Optimised Distillation for Nucleon Correlation Functions

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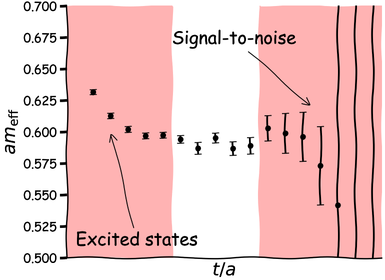
The Windowing Problem
Where to fit our plateaus?

- Extraction of physical quantities requires ground state dominance.
- Excited state contamination:

$$C_{2pt}(t) = \sum_n A_n e^{-E_n t}$$

- Signal-to-noise[1]:

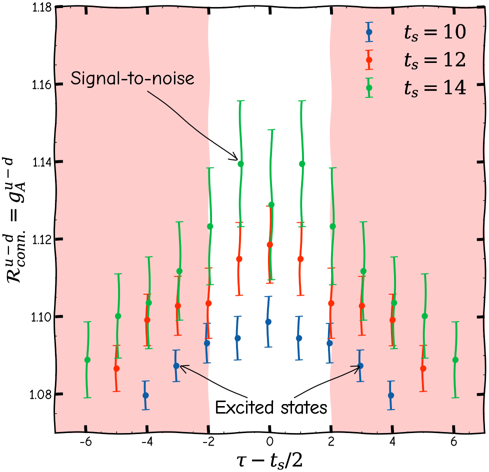
$$\frac{\langle C_{2pt}(t) \rangle}{\sigma_C(t)} \sim e^{-(E_N(\boldsymbol{p}) - \frac{3}{2} m_\pi)t}$$



[1] (Parisi. *Phys. Rept.* 1984; Lepage. 1989)

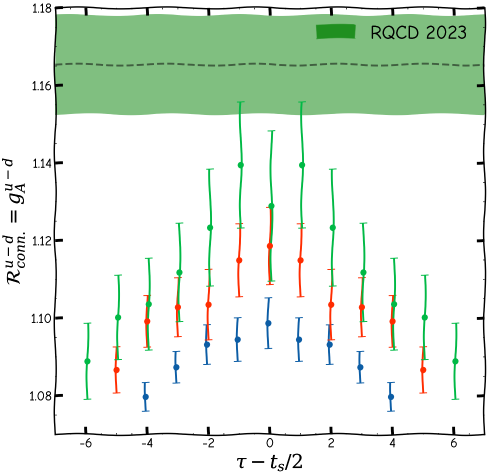
The Windowing Problem

Compounded problem for three-point functions



The Windowing Problem

Compounded problem for three-point functions



[1] (Bali, Collins, Heybrock, et al. *Phys. Rev. D*. 2023. arXiv: 2305.04717 [hep-lat])

Variational Methods for 3pt Functions

See L. Barca after the break!

$$C_{ij}(t) = \langle \mathcal{O}_i(t)\overline{\mathcal{O}}_j(0)\rangle; \qquad C^{\Gamma}_{ij}(\tau;t) = \langle \mathcal{O}_i(t)\mathcal{J}^{\Gamma}(\tau)\overline{\mathcal{O}}_j(0)\rangle$$

- Utilise a wide operator basis and perform a GEVP to get information about the spectrum.
  - Different smearing levels[2], different operator irreps + quark displacements[3].
- Requires the dominant excited state in the 3pt function to be captured in the single particle spectrum[4].

[2]Dragos et al. *Phys. Rev. D*. 2016. arXiv: 1606.03195 (hep-lat).
[3]Egerer, Richards, and Winter. *Phys. Rev. D*. 2019. arXiv: 1810.09991 (hep-lat).
[4]Barca. *Phys. Rev. D*. 2025. arXiv: 2508.09006 (hep-lat).

Distillation[5]

Simple smearing algorithm:

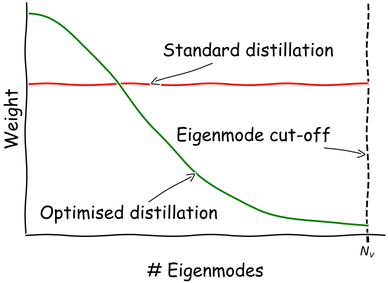
$$J_{\sigma,n_\sigma}(t) = \left(1 - \frac{\sigma\nabla^2(t)}{n_\sigma}\right)^{n_\sigma}$$

For large n_σ:

$$\lim_{n_\sigma\to\infty} J_{\sigma,n_\sigma}(t) = \exp(-\sigma\nabla^2(t))$$

Projector onto low eigenmodes:

$$\square(t) = \sum_{i=1}^{N_v} v^{(i)}[\boldsymbol{x},t]v^{(i)\dagger}[\boldsymbol{y},t]$$



[5] Peardon et al. *Phys. Rev. D*. 2009. arXiv: 0905.2160 (hep-lat).

