

Finite-Density Equation of State of Hot QCD using the Complex Langevin Equation

(based on arXiv:2604.19649 [hep-lat])

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with Dénes Sexty and Michael Mandl

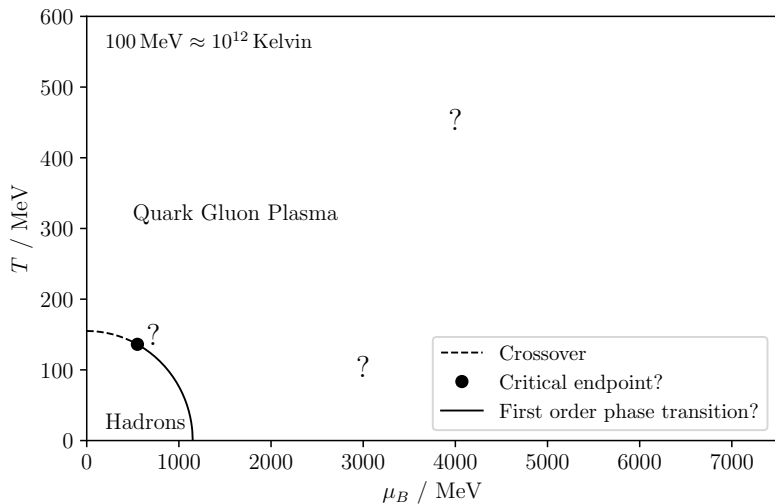
Lattice 2026
University of Maryland, College Park
30 July 2026



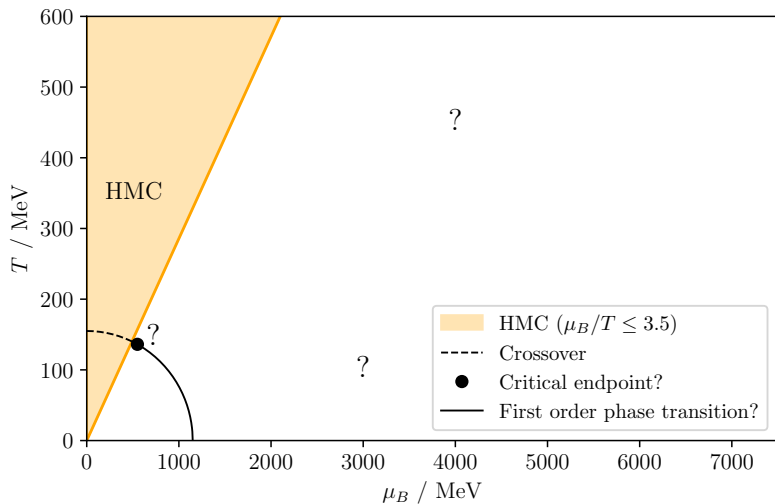
Motivation

- ▶ Quantum Chromodynamics (QCD)
- ▶ QCD phase diagram
- ▶ Equation of state: thermodynamic observables like baryon density, pressure, ...
- ▶ Neutron stars
- ▶ Early universe
- ▶ Heavy ion collisions

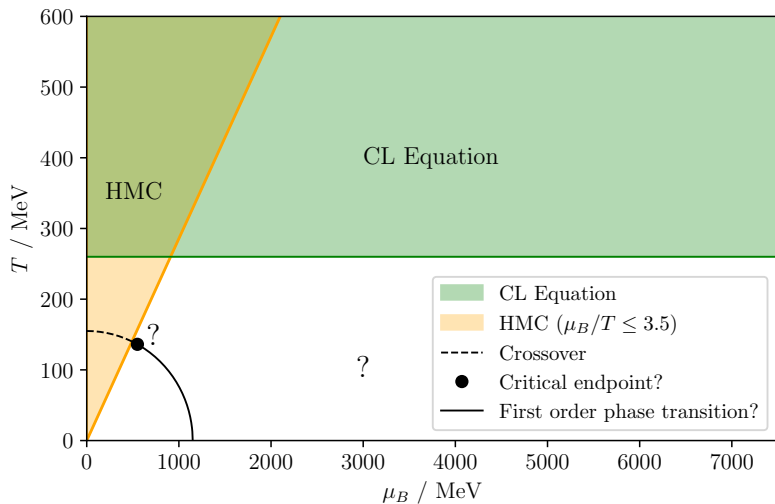
Hypothetical QCD Phase Diagram



Hypothetical QCD Phase Diagram



Hypothetical QCD Phase Diagram



Aim of this Study

- ▶ QCD at high temperatures and finite chemical potential
- ▶ Determination of the equation of state of 2+1+1-flavor QCD at the physical point
 - ▶ Two mass-degenerate light quark flavors: up and down
 - ▶ Strange quark
 - ▶ Charm quark
- ▶ Non-perturbative calculations from first-principles: Lattice QCD
- ▶ Circumvent the sign problem by using the complex Langevin (CL) equation

The Langevin Equation

- ▶ Stochastic differential equation:

$$\frac{dx(\tau)}{d\tau} = -\frac{\partial S(x)}{\partial x} + \eta(\tau) \quad (1)$$

- ▶ Euclidean action $S(x)$
- ▶ Gaussian noise η with $\langle \eta(\tau) \rangle = 0$, $\langle \eta(\tau) \eta(\tau') \rangle = 2 \delta(\tau - \tau')$
- ▶ Langevin time τ , fictitious evolution time used for the stochastic process

The Langevin Equation

- ▶ Solution $x(\tau)$ obeys the equilibrium distribution $P(x) \propto \exp(-S(x))$
- ▶ Computation of an observable F :

$$\langle F \rangle = \frac{1}{Z} \int dx e^{-S(x)} F(x) \quad \text{with} \quad Z = \int dx e^{-S(x)} \quad (2)$$

