Finite-Density Equation of State of Hot QCD using the Complex Langevin Equation

(based on arXiv:2604.19649 [hep-lat])

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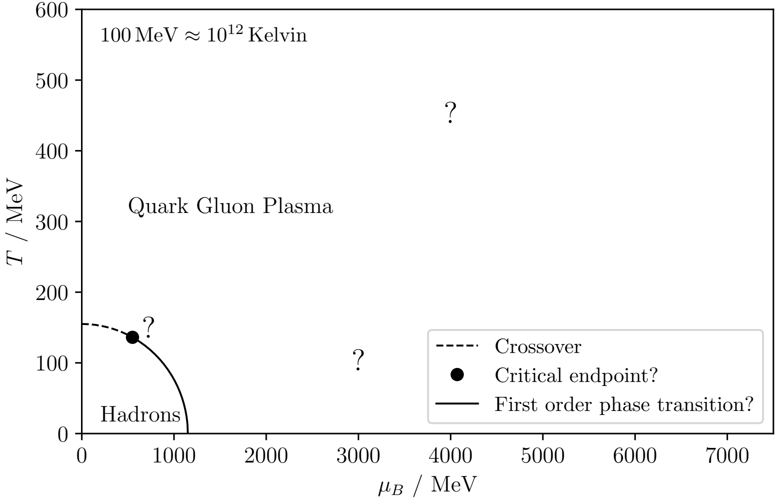
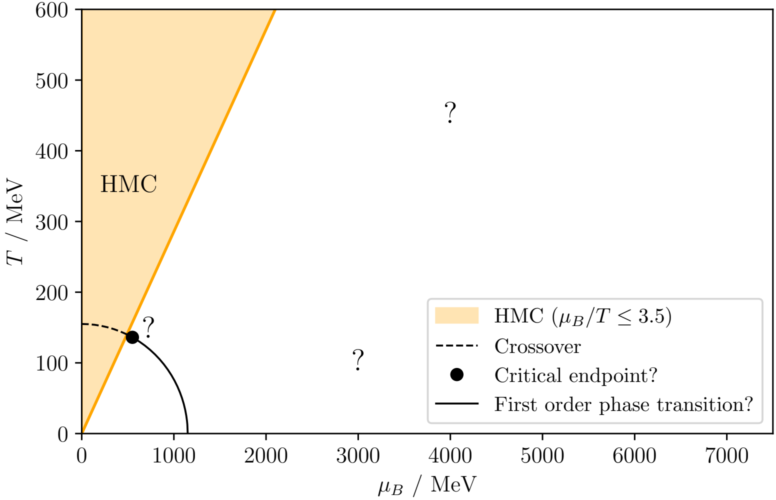
Motivation

- Quantum Chromodynamics (QCD)
- QCD phase diagram
- Equation of state: thermodynamic observables like baryon density, pressure, ...
- Neutron stars
- Early universe
- Heavy ion collisions

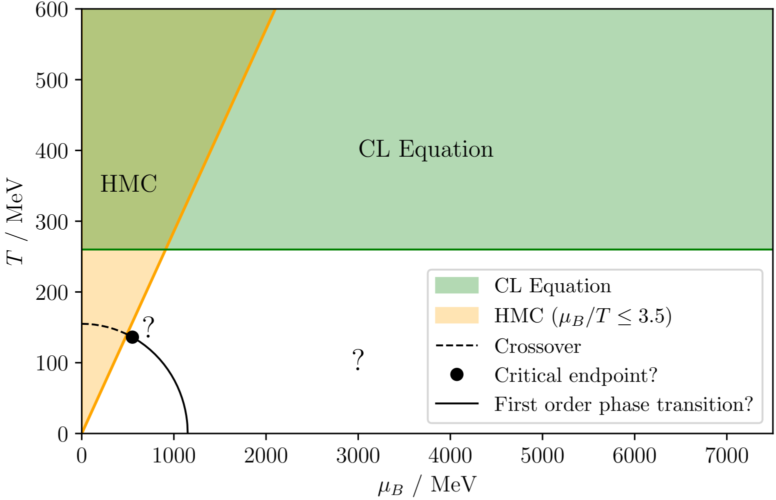
Hypothetical QCD Phase Diagram



Hypothetical QCD Phase Diagram



Hypothetical QCD Phase Diagram



Aim of this Study

- QCD at high temperatures and finite chemical potential
- Determination of the equation of state of 2+1+1-flavor QCD at the physical point
  - Two mass-degenerate light quark flavors: up and down
  - Strange quark
  - Charm quark
- Non-perturbative calculations from first-principles: Lattice QCD
- Circumvent the sign problem by using the complex Langevin (CL) equation

The Langevin Equation

- Stochastic differential equation:

$$\frac{dx(\tau)}{d\tau} = -\frac{\partial S(x)}{\partial x} + \eta(\tau) \tag{1}$$

- Euclidean action $S(x)$
- Gaussian noise η with $\langle \eta(\tau) \rangle = 0, \langle \eta(\tau)\eta(\tau') \rangle = 2\,\delta(\tau - \tau')$
- Langevin time τ, fictitious evolution time used for the stochastic process

The Langevin Equation

- Solution $x(\tau)$ obeys the equilibrium distribution $P(x) \propto \exp(-S(x))$

- Computation of an observable F:

$$\langle F \rangle = \frac{1}{Z} \int dx\, \mathrm{e}^{-S(x)} F(x) \quad \text{with} \quad Z = \int dx\, \mathrm{e}^{-S(x)} \tag{2}$$

- Using the Langevin equation:

$$\langle F \rangle = \int dx P(x) F(x) = \lim_{T\to\infty} \frac{1}{T - T_{\mathrm{therm}}} \int_{T_{\mathrm{therm}}}^{T} d\tau\, F\left(x(\tau)\right) \tag{3}$$

with some appropriate thermalization time T_{therm}

The complex Langevin Equation

- Complexification $x \in \mathbb{R} \to z = x + iy \in \mathbb{C}$ [1] [2]

$$\frac{dz(\tau)}{d\tau} = -\frac{\partial S(z)}{\partial z} + \eta(\tau) \tag{4}$$