Ambiguity in Choosing N_v

- N_v is a tunable smearing parameter.
- Too many: under-smeared sources, too few: over-smeared sources.
- More N_v required for accessing excited state spectrum.
- **Costly to recalculate perambulators if you choose too few!**

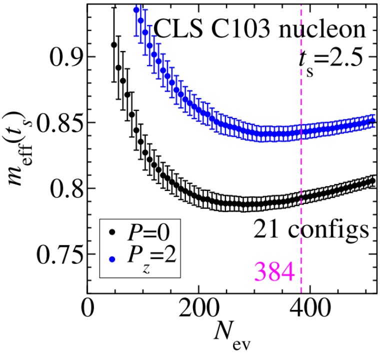


Figure: Nucleon mass as a function of N_v [6].

[6] (Bulava et al. *Phys. Rev. C*. 2026. arXiv: 2505.05547 [hep-lat])

Distillation

Advantages:

- Re-usable and shareable components.
- Improved overlaps with states of interest.
- Effective all-to-all propagators.

Figure: Credit: ChatGPT

Disadvantages:

- Inversion cost scaling for perambulators: $4 \times N_v \times N_T$.
- Volume scaling of required eigenvectors.
- Ambiguity in choosing N_v.

How to make the most of our investment?

Introducing Profiles

See also: J. Urrea-Niño Mon. 14:00

- Nucleon elemental:

$$\Phi^{ijk}_{\alpha\beta}[\boldsymbol{p},t] = \sum_{\boldsymbol{x}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \varepsilon_{abc} v^i_a[\boldsymbol{x},t] v^j_b[\boldsymbol{x},t] v^k_c[\boldsymbol{x},t] [C\gamma_5]_{\alpha\beta}$$

- Introduce a weight function for each eigenmode:

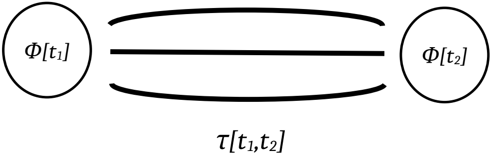
$$v^i_a[\boldsymbol{x},t] \to \exp\left(-\frac{\lambda_i^2}{2\sigma_u^2}\right) v^i_a[\boldsymbol{x},t]$$

- Additional index in elementals: $\Phi^{ijk}_{\alpha\beta}[\boldsymbol{p},t] \to \Phi^{(u)ijk}_{\alpha\beta}[\boldsymbol{p},t]$
- Correlation functions are now a matrix in this basis of distillation profiles[7].

[7] Urrea-Niño, Knechtli, et al. *Phys. Rev. D*. 2022. arXiv: 2205.11564 (hep-lat); Urrea-Niño, Höllwieser, et al. *Phys. Rev. D*. 2025. arXiv: 2506.22132 (hep-lat); Urrea-Niño, Knechtli, et al. *Phys. Rev. D*. 2026. arXiv: 2603.20178 (hep-lat).

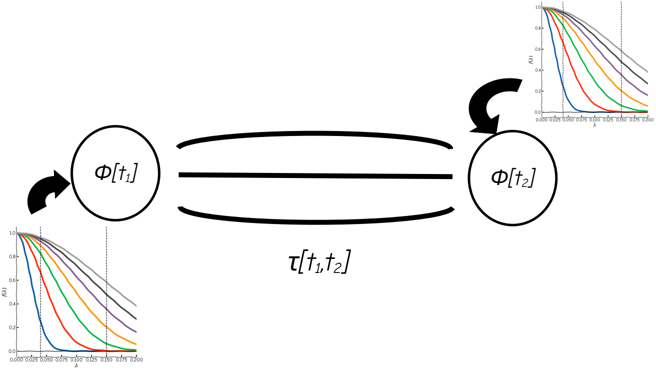
Correlator Construction

- In regular distillation, simply perform contractions between elementals and perambulators.