- ▶ Using the Langevin equation:

$$\langle F \rangle = \int dx P(x) F(x) = \lim_{T \rightarrow \infty} \frac{1}{T - T_{\text{therm}}} \int_{T_{\text{therm}}}^T d\tau F(x(\tau)) \quad (3)$$

with some appropriate thermalization time T_{therm}

The complex Langevin Equation

- Complexification $x \in \mathbb{R} \rightarrow z = x + iy \in \mathbb{C}$ ^{1 2}

$$\frac{dz(\tau)}{d\tau} = -\frac{\partial S(z)}{\partial z} + \eta(\tau) \quad (4)$$

- Expectation values are computed via

$$\begin{aligned} \langle F \rangle &= \int dx dy P(x, y) F(x + iy) = \\ &\lim_{T \rightarrow \infty} \frac{1}{T - T_{\text{therm}}} \int_{T_{\text{therm}}}^T d\tau F(z(\tau)) \end{aligned} \quad (5)$$

- Discretization of the complex Langevin equation

$$z(\tau + \Delta\tau) = z(\tau) - \Delta\tau \frac{\partial S(z)}{\partial z} + \sqrt{\Delta\tau} \eta(\tau) \quad (6)$$

¹Parisi 1983, *Physics Letters B*

²Klauder 1983, *Journal of Physics A: Mathematical and General*

The complex Langevin Equation for QCD ³

► $U_{n,\mu} \in \text{SU}(3) \rightarrow U_{n,\mu} \in \text{SL}(3, \mathbb{C})$

► Update scheme for QCD

$$U_{n,\mu}(\tau + \Delta\tau) = \exp \left(i \sum_a \lambda_a \left(\Delta\tau K_{an\mu} + \sqrt{\Delta\tau} \eta_{an\mu} \right) \right) U_{n,\mu}(\tau) \quad (7)$$

► Gell-Mann matrices λ_a

► Drift term

$$K_{an\mu} = -D_{an\mu} S_{\text{eff}}[U] \quad (8)$$

with the left derivative

$$D_{an\mu} f[U] = \left. \frac{\partial f[e^{i\varphi\lambda_a} U_{n,\mu}]}{\partial \varphi} \right|_{\varphi=0} \quad (9)$$

³Sexty 2014, *Physics Letters B*

The complex Langevin Equation for QCD

- ▶ Left derivative of the effective action

$$\begin{aligned} & -D_{an\mu}S_{\text{eff}}[U] \\ = & -D_{an\mu}S_G[U] + \frac{1}{4} \sum_{f=u,d,s,c} \text{tr} \left[M_f^{-1}[U] D_{an\mu} M_f[U] \right] \quad (10) \end{aligned}$$

- ▶ Tree-level improved Symanzik gauge action $S_G[U]$
- ▶ Staggered quarks with fermion matrix $M_f(\mu_f)[U]$; arbitrary numbers of quark flavours and flavour degeneracies, incorporated straightforwardly

The complex Langevin Equation for QCD

- ▶ Four steps of stout smearing, $\rho = 0.125$ ⁴⁵
- ▶ Gauge cooling for numerical stability: distance to the SU(3) manifold minimized via gauge transformations; 16 cooling steps per Langevin update ⁶
- ▶ Improved Langevin update scheme, partially $\mathcal{O}(\Delta\tau)$ -improved ⁷
- ▶ Adaptive step sizes $\Delta\tau$ ⁸

⁴Morningstar and Peardon 2004, *Physical Review D*

⁵Sexty 2019, *Physical Review D*

⁶Seiler, Sexty, and Stamatescu 2013, *Physics Letters B*

⁷Ukawa and Fukugita 1985, *Physical review letters*

⁸Aarts et al. 2010, *Physics Letters B*

The complex Langevin Equation for QCD

- ▶ The CL equation does not always sample the correct probability distribution⁹¹⁰:
- ▶ Appearing of boundary terms or different cycles
- ▶ For QCD at high temperatures above the crossover OK

⁹Aarts et al. 2011, *The European Physical Journal C*

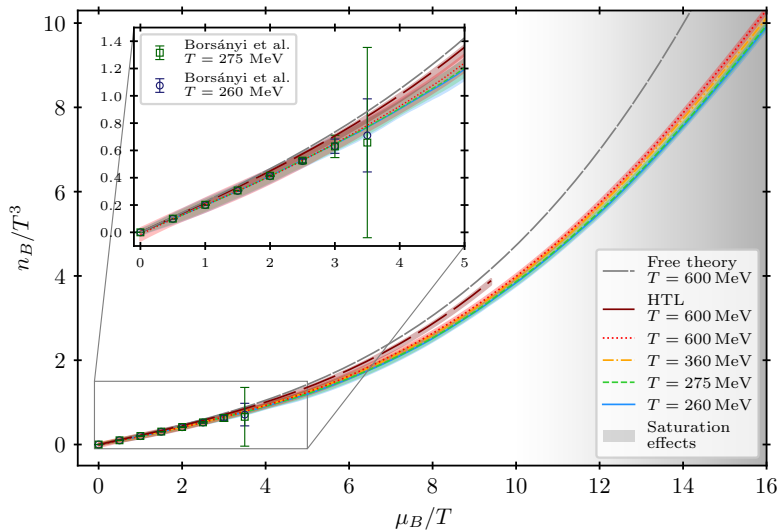
¹⁰Mandl, Seiler, and Sexty 2025, *Journal of Physics A*

