



Gauge-invariant renormalization of $\Delta F = 1$ four-quark operators

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Introduction - Motivation

- ▶ **Rare flavor-changing neutral-current decays are powerful probes of new physics** [M. Fedele, PoS (WIFAI2023) 022]
 - $b \rightarrow s(d)$ transitions: $B \rightarrow K^* \gamma$, $B \rightarrow K^{(*)} \ell^+ \ell^-$, $\bar{B}_s \rightarrow \gamma \ell^+ \ell^-$
- ▶ **Long-distance hadronic matrix elements involving four-quark operators**
 - Dominant theoretical uncertainty in B-decay phenomenology
- ▶ **First-principles nonperturbative determinations of complex Minkowski-space amplitudes via lattice QCD**
 - Spectral-density methods [R. Frezzotti et al., Phys. Rev. D 108 (2023) 074510]

$$i \int d^4x e^{ip \cdot x} \langle F | T[J_1(x) J_2(0)] | I \rangle$$

- ▶ **Spectral reconstruction makes the renormalization of four-quark operators even more important**
 - Complicated mixing pattern with operators of the same and/or lower dimension (UV power divergences).
 - Renormalization of contact (UV power) divergences in the time-ordered products of four-quark operators and electromagnetic currents

Introduction - Motivation

► **Gauge-invariant/coordinate-space renormalization scheme (GIRS):**

[V. Gimenez et al., PLB598 (2004) 227,
M. Costa et al., Phys. Rev. D 103 (2021) 094509]

- Consider vacuum expectation values of two or more gauge-invariant operators $\mathcal{O}_i$ at different spacetime points:

$$\langle \mathcal{O}_1(x_1) \mathcal{O}_2(x_2) \cdots \mathcal{O}_n(x_n) \rangle, \quad (x_1 \neq x_2 \neq \cdots \neq x_n)$$

- Integrate (sum, on the lattice) over time slices

$$\int d^3 \vec{x}_1 \cdots \int d^3 \vec{x}_{n-1} \langle \mathcal{O}_1(\vec{x}_1, t_1) \cdots \mathcal{O}_{n-1}(\vec{x}_{n-1}, t_{n-1}) \mathcal{O}_n(\vec{0}, 0) \rangle,$$

$$(t_1 \neq t_2 \neq \cdots \neq t_{n-1} \neq 0)$$

- **Good features:**

1. No need for gauge-fixing, **2.** No contact terms, **3.** No need to consider mixing with gauge-non-invariant operators

- **Challenges:**

1. When mixing occurs, $n > 2$ -point Green's functions are needed,
2. Calculation of Feynman diagrams with more loops at each perturbative order than Green's functions with external elementary fields,
3. Narrow renormalization window: $\Lambda_{\text{QCD}} \ll \mu \ll a^{-1}$, $\mu \sim |x_i - x_j|^{-1}$.

Operator Setup

- **Four-quark operators:** (Lorentz scalar and pseudoscalar)

$$(\mathcal{O}_{\Gamma\Gamma'})_{f_3,f_4}^{f_1,f_2} \equiv \left(\sum_a \sum_{\alpha,\beta} \bar{\psi}_\alpha^{a,f_1}(x) \Gamma_{\alpha\beta} \psi_{\beta,f_3}^a(x) \right) \left(\sum_{a'} \sum_{\alpha',\beta'} \bar{\psi}_{\alpha'}^{a',f_2}(x) \Gamma'_{\alpha'\beta'} \psi_{\beta',f_4}^{a'}(x) \right),$$

where $\Gamma \in \{\mathbf{1}, \gamma_5, \gamma_\mu, \gamma_\mu \gamma_5, \sigma_{\mu\nu}, \sigma_{\mu\nu} \gamma_5\} \equiv \{S, P, V, A, T, \tilde{T}\}$, $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$,
 $\Gamma' \in \{\Gamma, \gamma_5 \Gamma\}$

- We define operators with symmetric/antisymmetric flavor indices:

$$\begin{aligned} (\mathcal{O}_{\{\Gamma,\Gamma'\}}^+)_{f_3,f_4}^{f_1,f_2} &\equiv \frac{1}{2} \left[(\mathcal{O}_{\{\Gamma,\Gamma'\}})_{f_3,f_4}^{f_1,f_2} + (\mathcal{O}_{\{\Gamma,\Gamma'\}})_{f_4,f_3}^{f_1,f_2} \right], \\ (\mathcal{O}_{\{\Gamma,\Gamma'\}}^-)_{f_3,f_4}^{f_1,f_2} &\equiv \frac{1}{2} \left[(\mathcal{O}_{\{\Gamma,\Gamma'\}})_{f_3,f_4}^{f_1,f_2} - (\mathcal{O}_{\{\Gamma,\Gamma'\}})_{f_4,f_3}^{f_1,f_2} \right], \\ (\mathcal{O}_{[\Gamma,\Gamma']}^+)_{f_3,f_4}^{f_1,f_2} &\equiv \frac{1}{2} \left[(\mathcal{O}_{[\Gamma,\Gamma']})_{f_3,f_4}^{f_1,f_2} + (\mathcal{O}_{[\Gamma,\Gamma']})_{f_4,f_3}^{f_1,f_2} \right], \\ (\mathcal{O}_{[\Gamma,\Gamma']}^-)_{f_3,f_4}^{f_1,f_2} &\equiv \frac{1}{2} \left[(\mathcal{O}_{[\Gamma,\Gamma']})_{f_3,f_4}^{f_1,f_2} - (\mathcal{O}_{[\Gamma,\Gamma']})_{f_4,f_3}^{f_1,f_2} \right], \end{aligned}$$