Correlator Construction

- In optimised distillation, introduce distillation profiles at contraction level.



Extracting the Energies

- Effective masses

$$am_{\mathsf{eff}} = \log \frac{C(t)}{C(t+1)} \xrightarrow{t\to\infty} am_N + \frac{A_1}{A_0}(1-e^{-\Delta E})e^{-\Delta E t} + \dots$$

- Rotate the correlator to the optimal profile using eigenvectors, $\boldsymbol{v}$, from a GEVP

$$am'_{\mathsf{eff}} = \log \frac{\boldsymbol{v}^\dagger C_{\mathsf{2pt}}(t)\boldsymbol{v}}{\boldsymbol{v}^\dagger C_{\mathsf{2pt}}(t+1)\boldsymbol{v}}$$

$$\xrightarrow{t\to\infty} am_N + \frac{A'_1}{A'_0}(1-e^{-\Delta E})e^{-\Delta E t} + \dots$$

Excited State Suppression: $\frac{A'_1}{A'_0} \ll \frac{A_1}{A_0}$

Lattice Details[8]

Table: CLS lattice used in this analysis.

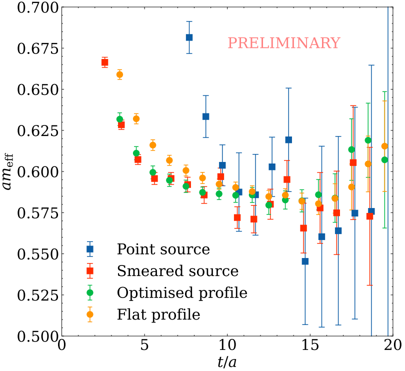
id	$L^3 \times T$	β	a (fm)	m_π (MeV)	N_{cfg}	N_v
A653	$24^3 \times 48$	3.34	0.098	429	132	50

- $N_f = 3$ non-perturbatively $\mathcal{O}(a)$-improved Wilson fermions.
- Tree-level Symanzik improved gauge action.

[8] Bali, Collins, Georg, et al. *JHEP*. 2023. arXiv: 2211.03744 (hep-lat).

Nucleon at Rest

- Compare optimised distillation with optimised Wuppertal smearing.
- Similar computational cost between smeared source and optimised distillation.
- **Compatible with optimised smearing parameters.**



Nucleon at Rest

- Can visualise the relative weights of each eigenmode.
- Huge favouring of low eigenmodes.
- **Flat profiles don't capture the optimal weight of each eigenmode!**

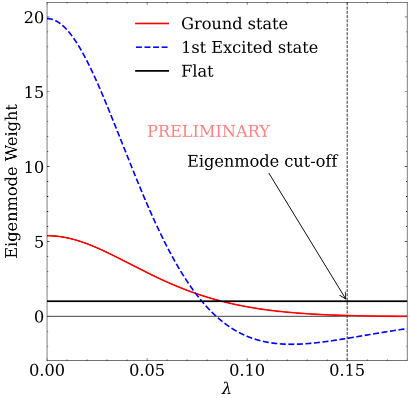


Figure: Optimised profiles on A653

Boosted Nucleon

- Introduce momentum through the elementals

$$\Phi^{ijk}_{\alpha\beta}[\boldsymbol{p},t] = \sum_{\boldsymbol{x}} e^{i\boldsymbol{p}\cdot\boldsymbol{x}} \varepsilon_{abc} v^i_a v^j_b v^k_c [C\gamma_5]$$

- Plateau region shortens drastically with each increase in momentum.

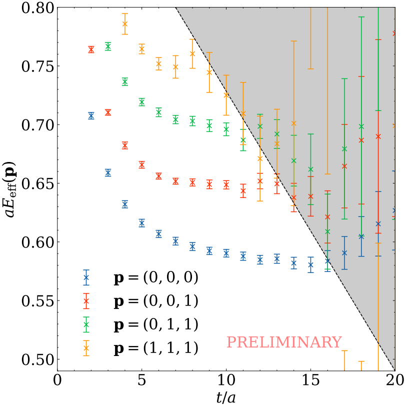


Figure: Flat profiles.

Boosted Nucleon

- Profiles offer some excited state suppression.
- Suppression worsens with increasing momentum.
- Momentum smearing could be used in conjunction with profiles[9].