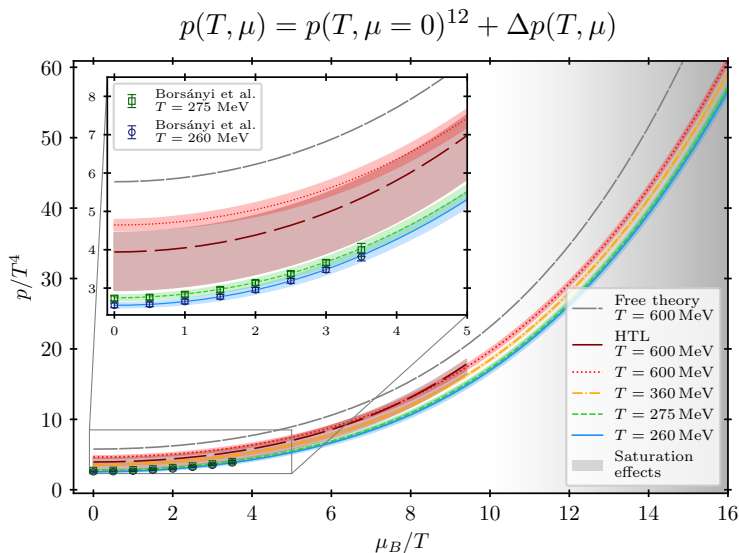
Simulation Setup

- ▶ Bare parameters β, m_f to stay on the line of constant physics (LCP) ¹¹
- ▶ $\mu_u = \mu_d \equiv \mu_l, \quad \mu_s = \mu_c = 0, \quad \mu_B = 3\mu_l$
- ▶ Perform simulations in the $T - \mu_B$ plane, at each point $N_t \in \{10, 12, 16\}$, $N_x = N_y = N_z = 2N_t$
- ▶ Continuum extrapolation: $1/N_t^2 \rightarrow 0$
- ▶ To obtain continuous results: interpolation in T and μ_B

¹¹Bellwied et al. 2015, *Physical Review D*

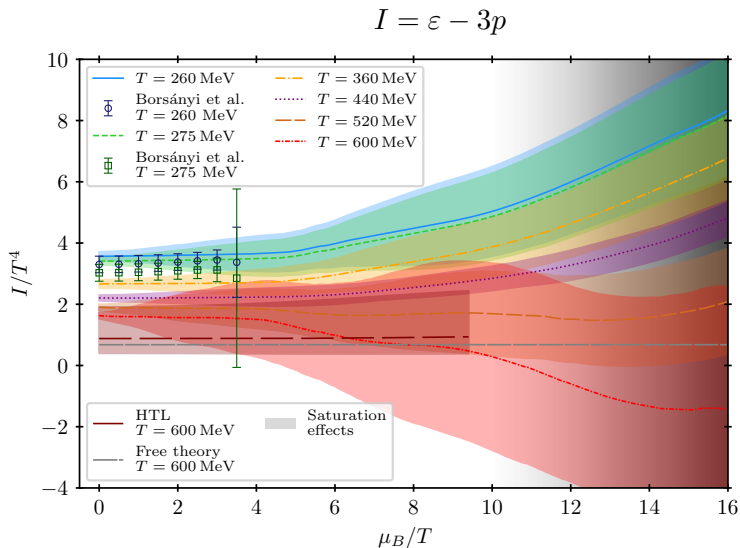
Results



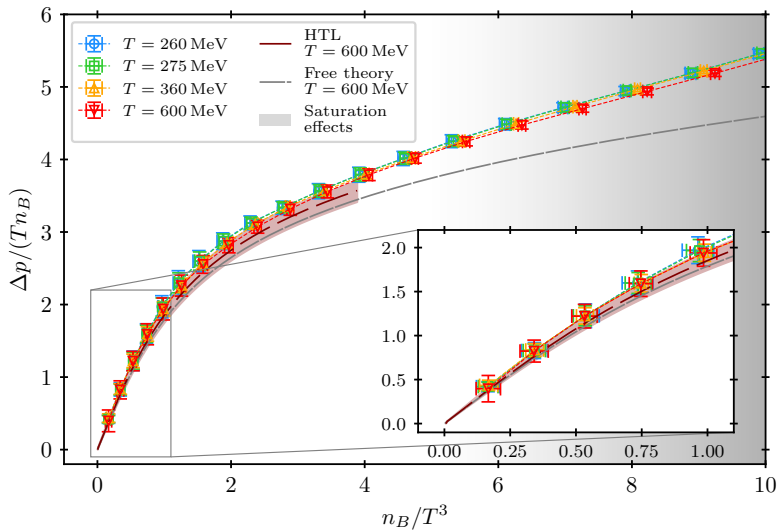


¹²Borsányi et al. 2016, *Nature*

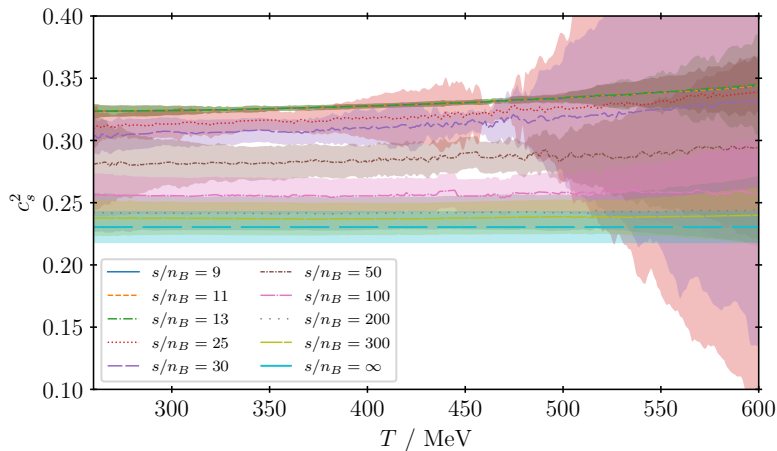
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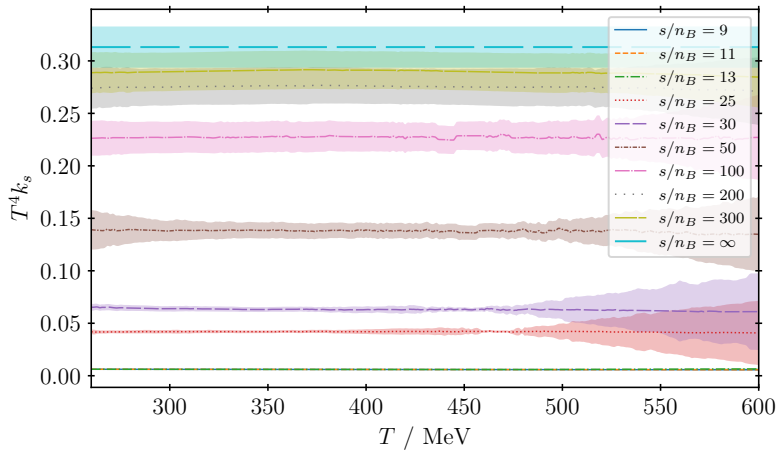
Results