- Expectation values are computed via

$$\begin{aligned} \langle F \rangle = \int dx\, dy\; P(x,y)\, F(x+iy) = \\ \lim_{T\to\infty} \frac{1}{T - T_{\text{therm}}} \int_{T_{\text{therm}}}^{T} d\tau\, F\left(z(\tau)\right) \end{aligned} \tag{5}$$

- Discretization of the complex Langevin equation

$$z(\tau + \Delta\tau) = z(\tau) - \Delta\tau \frac{\partial S(z)}{\partial z} + \sqrt{\Delta\tau}\, \eta(\tau) \tag{6}$$

[1] Parisi 1983, *Physics Letters B*
[2] Klauder 1983, *Journal of Physics A: Mathematical and General*

The complex Langevin Equation for QCD [3]

- $U_{n,\mu} \in \mathrm{SU}(3) \to U_{n,\mu} \in \mathrm{SL}(3, \mathbb{C})$
- Update scheme for QCD

$$U_{n,\mu}(\tau{+}\Delta\tau) = \exp\left(\mathrm{i}\sum_a \lambda_a \left(\Delta\tau\, K_{an\mu} + \sqrt{\Delta\tau}\, \eta_{an\mu}\right)\right) U_{n,\mu}(\tau) \tag{7}$$

- Gell-Mann matrices λ_a
- Drift term

$$K_{an\mu} = -D_{an\mu} S_{\mathrm{eff}}[U] \tag{8}$$

with the left derivative

$$D_{an\mu} f[U] = \left.\frac{\partial f[\mathrm{e}^{\mathrm{i}\varphi\lambda_a} U_{n,\mu}]}{\partial\varphi}\right|_{\varphi=0} \tag{9}$$

[3] Sexty 2014, *Physics Letters B*

The complex Langevin Equation for QCD

- Left derivative of the effective action

$$
\begin{aligned}
& -D_{an\mu}S_{\text{eff}}[U] \\
&= -D_{an\mu}S_G[U] + \frac{1}{4}\sum_{f=u,d,s,c} \operatorname{tr}\left[M_f^{-1}[U]D_{an\mu}M_f[U]\right]
\end{aligned} \tag{10}
$$

- Tree-level improved Symanzik gauge action $S_G[U]$
- Staggered quarks with fermion matrix $M_f(\mu_f)[U]$; arbitrary numbers of quark flavours and flavour degeneracies, incorporated straightforwardly

The complex Langevin Equation for QCD

- Four steps of stout smearing, $\rho = 0.125$ [4,5]
- Gauge cooling for numerical stability: distance to the SU(3) manifold minimized via gauge transformations; 16 cooling steps per Langevin update [6]
- Improved Langevin update scheme, partially $\mathcal{O}(\Delta\tau)$-improved [7]
- Adaptive step sizes $\Delta\tau$ [8]

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[4] Morningstar and Peardon 2004, *Physical Review D*
[5] Sexty 2019, *Physical Review D*
[6] Seiler, Sexty, and Stamatescu 2013, *Physics Letters B*
[7] Ukawa and Fukugita 1985, *Physical review letters*
[8] Aarts et al. 2010, *Physics Letters B*

The complex Langevin Equation for QCD

- The CL equation does not always sample the correct probability distribution[9][10]:
- Appearing of boundary terms or different cycles
- For QCD at high temperatures above the crossover OK

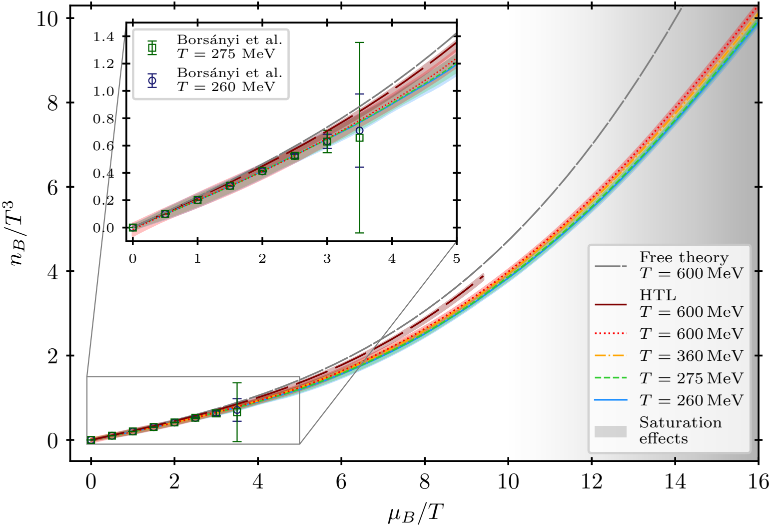
[9] Aarts et al. 2011, *The European Physical Journal C*
[10] Mandl, Seiler, and Sexty 2025, *Journal of Physics A*

Simulation Setup

- Bare parameters β, m_f to stay on the line of constant physics (LCP) [11]
- $\mu_u = \mu_d \equiv \mu_l, \quad \mu_s = \mu_c = 0, \quad \mu_B = 3\mu_l$
- Perform simulations in the $T - \mu_B$ plane, at each point $N_t \in \{10, 12, 16\}$, $N_x = N_y = N_z = 2N_t$
- Continuum extrapolation: $1/N_t^2 \to 0$
- To obtain continuous results: interpolation in T and μ_B