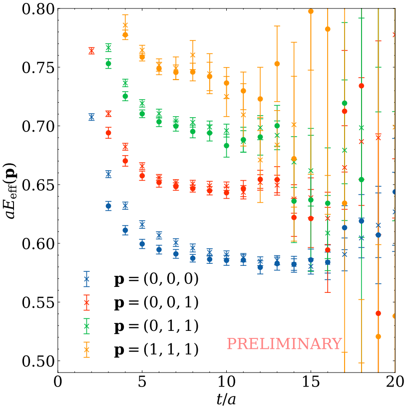


Figure: Comparison of flat and optimal profiles.

[9] (Bali, Lang, et al. *Phys. Rev. D*. 2016. arXiv: 1602.05525 [hep-lat]; Egerer, Edwards, et al. *Phys. Rev. D*. 2021. arXiv: 2009.10691 [hep-lat])

Boosted Nucleon

Discussion

- Higher momenta weight lower eigenmodes higher.
- But also require more information beyond the cut-off.
- Higher momentum $\rightarrow$ need more eigenvectors.

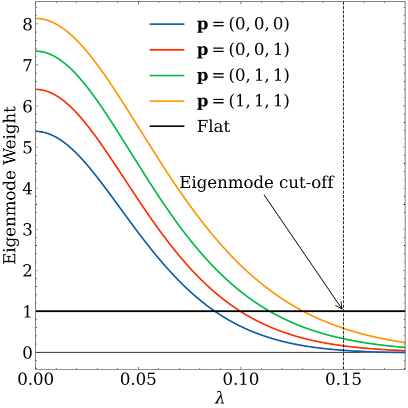
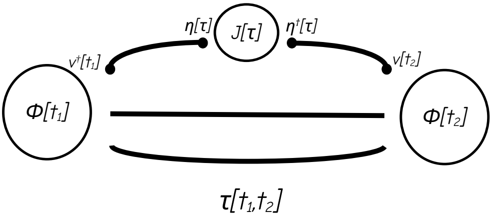


Figure: Optimal profiles for different momenta.

Three-Point Function Construction

- Estimate local current insertion stochastically with $N_s = 50$ Z_4 noise vectors.
  - **No additional inversions to construct generalised perambulators!**

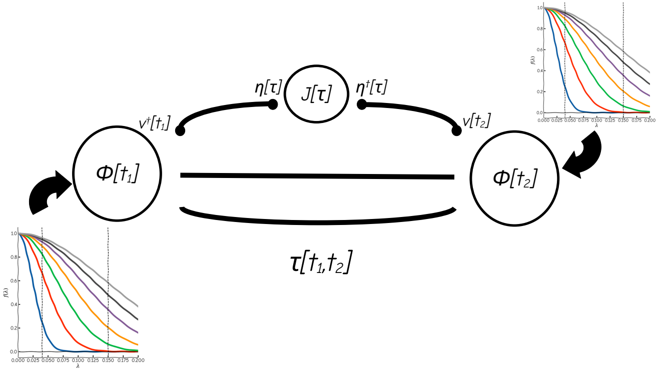


- For future: implement blending at current insertion[10].

[10] Hu et al. 2025. arXiv: 2505.01719 (hep-lat); Wang et al. 2025. arXiv: 2511.02326 (hep-lat).

Three-Point Function Construction

- Introduce distillation profiles during contractions.
- Rotate to the optimal profiles obtained from two-point functions.



Extracting the Matrix Element

- Form the ratio

$$\mathcal{R} = \frac{C_{\text{3pt}}(\Gamma; \tau, \mathbf{0}; \boldsymbol{p}, t_s)}{C_{\text{2pt}}(\boldsymbol{p}, t_s)} \xrightarrow[\tau - t_s \to \infty]{\tau \to \infty} g_\Gamma + \frac{A_1}{A_0} \left(g_\Gamma^{0 \to 1} e^{-\Delta E \tau} + g_\Gamma^{1 \to 0} e^{-\Delta E (t_s - \tau)} \right)$$

- Rotate both correlators to the optimised profile using eigenvectors, $\boldsymbol{v}$, from GEVP.