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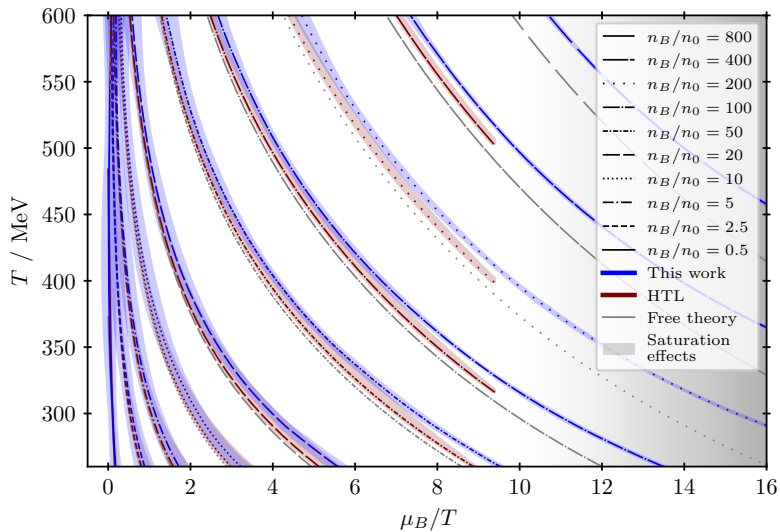
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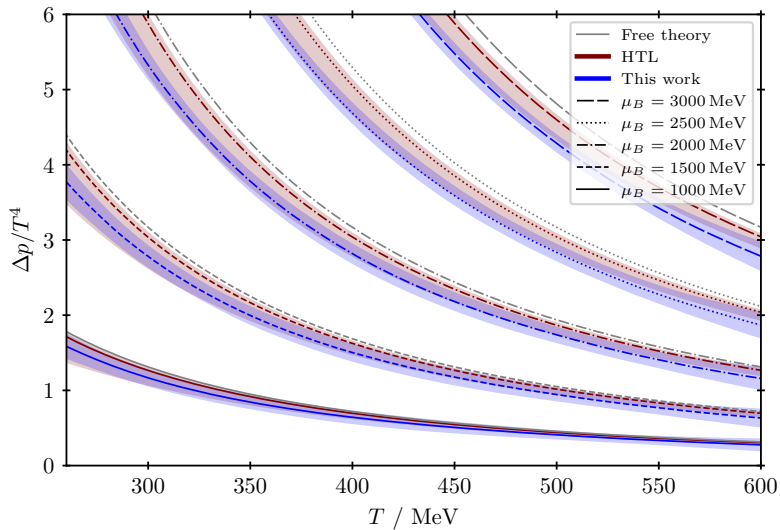
Summary and Outlook

- ▶ Equation of state of 2+1+1-flavor QCD at high temperatures and finite chemical potentials
- ▶ First results using the Complex Langevin equation at the physical point
- ▶ First results at such large chemical potentials
- ▶ Push to lower temperatures
- ▶ Isospin asymmetry