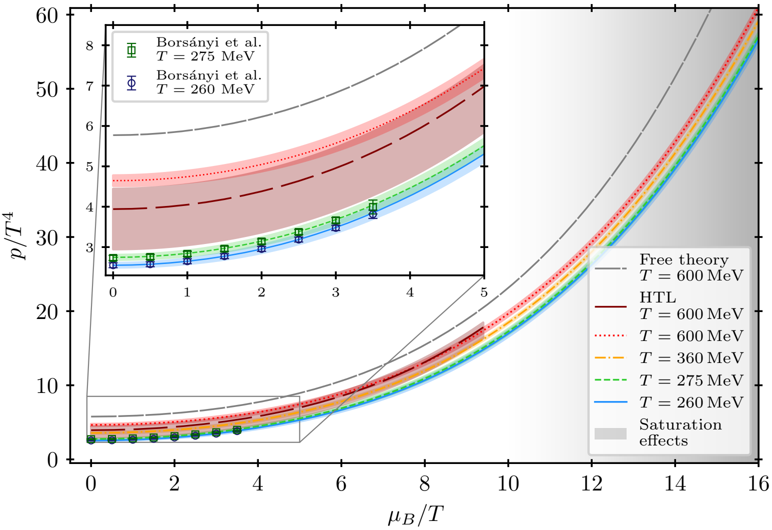
[11] Bellwied et al. 2015, *Physical Review D*

Results



Borsányi et al.
$T = 275$ MeV
Borsányi et al.
$T = 260$ MeV
Free theory
$T = 600\,\mathrm{MeV}$
HTL
$T = 600\,\mathrm{MeV}$
$T = 600\,\mathrm{MeV}$
$T = 360\,\mathrm{MeV}$
$T = 275\,\mathrm{MeV}$
$T = 260\,\mathrm{MeV}$
Saturation effects
n_B/T^3
μ_B/T

Results



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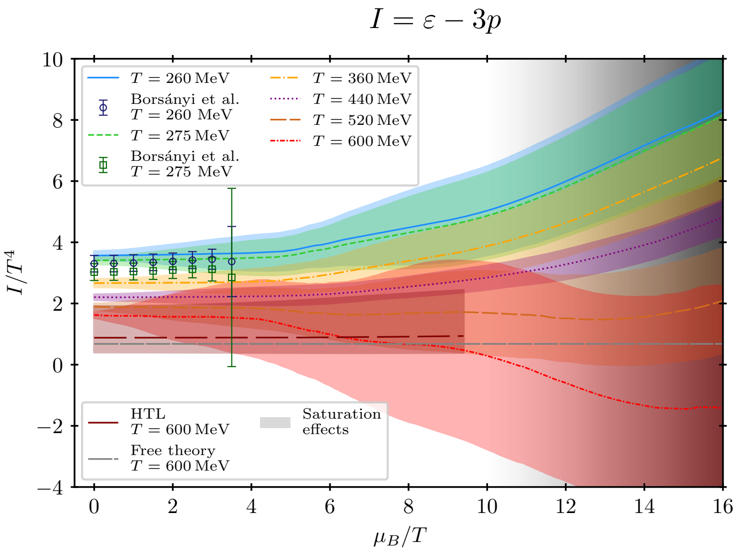
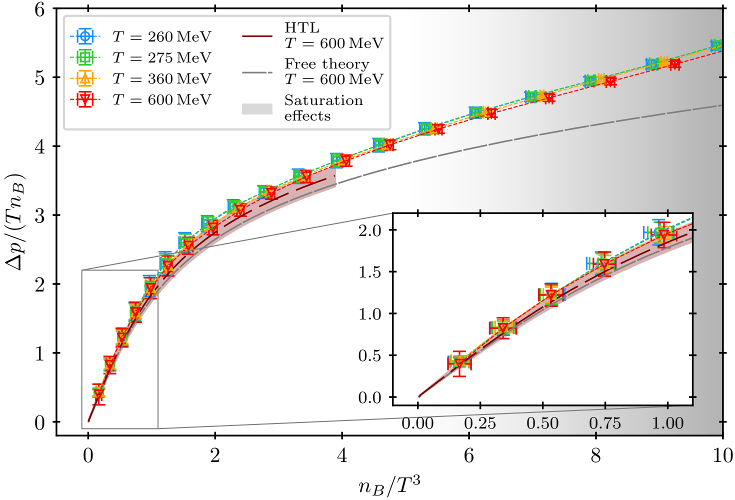
Saturation effects
p/T^4
μ_B/T

Results

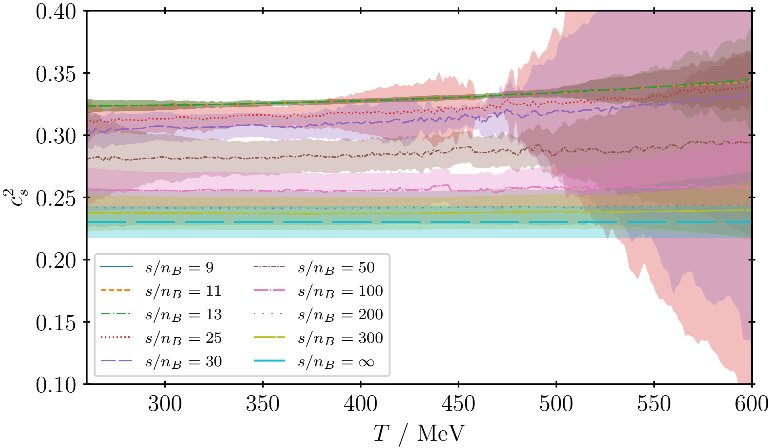

$I = \varepsilon - 3p$
I/T^4
μ_B/T
T = 260 MeV
Borsányi et al.
T = 260 MeV
T = 275 MeV
Borsányi et al.
T = 275 MeV
T = 360 MeV
T = 440 MeV
T = 520 MeV
T = 600 MeV
HTL
T = 600 MeV
Free theory
T = 600 MeV
Saturation effects
10
8
6
4
2
0
−2
−4
0
2
4
6
8
10
12
14
16