$$\mathcal{R}' = \frac{\boldsymbol{v}^\dagger C_{\text{3pt}}(\Gamma; \tau, \mathbf{0}; \boldsymbol{p}, t_s) \boldsymbol{v}}{\boldsymbol{v}^\dagger C_{\text{2pt}}(\boldsymbol{p}, t_s) \boldsymbol{v}}$$

$$\xrightarrow[\tau - t_s \to \infty]{\tau \to \infty} g_\Gamma + \frac{A_1'}{A_0'} \left(g_\Gamma^{0 \to 1} e^{-\Delta E \tau} + g_\Gamma^{1 \to 0} e^{-\Delta E (t_s - \tau)} \right)$$

Excited state suppression: $\frac{A_1'}{A_0'} \ll \frac{A_1}{A_0}$

Connected Three-Point Functions

See also: J. Koponen Fri. 14:30

- Axial current insertion:

$$\mathcal{R}^{u-d}_{A,\text{conn.}} \xrightarrow{\tau, t_s-\tau\to\infty} g^{u-d}_A$$

- Clear excited state contamination.
  - τ dependence of ratio.
  - t_s dependence of ratio.

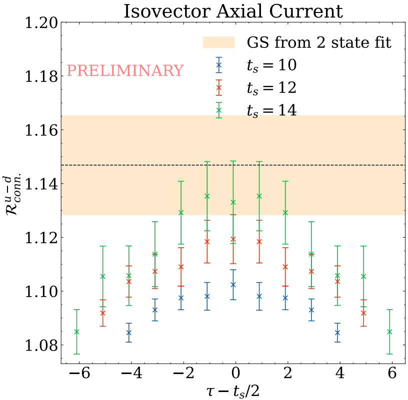


Figure: g^{u-d}_A ratio: Flat profiles.

Connected Three-Point Functions

- Axial current insertion

$$\mathcal{R}^{u-d}_{A,\text{conn.}} \xrightarrow{\tau, t_s-\tau\to\infty} g_A^{u-d}$$

- Suppression of excited states when using profiles.
- Plateau region at largest t_s consistent with ground state extracted from flat profiles.

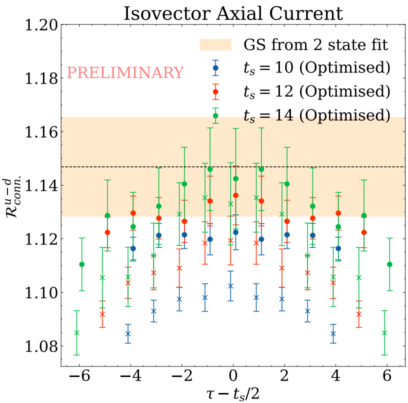


Figure: g_A^{u-d} ratio: Optimised profiles.

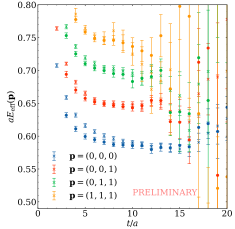
Summary

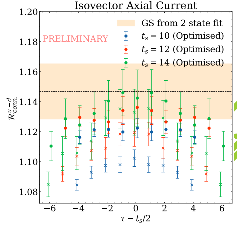
Optimised profiles: Cheap inclusion, better plateaus.

- No more fine-tuning of N_v.
- See gains in the plateau region on coarse lattices and heavy pion masses.
- Suppression in excited state contamination in three-point functions.

Outlook:

- Implement and study variance reduction from blending.
- Extend to off-forward matrix elements.





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Back Up I

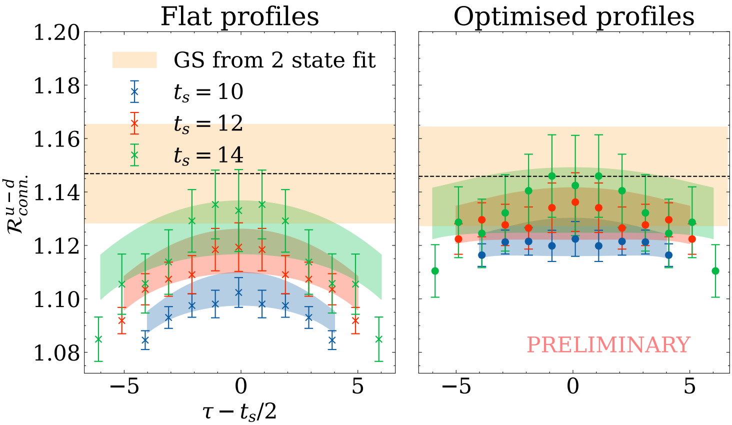

Figure: Comparison of g_A^{u-d} extraction.