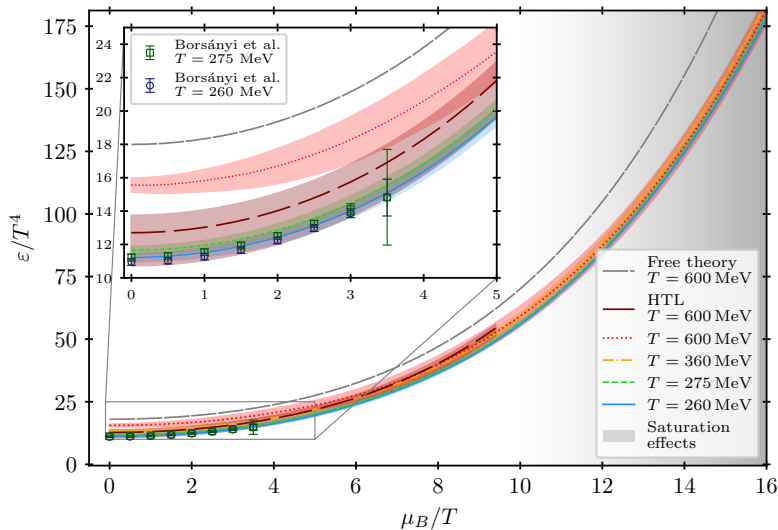
Further Results



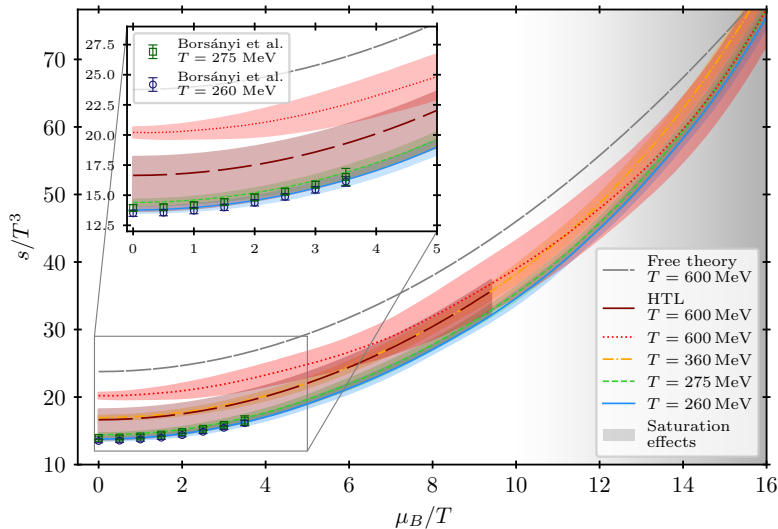
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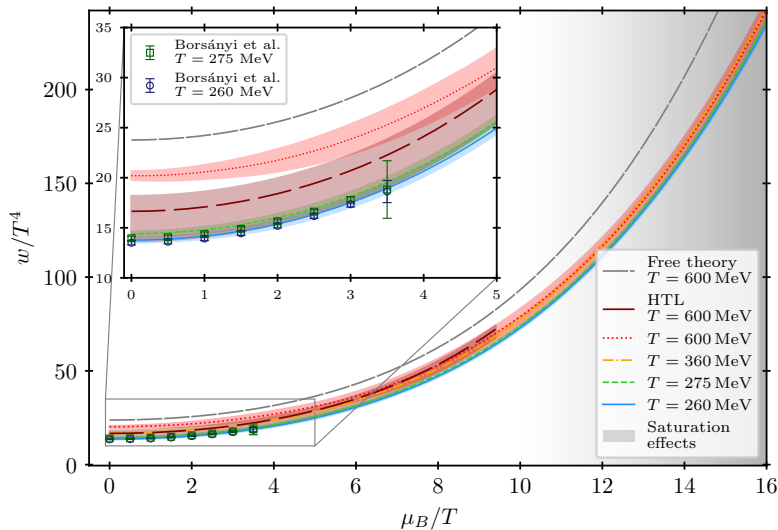
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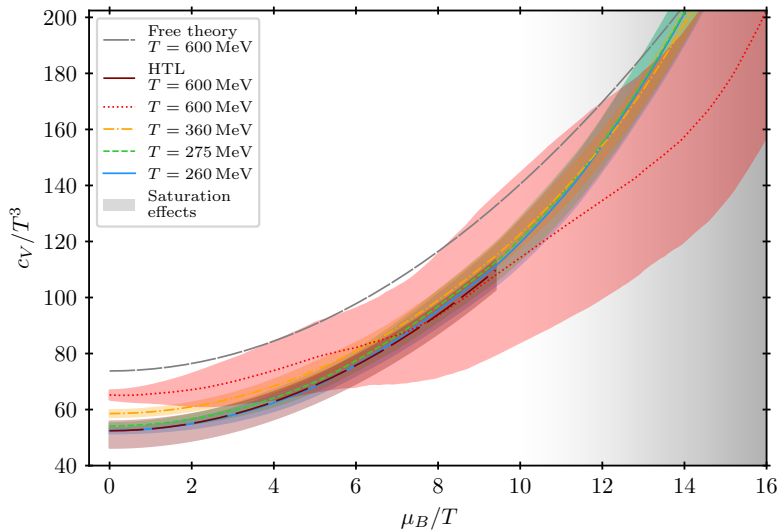
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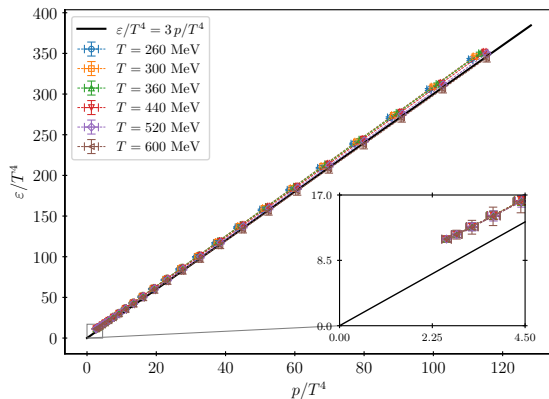
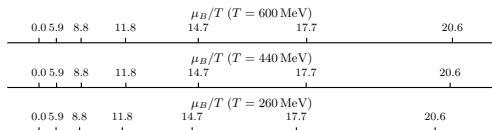
Further Results



Further Results



Further Results



$$\frac{\epsilon}{T^4} = \frac{I}{T^4} + 3\frac{p}{T^4}$$

Gauge Action

- ▶ Tree-level improved Symanzik gauge action

$$S_G[U] = \frac{\beta}{3} \sum_{n \in \Lambda} \sum_{\mu < \nu} (c_0 \operatorname{Re} (\operatorname{tr} [\mathbb{1} - U_{\mu\nu}(n)]) + c_1 \operatorname{Re} (\operatorname{tr} [\mathbb{1} - R_{\mu\nu}(n)])) \quad (11)$$

- ▶ Gauge coupling $\beta = 6/g^2$
- ▶ Plaquette $U_{\mu\nu}(n)$
- ▶ $R_{\mu\nu}(n)$ is the sum of the two planar 1×2 and 2×1 rectangular loops in the $\mu\nu$ -plane.
- ▶ Tree-level Symanzik coefficients $c_1 = -\frac{1}{12}, c_0 = 1 - 8c_1 = \frac{5}{3}$