where

$$\begin{aligned} (\mathcal{O}_{\{\Gamma,\Gamma'\}})_{f_3,f_4}^{f_1,f_2} &\equiv (\mathcal{O}_{\Gamma\Gamma'+\Gamma'\Gamma})_{f_3,f_4}^{f_1,f_2} \equiv (\mathcal{O}_{\Gamma\Gamma'})_{f_3,f_4}^{f_1,f_2} + (\mathcal{O}_{\Gamma'\Gamma})_{f_3,f_4}^{f_1,f_2}, \\ (\mathcal{O}_{[\Gamma,\Gamma']})_{f_3,f_4}^{f_1,f_2} &\equiv (\mathcal{O}_{\Gamma\Gamma'-\Gamma'\Gamma})_{f_3,f_4}^{f_1,f_2} \equiv (\mathcal{O}_{\Gamma\Gamma'})_{f_3,f_4}^{f_1,f_2} - (\mathcal{O}_{\Gamma'\Gamma})_{f_3,f_4}^{f_1,f_2}, \end{aligned}$$

Operator Setup

- **Classification into irreps of $SU(N_f)$** (for mass-degenerate quarks)

$$N_f - 1 \left\{ \begin{array}{c} \square \\ \vdots \\ \square \end{array} \right\} \otimes N_f - 1 \left\{ \begin{array}{c} \square \\ \vdots \\ \square \end{array} \right\} \otimes \square \otimes \square =$$

$$N_f - 1 \left\{ \begin{array}{c} \square \square \square \\ \square \square \square \\ \vdots \\ \square \square \square \end{array} \right\} \oplus N_f \left\{ \begin{array}{c} \square \square \square \\ \square \square \square \\ \vdots \\ \square \square \square \end{array} \right\} \oplus N_f \left\{ \begin{array}{c} \square \square \\ \square \square \square \\ \vdots \\ \square \square \square \end{array} \right\} \oplus N_f - 1 \left\{ \begin{array}{c} \square \square \square \\ \square \square \square \\ \vdots \\ \square \square \square \end{array} \right\} \oplus N_f \left\{ \begin{array}{c} \square \square \square \\ \square \square \square \\ \vdots \\ \square \square \square \end{array} \right\} \oplus$$

$$(\widetilde{\mathcal{O}}^+_{\{\Gamma, \Gamma'\}})^{f_1, f_2}_{f_3, f_4}$$

$$(\overline{\mathcal{O}}^+_{[\Gamma, \Gamma']})^{f_1, f_2}_{f_3, f_4}$$

$$(\widehat{\mathcal{O}}^+_{\{\Gamma, \Gamma'\}})^{f_1, f_2}_{f_3, f_4}$$

$$(\widetilde{\mathcal{O}}^-_{[\Gamma, \Gamma']})^{f_1, f_2}_{f_3, f_4}$$

$$(\overline{\mathcal{O}}^-_{[\Gamma, \Gamma']})^{f_1, f_2}_{f_3, f_4}$$

$$N_f \left\{ \begin{array}{c} \square \square \square \\ \square \square \square \\ \vdots \\ \square \square \square \end{array} \right\} \oplus N_f \left\{ \begin{array}{c} \square \square \square \\ \square \square \square \\ \vdots \\ \square \square \square \end{array} \right\} \oplus N_f \left\{ \begin{array}{c} \square \square \square \\ \square \square \square \\ \vdots \\ \square \square \square \end{array} \right\} \oplus N_f \left\{ \begin{array}{c} \square \square \square \\ \square \square \square \\ \vdots \\ \square \square \square \end{array} \right\} \oplus N_f \left\{ \begin{array}{c} \square \square \square \\ \square \square \square \\ \vdots \\ \square \square \square \end{array} \right\} \oplus$$

$$(\widetilde{\mathcal{O}}^+_{[\Gamma, \Gamma']})^{f_1, f_2}_{f_3, f_4}$$

$$(\overline{\mathcal{O}}^+_{[\Gamma, \Gamma']})^{f_1, f_2}_{f_3, f_4}$$

$$(\widetilde{\mathcal{O}}^-_{\{\Gamma, \Gamma'\}})^{f_1, f_2}_{f_3, f_4}$$

$$(\overline{\mathcal{O}}^-_{\{\Gamma, \Gamma'\}})^{f_1, f_2}_{f_3, f_4}$$

$$(\widehat{\mathcal{O}}^-_{[\Gamma, \Gamma']})^{f_1, f_2}_{f_3, f_4}$$

Operator Setup

- **Classification into irreps of $SU(N_f)$** (for mass-degenerate quarks)

$$(\check{\mathcal{O}}_{[\Gamma, \Gamma']}^{\pm})_{f_3, f_4}^{f_1, f_2} \equiv (\mathcal{O}_{[\Gamma, \Gamma']}^{\pm})_{f_3, f_4}^{f_1, f_2} - \frac{1}{(N_f \pm 2)} (\overline{\mathcal{O}}_{[\Gamma, \Gamma']}^{\pm})_{f_3, f_4}^{f_1, f_2} - \frac{1}{N_f(N_f \pm 1)} (\hat{\mathcal{O}}_{[\Gamma, \Gamma']}^{\pm})_{f_3, f_4}^{f_1, f_2},$$