Results

$$p(T, \mu/T) = p(T, \mu/T = 0) + \Delta p(T, \mu/T)$$

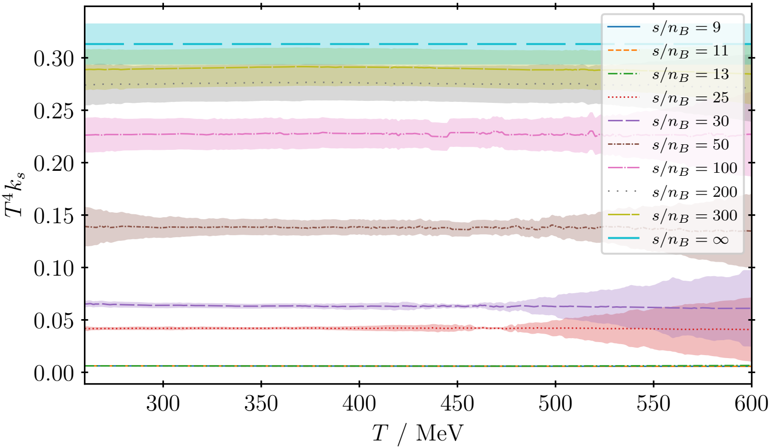


Results



0.40
0.35
0.30
0.25
0.20
0.15
0.10
c_s^2
300
350
400
450
500
550
600
T / MeV
$s/n_B = 9$
$s/n_B = 11$
$s/n_B = 13$
$s/n_B = 25$
$s/n_B = 30$
$s/n_B = 50$
$s/n_B = 100$
$s/n_B = 200$
$s/n_B = 300$
$s/n_B = \infty$

Results



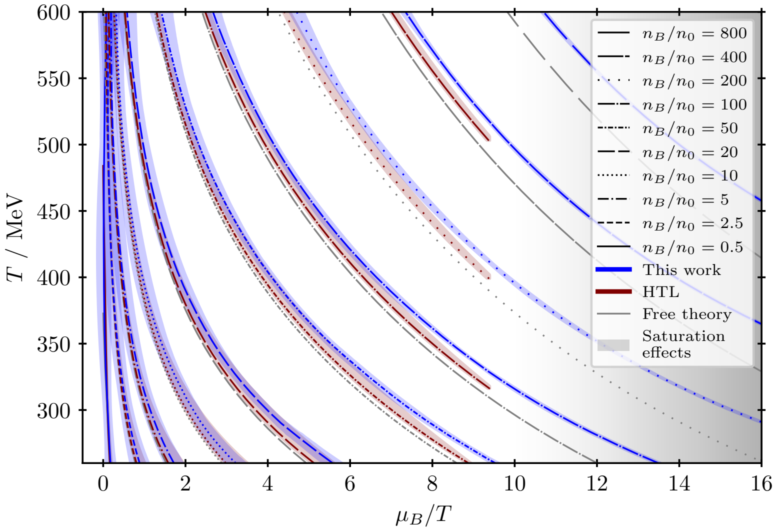
$s/n_B = 9$
$s/n_B = 11$
$s/n_B = 13$
$s/n_B = 25$
$s/n_B = 30$
$s/n_B = 50$
$s/n_B = 100$
$s/n_B = 200$
$s/n_B = 300$
$s/n_B = \infty$
0.30
0.25
0.20
0.15
0.10
0.05
0.00
$T^4 k_s$
300
350
400
450
500
550
600
T / MeV

Summary and Outlook

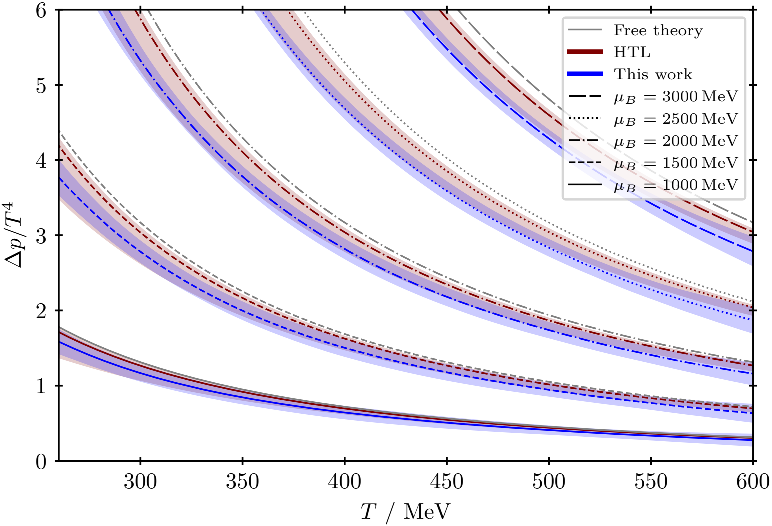
- Equation of state of 2+1+1-flavor QCD at high temperatures and finite chemical potentials
- First results using the Complex Langevin equation at the physical point
- First results at such large chemical potentials

- Push to lower temperatures
- Isospin asymmetry

Further Results



Further Results



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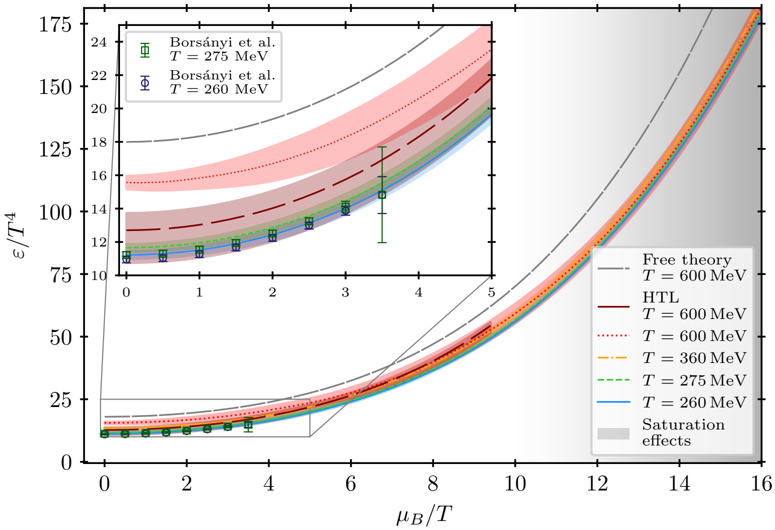