Fermions

- Fermion matrix for staggered fermions

$$M_f(\mu_f, U)_{nm} = \sum_{\nu=1}^4 \eta_{\nu}(n) \frac{e^{\delta_{\nu 4} \mu_f} U_{\nu}(n) \delta_{n+\hat{\nu}, m} - e^{-\delta_{\nu 4} \mu_f} U_{-\nu}(n) \delta_{n-\hat{\nu}, m}}{2a} + m_f \delta_{n, m} \quad (12)$$

- Staggered sign functions

$$\begin{aligned} \eta_1(n) &= 1, \quad \eta_2(n) = (-1)^{n_1}, \\ \eta_3(n) &= (-1)^{n_1+n_2}, \quad \eta_4(n) = (-1)^{n_1+n_2+n_3} \end{aligned} \quad (13)$$

- ▶ Quark density in lattice units¹³:

$$\begin{aligned}\langle a^3 n_l \rangle &= \frac{1}{N_s^3 N_t} \frac{\partial \ln Z}{\partial (a\mu_l)} = \\ \frac{1}{N_s^3 N_t} \langle \text{tr}[M_l^{-1} \partial_{(a\mu_l)} M_l] \rangle &\equiv a^3 n_l(T, a\mu_l)\end{aligned}\tag{14}$$

- ▶ Convert to dimensionless physical quantities using the temperature relation $T = 1/aN_t$:

$$\frac{n_l(T, \mu_l/T)}{T^3} = N_t^3 a^3 n_l$$

¹³Sexty 2019, *Physical Review D*

Thermodynamics

- Pressure difference:

$$\frac{\Delta p}{T^4} = N_t^4 \int_0^{a\mu_l} d(a\mu'_l) a^3 n_l(a\mu'_l)$$

$$\frac{p(T, \mu/T)}{T^4} = \frac{p(T, \mu/T = 0)}{T^4} + \frac{\Delta p(T, \mu/T)}{T^4}$$

- Using thermodynamic relations $\rightarrow$ computation of further quantities

¹⁴Borsányi et al. 2016, *Nature*

Error Analysis

- ▶ Statistical uncertainties: jackknife routine with blocking, block size ≥ 6 times the integrated autocorrelation time
- ▶ Systematic error: varying several parameters:
- ▶ Thermalization cut: 7, 8, 9, 10 units in Langevin time
- ▶ Interpolation method in the μ_B/T direction: cubic spline or ninth-degree polynomial containing only odd powers due to symmetry constraints (for the density)
- ▶ For each fixed temperature, 16 different values of μ_B/T are available: different subset of data points used for interpolation (all, even, or odd)

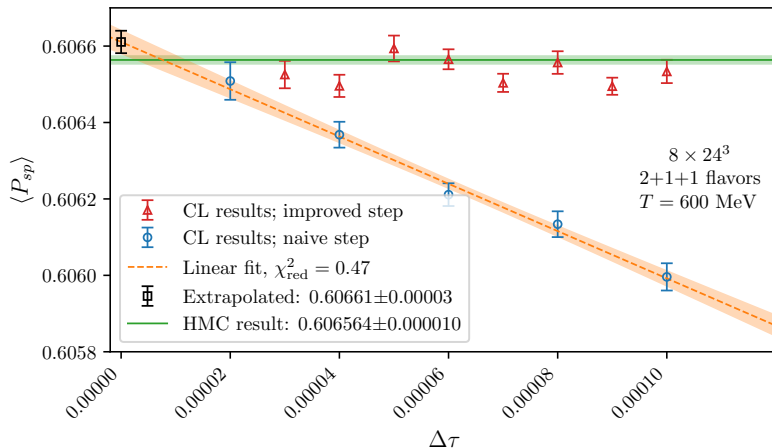
Finite-Volume Effects

- ▶ For $N_t = 10$, simulations were performed with larger aspect ratios $N_s/N_t = 3$ and 4
- ▶ The averages of observables (densities, Polyakov loop, chiral condensates) remain unchanged within statistical uncertainties.
- ▶ Simulations were carried out at various temperatures:
 $T = 300, 360, 440, 520, 600$ MeV
- ▶ Finite-volume effects are smaller than the statistical uncertainties

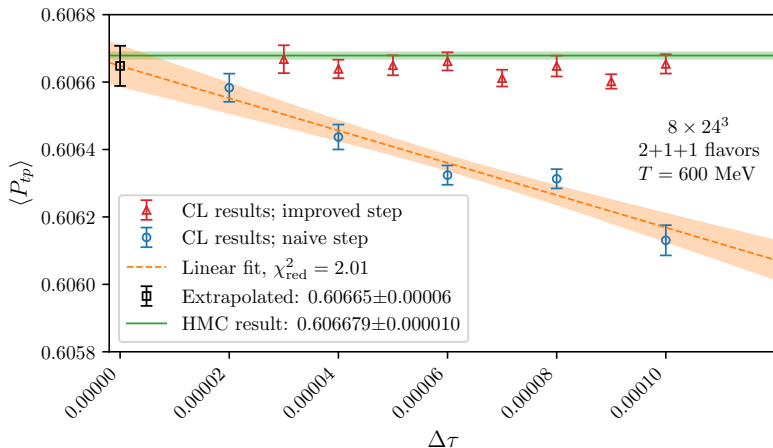
Some Side Remarks

- ▶ $\approx 520,000$ samples generated
- ▶ $\approx 270,000$ hours of GPU time

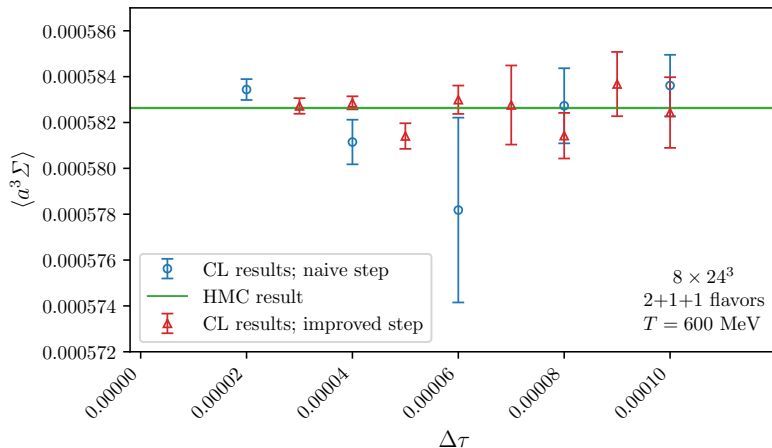
Continuum Extrapolation in Langevin Time