$$\begin{aligned} (\overline{\mathcal{O}}_{[\Gamma, \Gamma']}^{\pm})_{f_3, f_4}^{f_1, f_2} \equiv & \sum_{f'} \left(\delta^{f_1} f_3 (\mathcal{O}_{[\Gamma, \Gamma']}^{\pm})_{f', f_4}^{f', f_2} + \delta^{f_2} f_3 (\mathcal{O}_{[\Gamma, \Gamma']}^{\pm})_{f', f_4}^{f_1, f'} \right. \\ & \left. + \delta^{f_1} f_4 (\mathcal{O}_{[\Gamma, \Gamma']}^{\pm})_{f_3, f'}^{f', f_2} + \delta^{f_2} f_4 (\mathcal{O}_{[\Gamma, \Gamma']}^{\pm})_{f_3, f'}^{f_1, f'} \right) - \frac{2}{N_f} (\hat{\mathcal{O}}_{[\Gamma, \Gamma']}^{\pm})_{f_3, f_4}^{f_1, f_2}, \end{aligned}$$

$$(\hat{\mathcal{O}}_{[\Gamma, \Gamma']}^{\pm})_{f_3, f_4}^{f_1, f_2} \equiv (\delta^{f_1} f_3 \delta^{f_2} f_4 \pm \delta^{f_1} f_4 \delta^{f_2} f_3) \sum_{f', f''} (\mathcal{O}_{[\Gamma, \Gamma']}^{\pm})_{f', f''}^{f', f''},$$

$$(\check{\mathcal{O}}_{[\Gamma, \Gamma']}^{\pm})_{f_3, f_4}^{f_1, f_2} \equiv (\mathcal{O}_{[\Gamma, \Gamma']}^{\pm})_{f_3, f_4}^{f_1, f_2} - \frac{1}{N_f} (\overline{\mathcal{O}}_{[\Gamma, \Gamma']}^{\pm})_{f_3, f_4}^{f_1, f_2},$$

$$\begin{aligned} (\overline{\mathcal{O}}_{[\Gamma, \Gamma']}^{\pm})_{f_3, f_4}^{f_1, f_2} \equiv & \sum_{f'} \left(\delta^{f_1} f_3 (\mathcal{O}_{[\Gamma, \Gamma']}^{\pm})_{f', f_4}^{f', f_2} + \delta^{f_2} f_3 (\mathcal{O}_{[\Gamma, \Gamma']}^{\pm})_{f', f_4}^{f_1, f'} \right. \\ & \left. + \delta^{f_1} f_4 (\mathcal{O}_{[\Gamma, \Gamma']}^{\pm})_{f_3, f'}^{f', f_2} + \delta^{f_2} f_4 (\mathcal{O}_{[\Gamma, \Gamma']}^{\pm})_{f_3, f'}^{f_1, f'} \right), \end{aligned}$$

$\check{\mathcal{O}}^{\pm}$: completely traceless, $\hat{\mathcal{O}}^{\pm}$: pure trace (flavor singlets), $\overline{\mathcal{O}}^{\pm}$: support the adjoint representation of the flavor group.

Operator Setup

► Operators for flavor-changing processes

- $\underline{\Delta F = 2}$: $(f_1 = f_2 \text{ and } f_1 \notin \{f_3, f_4\}) \text{ or } (f_3 = f_4 \text{ and } f_3 \notin \{f_1, f_2\})$

⇒ Only traceless operators:

$$\check{\mathcal{O}}_{\{\Gamma, \Gamma'\}}^{\pm}, \check{\mathcal{O}}_{[\Gamma, \Gamma']}^{\pm}$$

- $\underline{\Delta F = 1}$:

1. (f_1, f_2, f_3, f_4) are all different

⇒ Only traceless operators:

$$\check{\mathcal{O}}_{\{\Gamma, \Gamma'\}}^{\pm}, \check{\mathcal{O}}_{[\Gamma, \Gamma']}^{\pm}$$

2. $(f_1 \in \{f_3, f_4\} \text{ and } f_2 \notin \{f_3, f_4\}) \text{ or } (f_1 \notin \{f_3, f_4\} \text{ and } f_2 \in \{f_3, f_4\})$

⇒ Operators transforming under traceless or adjoint reps:

$$\check{\mathcal{O}}_{\{\Gamma, \Gamma'\}}^{\pm}, \check{\mathcal{O}}_{[\Gamma, \Gamma']}^{\pm}, \overline{\mathcal{O}}_{\{\Gamma, \Gamma'\}}^{\pm}, \overline{\mathcal{O}}_{[\Gamma, \Gamma']}^{\pm}$$

- $\underline{\Delta F = 0}$: (f_1, f_2) are pairwise equal to (f_3, f_4)

⇒ All classes of operators:

$$\check{\mathcal{O}}_{\{\Gamma, \Gamma'\}}^{\pm}, \check{\mathcal{O}}_{[\Gamma, \Gamma']}^{\pm}, \overline{\mathcal{O}}_{\{\Gamma, \Gamma'\}}^{\pm}, \overline{\mathcal{O}}_{[\Gamma, \Gamma']}^{\pm}, \hat{\mathcal{O}}_{\{\Gamma, \Gamma'\}}^{\pm}$$