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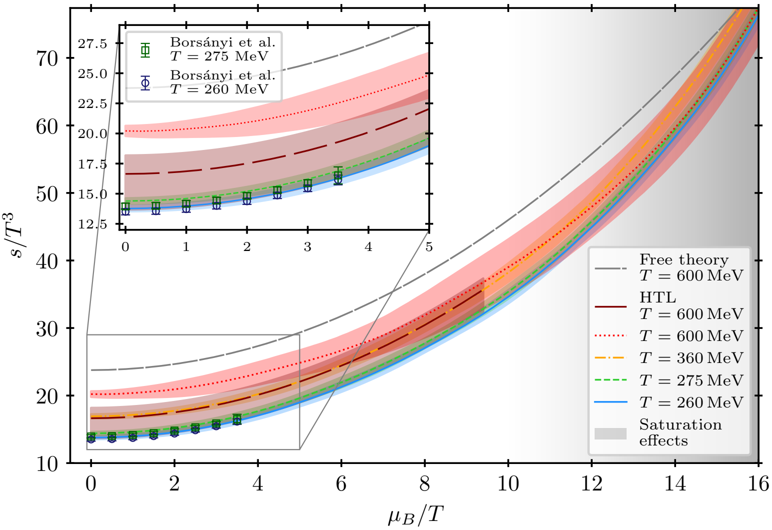

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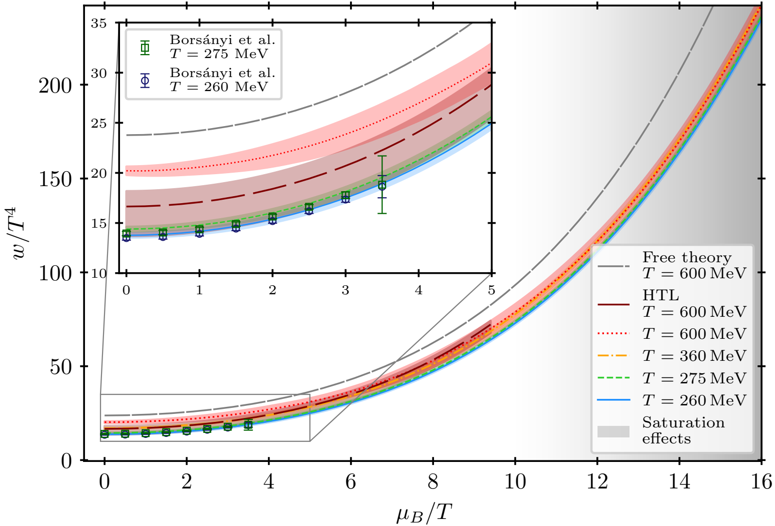

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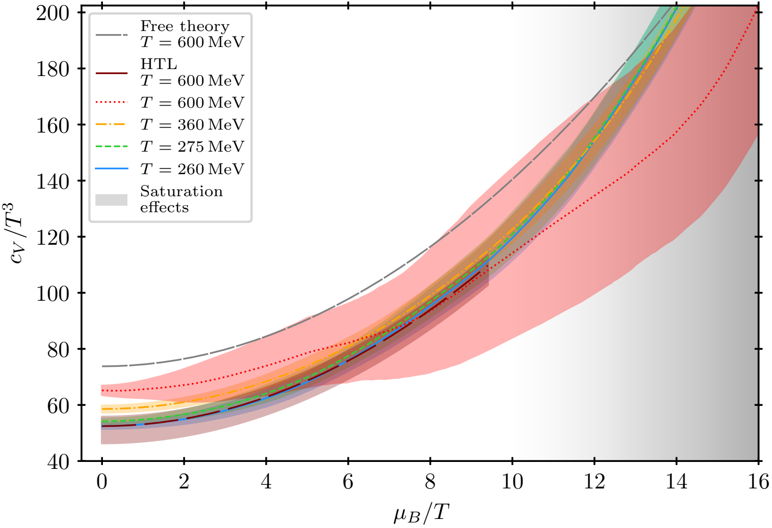

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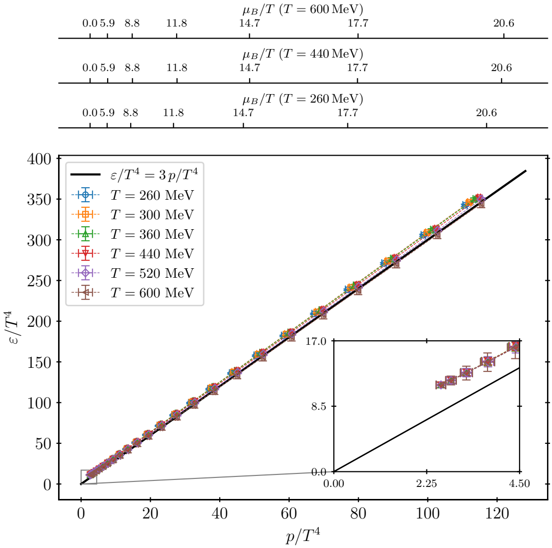


$$\frac{\varepsilon}{T^4} = \frac{I}{T^4} + 3\frac{p}{T^4}$$

Gauge Action

- Tree-level improved Symanzik gauge action

$$S_G[U] = \frac{\beta}{3}\sum_{n\in\Lambda}\sum_{\mu<\nu}\left(c_0\,\mathrm{Re}\left(\mathrm{tr}\left[\mathbb{1} - U_{\mu\nu}(n)\right]\right) + c_1\,\mathrm{Re}\left(\mathrm{tr}\left[\mathbb{1} - R_{\mu\nu}(n)\right]\right)\right) \tag{11}$$