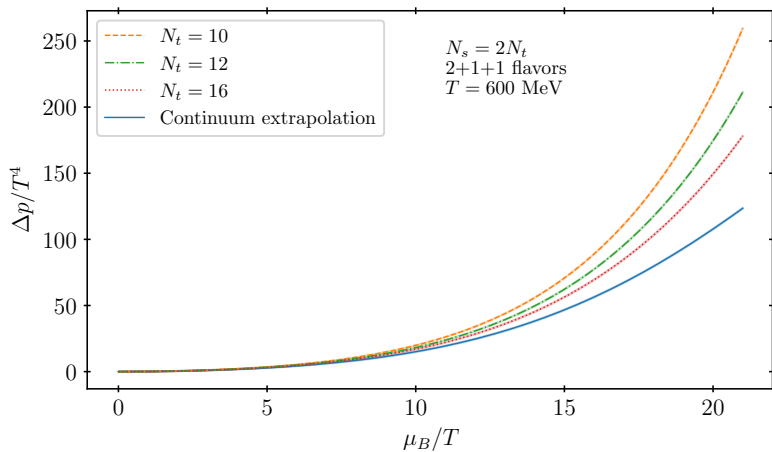
Continuum Extrapolation in Langevin Time



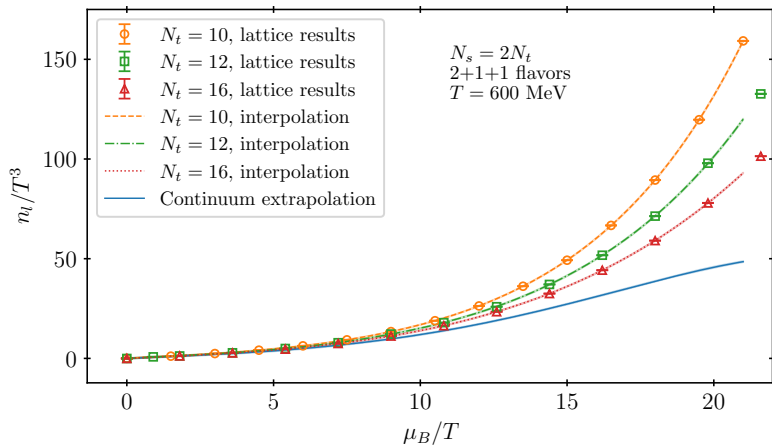
Continuum Extrapolation in Langevin Time