$\Delta F = 1$ traceless operators

- **Operator basis** [Frezzotti and Rossi, JHEP 10 (2004) 070]

Parity even

$$Q_1^\pm \equiv \check{O}_{VV+AA}^\pm,$$

$$Q_2^\pm \equiv \check{O}_{VV-AA}^\pm,$$

$$Q_3^\pm \equiv \check{O}_{SS-PP}^\pm,$$

$$Q_4^\pm \equiv \check{O}_{SS+PP}^\pm,$$

$$Q_5^\pm \equiv \check{O}_{TT}^\pm,$$

Parity odd

$$\mathcal{Q}_1^\pm \equiv \check{O}_{VA+AV}^\pm,$$

$$\mathcal{Q}_2^\pm \equiv \check{O}_{VA-AV}^\pm,$$

$$\mathcal{Q}_3^\pm \equiv \check{O}_{PS-SP}^\pm,$$

$$\mathcal{Q}_4^\pm \equiv \check{O}_{PS+SP}^\pm,$$

$$\mathcal{Q}_5^\pm \equiv \check{O}_{T\tilde{T}}^\pm.$$

- **Mixing sets:** Consider also CP symmetries [Spanoudes et al., Phys.Rev.D 113 (2026) 034504]

$$S_1^\pm : \{Q_1^\pm, Q_2^\pm, Q_3^\pm, Q_4^\pm, Q_5^\pm\},$$

$$S_{2a}^\pm : \{\mathcal{Q}_1^\pm\},$$

$$S_{2b}^\pm : \{\mathcal{Q}_4^\pm, \mathcal{Q}_5^\pm\},$$

$$S_3^\pm : \{\mathcal{Q}_2^\pm, \mathcal{Q}_3^\pm\},$$

Renormalization Setup

► Mixing matrices

$$Q_l^{\pm,R} = \sum_{m=1}^5 Z_{lm}^{\pm} \cdot Q_m^{\pm}, \quad \mathcal{Q}_l^{\pm,R} = \sum_{m=1}^5 \mathcal{Z}_{lm}^{\pm} \cdot \mathcal{Q}_m^{\pm}, \quad l = 1, \dots, 5$$

$$Z^{\pm} = \begin{pmatrix} Z_{11}^{\pm} & Z_{12}^{\pm} & Z_{13}^{\pm} & Z_{14}^{\pm} & Z_{15}^{\pm} \\ Z_{21}^{\pm} & Z_{22}^{\pm} & Z_{23}^{\pm} & Z_{24}^{\pm} & Z_{25}^{\pm} \\ Z_{31}^{\pm} & Z_{32}^{\pm} & Z_{33}^{\pm} & Z_{34}^{\pm} & Z_{35}^{\pm} \\ Z_{41}^{\pm} & Z_{42}^{\pm} & Z_{43}^{\pm} & Z_{44}^{\pm} & Z_{45}^{\pm} \\ Z_{51}^{\pm} & Z_{52}^{\pm} & Z_{53}^{\pm} & Z_{54}^{\pm} & Z_{55}^{\pm} \end{pmatrix}, \quad \mathcal{Z}^{\pm} = \begin{pmatrix} \mathcal{Z}_{11}^{\pm} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{Z}_{22}^{\pm} & \mathcal{Z}_{23}^{\pm} & 0 & 0 \\ 0 & \mathcal{Z}_{32}^{\pm} & \mathcal{Z}_{33}^{\pm} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{Z}_{44}^{\pm} & \mathcal{Z}_{45}^{\pm} \\ 0 & 0 & 0 & \mathcal{Z}_{54}^{\pm} & \mathcal{Z}_{55}^{\pm} \end{pmatrix}.$$

- To determine all mixing coefficients (diagonal+nondiagonal), we need to define 25 conditions for Z^+ and 9 conditions for $\mathcal{Z}^+$. Similarly, for Z^- and $\mathcal{Z}^-$.

Renormalization Setup

► Green's functions in GIRS:

- The determination of the mixing matrices requires the calculation of both two-point and three-point functions.
- For the parity-even $Q^\pm$, we consider:

$$\begin{aligned} G_{Q_i^\pm; Q_j^\pm}^{2\text{pt}}(t) &\equiv \int d^3\vec{x} \left\langle Q_i^\pm(\vec{x}, t) \left[Q_j^\pm(\vec{0}, 0) \right]^\dagger \right\rangle, \\ G_{\mathcal{O}_\Gamma; Q_i^\pm; \mathcal{O}_\Gamma}^{3\text{pt}}(t, t') &\equiv \int d^3\vec{x} \int d^3\vec{x}' \left\langle \mathcal{O}_\Gamma(\vec{x}, t) Q_i^\pm(\vec{0}, 0) \mathcal{O}_\Gamma(-\vec{x}', -t') \right\rangle, \end{aligned}$$

where $(\mathcal{O}_\Gamma)^{f_1}_{f_2}(x) \equiv \bar{\psi}^{f_1}(x) \Gamma \psi_{f_2}(x)$ is a quark bilinear operator.