- Gauge coupling $\beta = 6/g^2$
- Plaquette $U_{\mu\nu}(n)$
- $R_{\mu\nu}(n)$ is the sum of the two planar 1×2 and 2×1 rectangular loops in the $\mu\nu$-plane.
- Tree-level Symanzik coefficients $c_1 = -\frac{1}{12}, c_0 = 1 - 8c_1 = \frac{5}{3}$

Fermions

- Fermion matrix for staggered fermions

$$M_f(\mu_f, U)_{nm} =$$
$$\sum_{\nu=1}^{4} \eta_\nu(n) \frac{e^{\delta_{\nu 4}\mu_f} U_\nu(n)\delta_{n+\hat{\nu},m} - e^{-\delta_{\nu 4}\mu_f} U_{-\nu}(n)\delta_{n-\hat{\nu},m}}{2a} + m_f \delta_{n,m} \tag{12}$$

- Staggered sign functions

$$\begin{gathered} \eta_1(n) = 1,\ \eta_2(n) = (-1)^{n_1}, \\ \eta_3(n) = (-1)^{n_1+n_2},\ \eta_4(n) = (-1)^{n_1+n_2+n_3} \end{gathered} \tag{13}$$

Thermodynamics

- Quark density in lattice units[12]:

$$\langle a^3 n_l \rangle = \frac{1}{N_s^3 N_t} \frac{\partial \ln Z}{\partial (a\mu_l)} = \frac{1}{N_s^3 N_t} \langle \mathrm{tr}[M_l^{-1} \partial_{(a\mu_l)} M_l] \rangle \equiv a^3 n_l(T, a\mu_l) \tag{14}$$

- Convert to dimensionless physical quantities using the temperature relation $T = 1/aN_t$:

$$\frac{n_l(T, \mu_l/T)}{T^3} = N_t^3 a^3 n_l$$

[12] Sexty 2019, *Physical Review D*

Thermodynamics

- Pressure difference:

$$\frac{\Delta p}{T^4} = N_t^4 \int_0^{a\mu_l} \mathrm{d}(a\mu_l')\, a^3 n_l(a\mu_l')$$

$$\frac{p(T,\mu/T)}{T^4} = \frac{p(T,\mu/T=0)}{T^4}^{13} + \frac{\Delta p(T,\mu/T)}{T^4}$$

- Using thermodynamic relations → computation of further quantities

[13] Borsányi et al. 2016, *Nature*

Error Analysis

- Statistical uncertainties: jackknife routine with blocking, block size ≥ 6 times the integrated autocorrelation time

- Systematic error: varying several parameters:

- Thermalization cut: 7, 8, 9, 10 units in Langevin time

- Interpolation method in the μ_B/T direction: cubic spline or ninth-degree polynomial containing only odd powers due to symmetry constraints (for the density)

- For each fixed temperature, 16 different values of μ_B/T are available: different subset of data points used for interpolation (all, even, or odd)

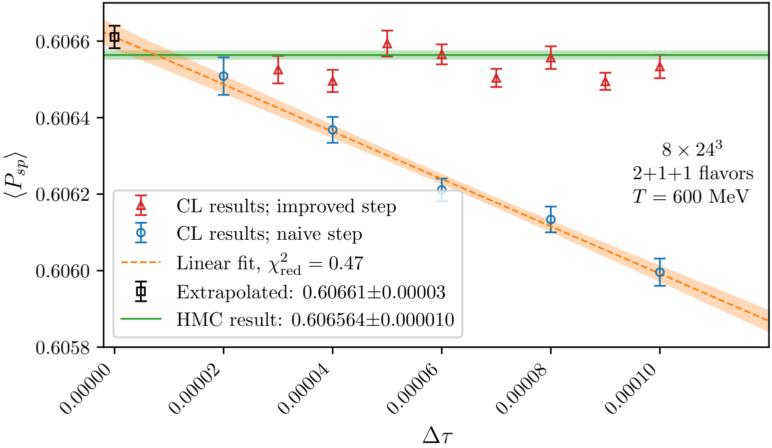
Finite-Volume Effects

- For $N_t = 10$, simulations were performed with larger aspect ratios $N_s/N_t = 3$ and 4
- The averages of observables (densities, Polyakov loop, chiral condensates) remain unchanged within statistical uncertainties.
- Simulations were carried out at various temperatures: $T = 300, 360, 440, 520, 600\,\mathrm{MeV}$
- Finite-volume effects are smaller than the statistical uncertainties

Some Side Remarks

- $\approx 520{,}000$ samples generated
- $\approx 270{,}000$ hours of GPU time

Continuum Extrapolation in Langevin Time



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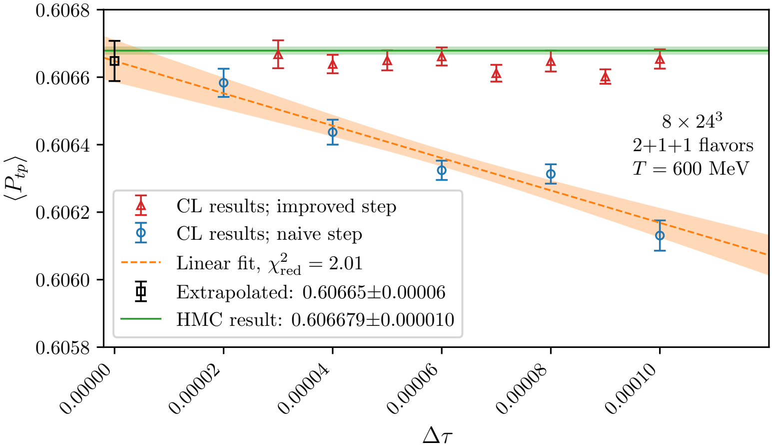