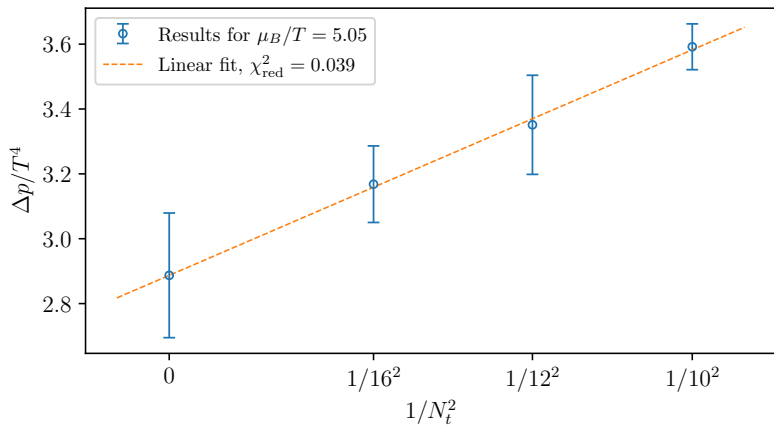
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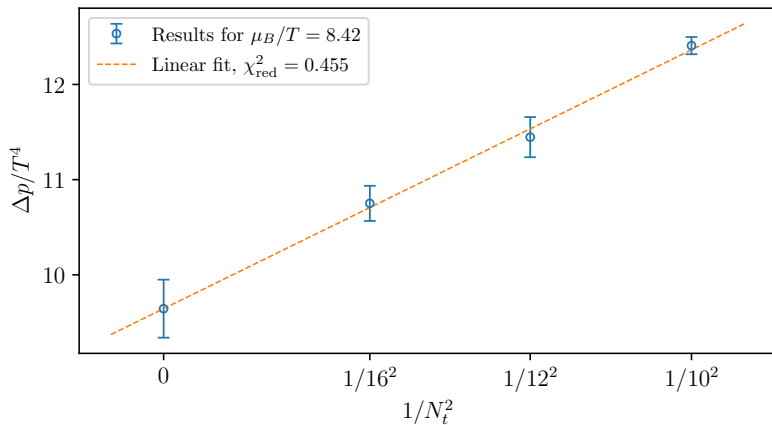
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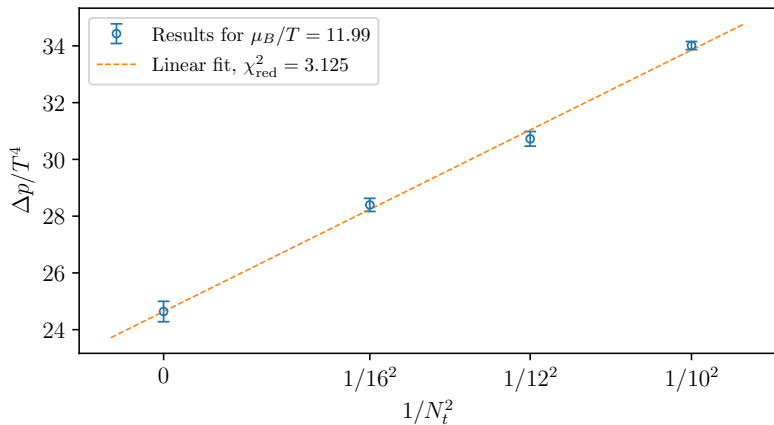
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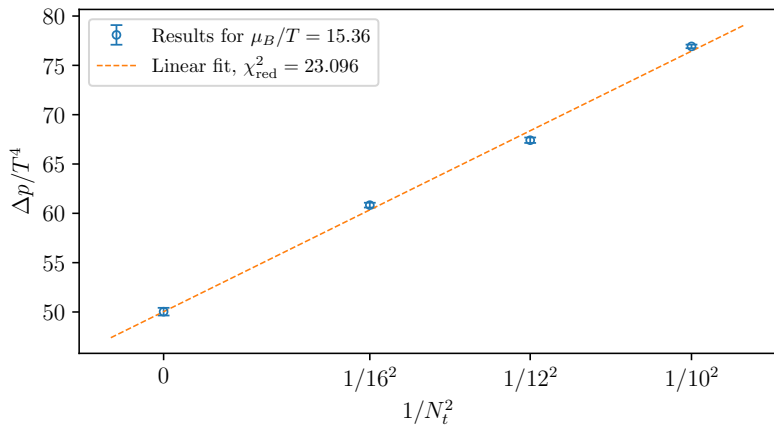
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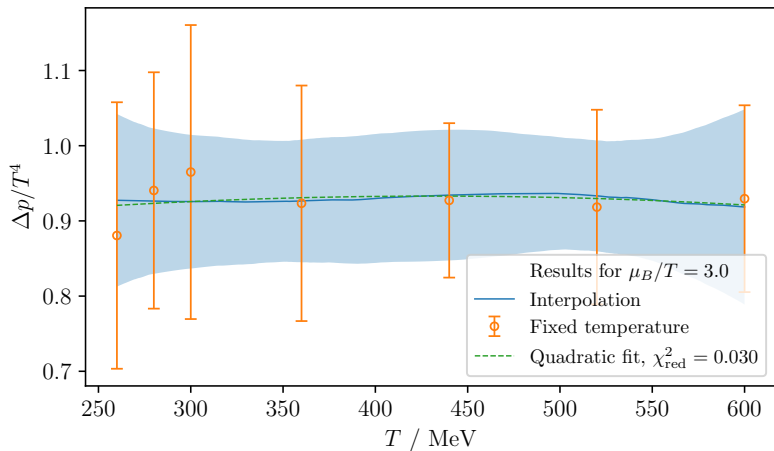
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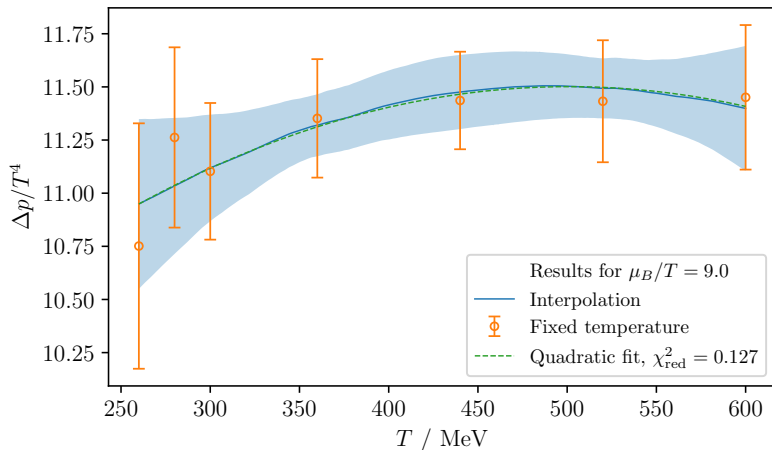
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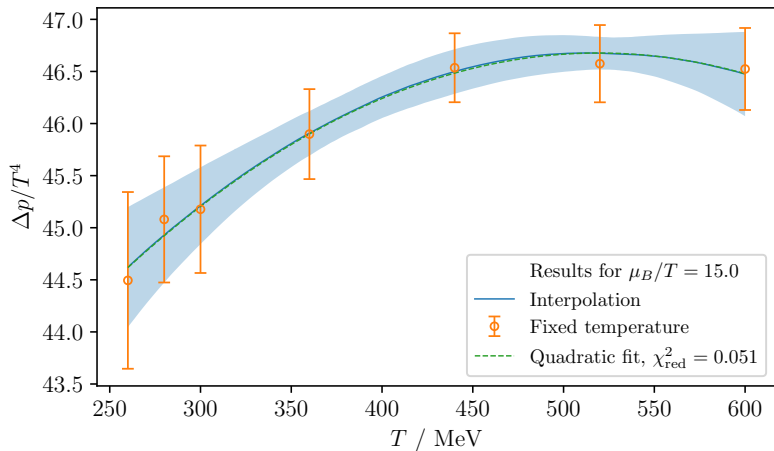
Interpolation in the Temperature-direction



Interpolation in the Temperature-direction



Interpolation in the Temperature-direction



Crosscheck of the Trace Anomaly