- Similarly, for the parity-odd $\mathcal{Q}^\pm$, we consider:

$$\begin{aligned} G_{\mathcal{Q}_i^\pm; \mathcal{Q}_j^\pm}^{2\text{pt}}(t) &\equiv \int d^3\vec{x} \left\langle \mathcal{Q}_i^\pm(\vec{x}, t) \left[\mathcal{Q}_j^\pm(\vec{0}, 0) \right]^\dagger \right\rangle, \\ G_{\mathcal{O}_\Gamma; \mathcal{Q}_i^\pm; \mathcal{O}_{\Gamma\gamma_5}}^{3\text{pt}}(t, t') &\equiv \int d^3\vec{x} \int d^3\vec{x}' \left\langle \mathcal{O}_\Gamma(\vec{x}, t) \mathcal{Q}_i^\pm(\vec{0}, 0) \mathcal{O}_{\Gamma\gamma_5}(-\vec{x}', -t') \right\rangle, \end{aligned}$$

Renormalization Setup

► **Renormalization conditions in GIRS:**

- For the parity-even operators $Q^\pm$, the **25** conditions can be obtained:
 * **15** conditions from two-point functions:

$$[G_{Q_i^\pm; Q_j^\pm}^{2\text{pt}}(t)]^{\text{GIRS}} \equiv \sum_{k,l=1}^5 (Z_{ik}^\pm)^{\text{GIRS}} (Z_{jl}^\pm)^{\text{GIRS}} G_{Q_k^\pm; Q_l^\pm}^{2\text{pt}}(t) = [G_{Q_i^\pm; Q_j^\pm}^{2\text{pt}}(t)]^{\text{free}}, \quad i \leq j,$$

- * **10** conditions from three-point functions: **30!/(10! 20!) possible choices!**

$$[G_{\mathcal{O}_\Gamma; Q_i^\pm; \mathcal{O}_\Gamma}^{3\text{pt}}(t, t)]^{\text{GIRS}} \equiv (Z_{\mathcal{O}_\Gamma}^{\text{GIRS}})^2 \sum_{k=1}^5 (Z_{ik}^\pm)^{\text{GIRS}} G_{\mathcal{O}_\Gamma; Q_k^\pm; \mathcal{O}_\Gamma}^{3\text{pt}}(t, t) = [G_{\mathcal{O}_\Gamma; Q_i^\pm; \mathcal{O}_\Gamma}^{3\text{pt}}(t, t)]^{\text{free}},$$

where $i \in [1, 5]$, $\Gamma \in \{\mathbf{1}, \gamma_5, \gamma_m, \gamma_m \gamma_5, \sigma_{mn}, \sigma_{m4}\}$ (m, n are spatial directions).

Renormalization Setup

► Renormalization conditions in GIRS:

- For the parity-odd operators $Q^\pm$, the **9** conditions can be obtained:
- * **7** conditions from two-point functions: (1 + 3 + 3 for each block, respectively)

$$[G_{Q_1^\pm; Q_1^\pm}^{2pt}(t)]^{GIRS} \equiv [(Z_{11}^\pm)^{GIRS}]^2 G_{Q_1^\pm; Q_1^\pm}^{2pt}(t) = [G_{Q_1^\pm; Q_1^\pm}^{2pt}(t)]^{free},$$

$$[G_{Q_i^\pm; Q_j^\pm}^{2pt}(t)]^{GIRS} \equiv \sum_{k,l=2}^3 (Z_{ik}^\pm)^{GIRS} (Z_{jl}^\pm)^{GIRS} G_{Q_k^\pm; Q_l^\pm}^{2pt}(t) = [G_{Q_i^\pm; Q_j^\pm}^{2pt}(t)]^{free}, \quad (i, j = 2, 3),$$

$$[G_{Q_i^\pm; Q_j^\pm}^{2pt}(t)]^{GIRS} \equiv \sum_{k,l=4}^5 (Z_{ik}^\pm)^{GIRS} (Z_{jl}^\pm)^{GIRS} G_{Q_k^\pm; Q_l^\pm}^{2pt}(t) = [G_{Q_i^\pm; Q_j^\pm}^{2pt}(t)]^{free}, \quad (i, j = 4, 5),$$

where $i \leq j$.

- * **2** conditions from three-point functions: **6 possible choices per block!**

$$\begin{aligned} [G_{O_\Gamma; Q_i^\pm; O_{\Gamma\gamma_5}}^{3pt}(t, t')]^{GIRS} &\equiv Z_{O_\Gamma}^{GIRS} Z_{O_{\Gamma\gamma_5}}^{GIRS} \sum_{k=2}^3 (Z_{ik}^\pm)^{GIRS} G_{O_\Gamma; Q_k^\pm; O_{\Gamma\gamma_5}}^{3pt}(t, t) \\ &= [G_{O_\Gamma; Q_i^\pm; O_{\Gamma\gamma_5}}^{3pt}(t, t)]^{free}, \quad (i = 2 \text{ or } 3), \end{aligned}$$

$$\begin{aligned} [G_{O_\Gamma; Q_i^\pm; O_{\Gamma\gamma_5}}^{3pt}(t, t)]^{GIRS} &\equiv Z_{O_\Gamma}^{GIRS} Z_{O_{\Gamma\gamma_5}}^{GIRS} \sum_{k=4}^5 (Z_{ik}^\pm)^{GIRS} G_{O_\Gamma; Q_k^\pm; O_{\Gamma\gamma_5}}^{3pt}(t, t) \\ &= [G_{O_\Gamma; Q_i^\pm; O_{\Gamma\gamma_5}}^{3pt}(t, t)]^{free}, \quad (i = 4 \text{ or } 5), \end{aligned}$$

where $\Gamma \in \{\mathbf{1}, \gamma_m, \sigma_{mn}\}$ (m, n are spatial directions).