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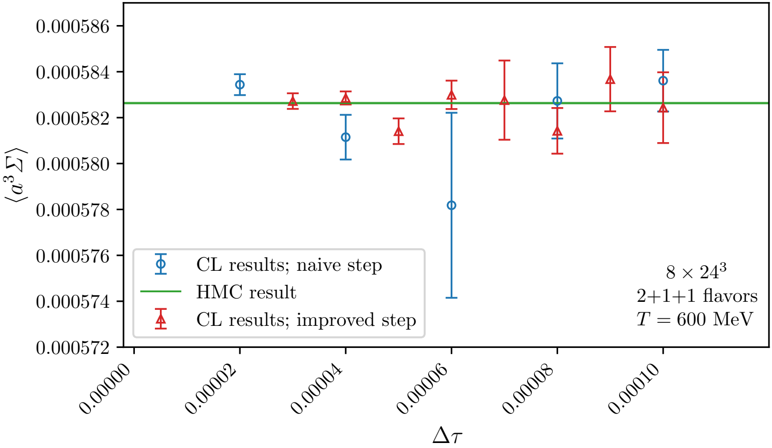

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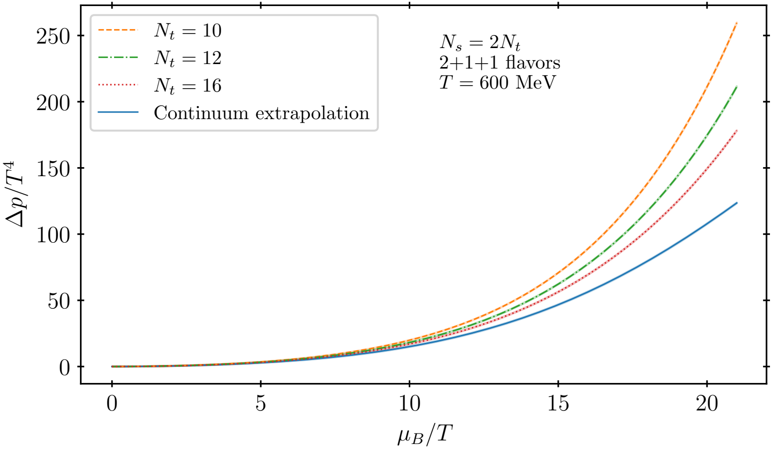

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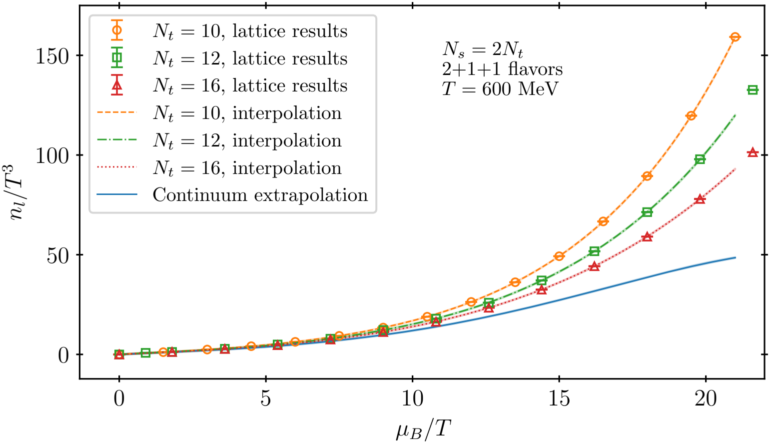

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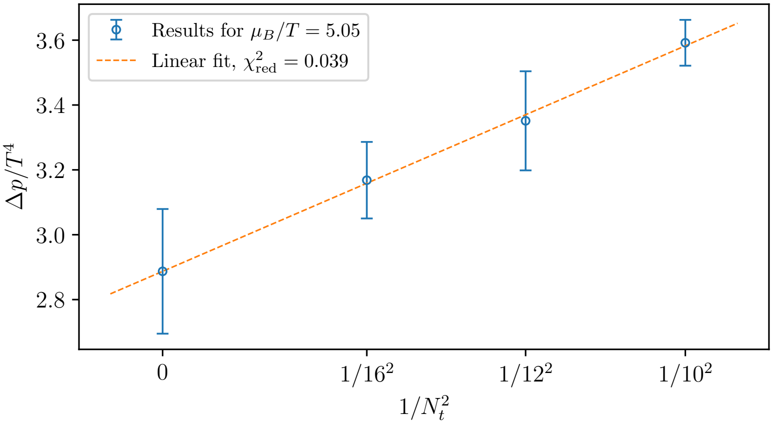

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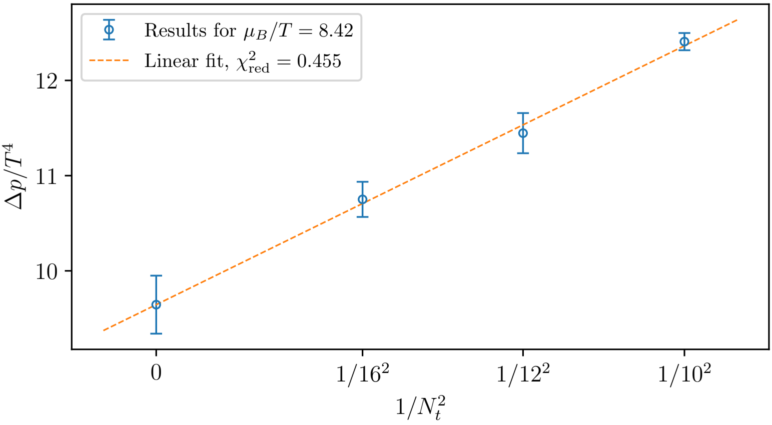

