2m_M} \left[e^{-m_M t} + e^{-m_M (T-t)} \right]$$