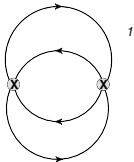
Calculation

- **NLO calculation in dimensional regularization:** [M. Constantinou et al., Phys.Rev.D 110 (2024) 074506, G. Spanoudes et al., Phys.Rev.D 113 (2026) 034504]

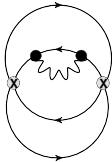
• **Two-point functions:**

$$G_{Q_i^\pm; Q_j^\pm}^{2\text{pt}}(t) = \frac{N_c}{\pi^6 |t|^9} (\delta_{f_3}^{f_1} \delta_{f_4}^{f_2} \pm \delta_{f_4}^{f_1} \delta_{f_3}^{f_2}) \times \\ (\delta_{f_3}^{f_1'} \delta_{f_4}^{f_2'} \pm \delta_{f_3}^{f_2'} \delta_{f_4}^{f_1'}) (a_{ij;0}^\pm + a_{ij;1}^\pm N_c),$$

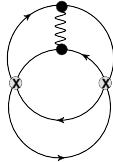
$$G_{\bar{Q}_i^\pm; \bar{Q}_j^\pm}^{2\text{pt}}(t) = \frac{N_c}{\pi^6 |t|^9} (\delta_{f_3}^{f_1} \delta_{f_4}^{f_2} \pm \delta_{f_4}^{f_1} \delta_{f_3}^{f_2}) \times \\ ((-1)^{\delta_{i2}+\delta_{i3}} \delta_{f_3}^{f_1'} \delta_{f_4}^{f_2'} \pm \delta_{f_3}^{f_2'} \delta_{f_4}^{f_1'}) (\tilde{a}_{ij;0}^\pm + \tilde{a}_{ij;1}^\pm N_c)$$



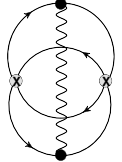
1



2



3



4

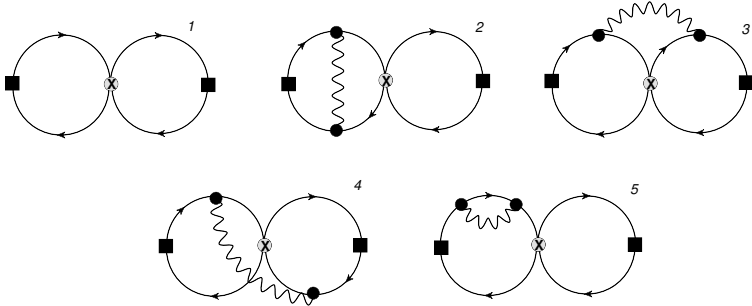
$\otimes = Q_i^\pm$ or $\bar{Q}_i^\pm$. Wavy (solid) lines represent gluons (quarks). Diagrams 2 and 4 have also mirror variants.

Calculation

- Three-point functions:**

$$G_{\mathcal{O}_\Gamma; Q_i^\pm; \mathcal{O}_\Gamma}^{3\text{pt}}(t, t) = \frac{N_c}{\pi^4 t^6} (\delta_{f_3}^{f_1} \delta_{f_4}^{f_2} \pm \delta_{f_4}^{f_1} \delta_{f_3}^{f_2}) \times \\ (\delta_{f_3}^{f_1'} \delta_{f_4}^{f_2'} \pm \delta_{f_3}^{f_2'} \delta_{f_4}^{f_1'}) (d_{i\Gamma;0}^\pm + d_{i\Gamma;1}^\pm N_c),$$

$$G_{\mathcal{O}_\Gamma; \mathcal{Q}_i^\pm; \mathcal{O}_{\Gamma\gamma_5}}^{3\text{pt}}(t, t) = \frac{N_c}{\pi^4 t^6} (\delta_{f_3}^{f_1} \delta_{f_4}^{f_2} \pm \delta_{f_4}^{f_1} \delta_{f_3}^{f_2}) \times \\ ((-1)^{\delta_{i2}+\delta_{i3}} \delta_{f_3}^{f_1'} \delta_{f_4}^{f_2'} \pm \delta_{f_3}^{f_2'} \delta_{f_4}^{f_1'}) (\bar{d}_{i\Gamma;0}^\pm + \bar{d}_{i\Gamma;1}^\pm N_c)$$



$\otimes = Q_i^\pm$ or $\mathcal{Q}_i^\pm$, $\blacksquare = \mathcal{O}_\Gamma$ or $\mathcal{O}_{\Gamma\gamma_5}$. Wavy (solid) lines represent gluons (quarks). Diagrams 2-5 have also mirror variants.