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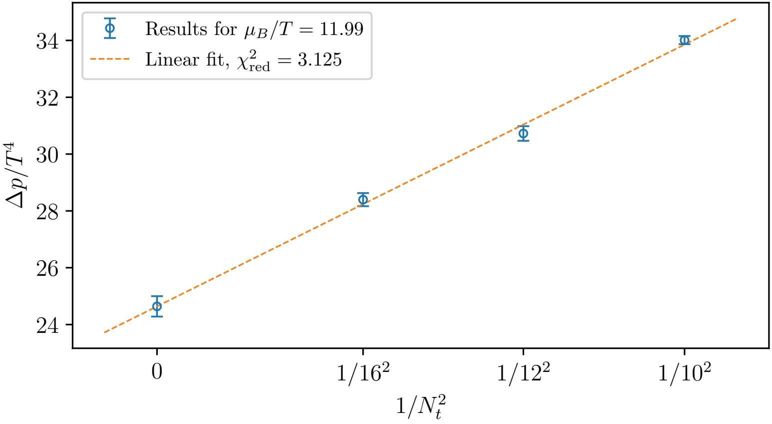

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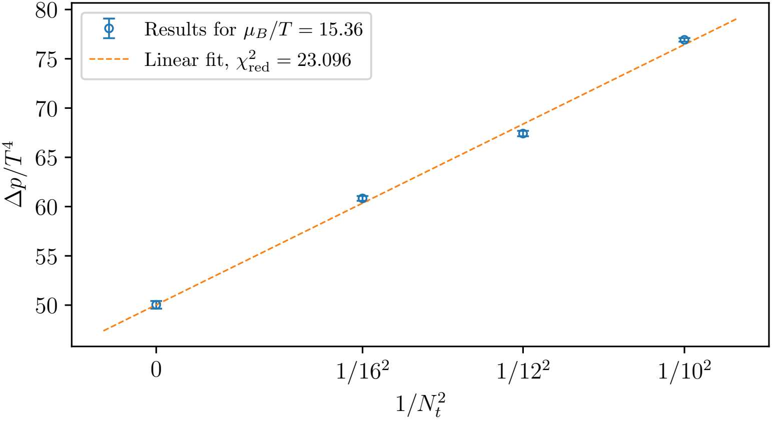
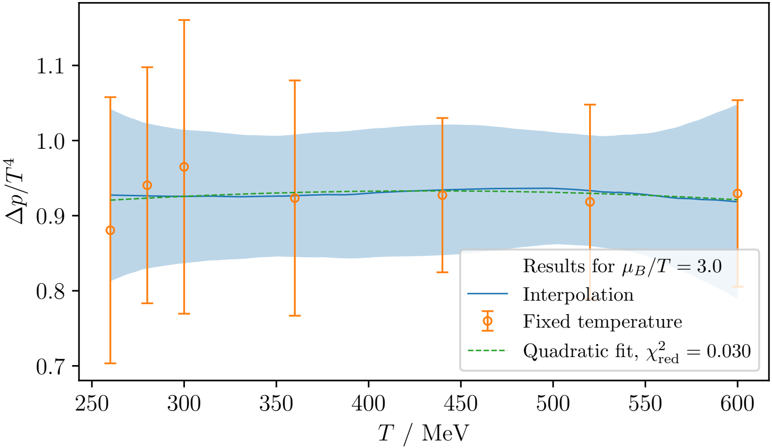


Interpolation in the Temperature-direction



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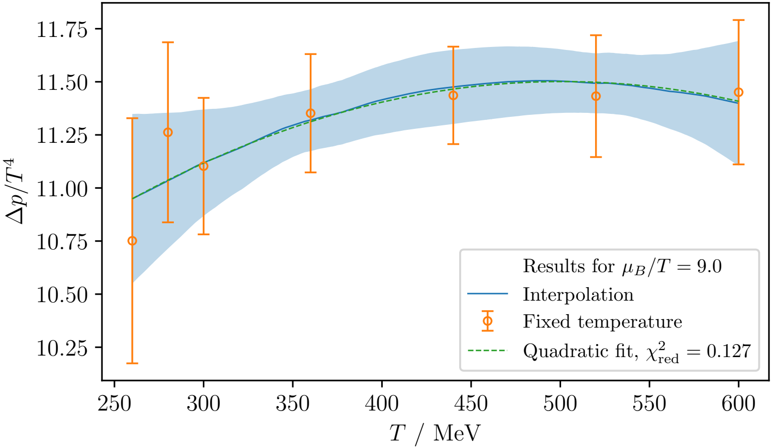

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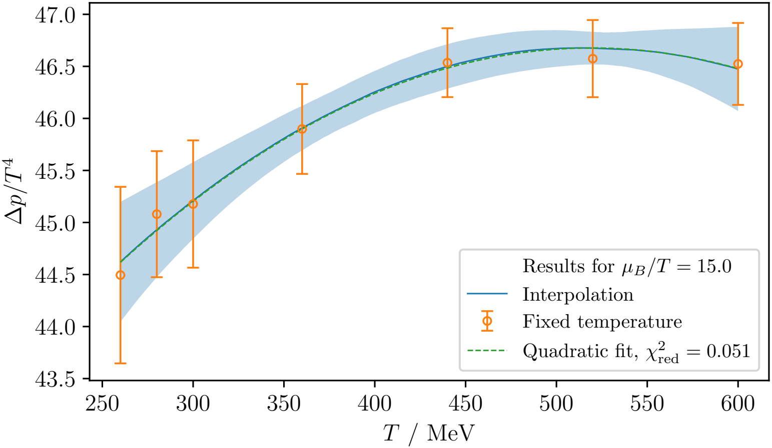

Crosscheck of the Trace Anomaly