Conversion matrices

► **Conversion matrices from GIRS to $\overline{\text{MS}}$:**

$$(Z^\pm)^{\overline{\text{MS}}} = (C^\pm)^{\overline{\text{MS}}, \text{GIRS}} (Z^\pm)^{\text{GIRS}}, \quad (\mathcal{Z}^\pm)^{\overline{\text{MS}}} = (\tilde{C}^\pm)^{\overline{\text{MS}}, \text{GIRS}} (\mathcal{Z}^\pm)^{\text{GIRS}}.$$

► We explore a wide range of GIRS variants and identify those with reduced mixing effects. For a promising variant, we have extracted the NLO conversion matrices to $\overline{\text{MS}}$ (see next slide).

• **Selected 3pt functions for $Q^\pm$:**

$$G_{S;Q_1^\pm;S}^{3\text{pt}}(t,t), G_{P;Q_1^\pm;P}^{3\text{pt}}(t,t), G_{V_i;Q_1^\pm;V_i}^{3\text{pt}}(t,t), G_{S;Q_2^\pm;S}^{3\text{pt}}(t,t), G_{P;Q_2^\pm;P}^{3\text{pt}}(t,t), \\ G_{S;Q_3^\pm;S}^{3\text{pt}}(t,t), G_{S;Q_5^\pm;S}^{3\text{pt}}(t,t), G_{P;Q_5^\pm;P}^{3\text{pt}}(t,t), G_{V_i;Q_5^\pm;V_i}^{3\text{pt}}(t,t), G_{A_i;Q_5^\pm;A_i}^{3\text{pt}}(t,t),$$

• **Selected 3pt functions for $Q^\pm$:**

$$G_{S;Q_2^\pm;P}^{3\text{pt}}(t,t), G_{S;Q_5^\pm;P}^{3\text{pt}}(t,t).$$

► Alternative variants: a democratic version that treats all mixing operators uniformly (see [\[Spanoudes et al., Phys.Rev.D 113 \(2026\) 034504\]](#))

Conversion matrices

► Conversion matrices from GIRS to $\overline{\text{MS}}$:

$$(C_{ij}^{\pm})^{\overline{\text{MS}},\text{GIRS}} = \delta_{ij} + \frac{g_{\overline{\text{MS}}}^2(\bar{\mu})}{16\pi^2} \sum_{k=-1}^{+1} \left[g_{ij;k}^{\pm} + \left(\ln(\bar{\mu}^2 t^2) + 2\gamma_E \right) h_{ij;k}^{\pm} \right] N_c^k + \mathcal{O}(g_{\overline{\text{MS}}}^4(\bar{\mu})),$$

$$(\tilde{C}_{ij}^{\pm})^{\overline{\text{MS}},\text{GIRS}} = \delta_{ij} + \frac{g_{\overline{\text{MS}}}^2(\bar{\mu})}{16\pi^2} \sum_{k=-1}^{+1} \left[\tilde{g}_{ij;k}^{\pm} + \left(\ln(\bar{\mu}^2 t^2) + 2\gamma_E \right) \tilde{h}_{ij;k}^{\pm} \right] N_c^k + \mathcal{O}(g_{\overline{\text{MS}}}^4(\bar{\mu})),$$

► More results on [\[Spanoudes et al., Phys.Rev.D 113 \(2026\) 034504\]](#)

i	j	$\bar{g}_{ij;-1}^{\pm}$	$\bar{g}_{ij;0}^{\pm}$	$\bar{g}_{ij;+1}^{\pm}$	$\tilde{h}_{ij;-1}^{\pm}$	$\tilde{h}_{ij;0}^{\pm}$	$\tilde{h}_{ij;+1}^{\pm}$
1	1	-869/140	$\pm 379/140$	7/2	3	∓ 3	0
2	2	-9/2	0	7/2	-3	0	0
2	3	0	∓ 2	0	0	∓ 6	0
3	2	0	$\pm 99/280$	0	0	0	0
3	3	-38/35	0	251/140	-3	0	3
4	4	$-307/112 + 3 \ln(2)$	$\pm 169/140$	251/140	-3	∓ 3	3
4	5	$-269/480 + 1/2 \ln(2)$	$\pm (869/1680 - \ln(2))$	0	1	$\mp 1/2$	0
5	4	$-269/40 + 6 \ln(2)$	$\mp (29/140 - 12 \ln(2))$	0	12	± 6	0
5	5	$-1229/240 - 3 \ln(2)$	$\pm 309/140$	1709/420	1	∓ 3	-1

$\Delta F = 1$ operators in the adjoint representations

► Operator basis

Parity even

$$\overline{\mathcal{O}}_{VV+AA}^{\pm},$$

$$\overline{\mathcal{O}}_{VV-AA}^{\pm},$$

$$\overline{\mathcal{O}}_{SS-PP}^{\pm},$$

$$\overline{\mathcal{O}}_{SS+PP}^{\pm},$$

$$\overline{\mathcal{O}}_{TT}^{\pm},$$

Parity odd

$$\overline{\mathcal{O}}_{VA+AV}^{\pm},$$

$$\overline{\mathcal{O}}_{VA-AV}^{\pm},$$

$$\overline{\mathcal{O}}_{PS-SP}^{\pm},$$

$$\overline{\mathcal{O}}_{PS+SP}^{\pm},$$

$$\overline{\mathcal{O}}_{T\bar{T}}^{\pm}.$$

- We focus on the weak current-current operators entering the effective Hamiltonian for $b \rightarrow s$ decays: [\[Frezzotti et al., Phys.Rev.D 113 \(2026\) 034509\]](#)

$$\left(\bar{\psi}_s^i \gamma_{\rho} P_L \psi_b^i\right) \left(\bar{\psi}_c^j \gamma_{\rho} P_L \psi_c^j\right), \quad \left(\bar{\psi}_s^i \gamma_{\rho} P_L \psi_c^i\right) \left(\bar{\psi}_c^j \gamma_{\rho} P_L \psi_b^j\right), \quad P_L = (1 - \gamma_5)/2.$$

- Mix with lower-dimensional operators (power-divergent mixing on the lattice)
Nonperturbative renormalization strategy by [Frezzotti et al., \[Phys.Rev.D 113 \(2026\) 034509\]](#)
- To determine the full operator mixing set, we further investigate relevant spurionic symmetries on the lattice for mass-nongenerate quarks.
- A preliminary study in lattice perturbation theory, using Osterwalder-Seiler (OS) fermions, is ongoing.

$\Delta F = 1$ operators in the adjoint representations

- Candidate operators of dimensions ≤ 6 that can mix with the four-quark operators under study (in addition to the four-quark operators):

$$\mathcal{O}_1^{2F;d=3} = \bar{\psi}_s(x)\psi_b(x)$$

$$\mathcal{O}_2^{2F;d=3} = \bar{\psi}_s(x)\gamma_5\psi_b(x)$$

$$\mathcal{O}_1^{2F;d=5} = g_0 \bar{\psi}_s(x) \sigma_{\mu\nu} G_{\mu\nu}(x) \psi_b(x)$$

$$\mathcal{O}_2^{2F;d=5} = g_0 \bar{\psi}_s(x) \gamma_5 \sigma_{\mu\nu} G_{\mu\nu}(x) \psi_b(x)$$

$$\mathcal{O}_3^{2F;d=5} = \square(\bar{\psi}_s(x)\psi_b(x))$$

$$\mathcal{O}_4^{2F;d=5} = \square(\bar{\psi}_s(x)\gamma_5\psi_b(x))$$

$$\mathcal{O}_1^{2F;d=6} = g_0(\bar{\psi}_s(x)\gamma_\nu G_{\mu\nu}\vec{D}_\mu\psi_b(x) - \bar{\psi}_s(x)\vec{D}_\mu\gamma_\nu G_{\mu\nu}(x)\psi_b(x))$$

$$\mathcal{O}_2^{2F;d=6} = g_0(\bar{\psi}_s(x)\gamma_5\gamma_\nu G_{\mu\nu}\vec{D}_\mu\psi_b(x) - \bar{\psi}_s(x)\vec{D}_\mu\gamma_5\gamma_\nu G_{\mu\nu}(x)\psi_b(x))$$

$$\mathcal{O}_3^{2F;d=6} = g_0 \partial_\mu(\bar{\psi}_s\gamma_\nu G_{\mu\nu}\psi_b)$$

$$\mathcal{O}_4^{2F;d=6} = g_0 \partial_\mu(\bar{\psi}_s\gamma_5\gamma_\nu G_{\mu\nu}\psi_b)$$

$$\mathcal{O}_5^{2F;d=6} = \partial_\mu\partial_\mu\partial_\mu(\bar{\psi}_s(x)\gamma_\mu\psi_b(x))$$

$$\mathcal{O}_6^{2F;d=6} = \partial_\mu\partial_\mu\partial_\mu(\bar{\psi}_s(x)\gamma_5\gamma_\mu\psi_b(x))$$

$\Delta F = 1$ operators in the adjoint representations

$$\begin{aligned}
 \mathcal{O}_7^{2F;d=6} &= \partial_\mu \partial_\mu (\bar{\psi}_s(x) \gamma_\mu \overleftrightarrow{D}_\mu \psi_b(x)) \\
 \mathcal{O}_8^{2F;d=6} &= \partial_\mu \partial_\mu (\bar{\psi}_s(x) \gamma_5 \gamma_\mu \overleftrightarrow{D}_\mu \psi_b(x)) \\
 \mathcal{O}_9^{2F;d=6} &= \partial_\mu (\bar{\psi}_s(x) \overleftarrow{D}_\mu \gamma_\mu \overrightarrow{D}_\mu \psi_b(x)) \\
 \mathcal{O}_{10}^{2F;d=6} &= \partial_\mu (\bar{\psi}_s(x) \overleftarrow{D}_\mu \gamma_5 \gamma_\mu \overrightarrow{D}_\mu \psi_b(x)) \\
 \mathcal{O}_{11}^{2F;d=6} &= \bar{\psi}_s(x) \overleftarrow{D}_\mu \gamma_\mu \overleftrightarrow{D}_\mu \overrightarrow{D}_\mu \psi_b(x) \\
 \mathcal{O}_{12}^{2F;d=6} &= \bar{\psi}_s(x) \overleftarrow{D}_\mu \gamma_5 \gamma_\mu \overleftrightarrow{D}_\mu \overrightarrow{D}_\mu \psi_b(x)
 \end{aligned}$$

- The list has been substantially reduced when considering GIRS. Gauge-noninvariant operators and gauge-invariant operators that vanish by the equations of motion (~ 30) have been excluded.

Summary and outlook

- ▶ We classify the four-quark operators into traceless, singlets and operators that support the adjoint representation of the flavor group.
- ▶ We study the renormalization of traceless operators using GIRS
- ▶ We explore a wide range of GIRS variants in continuum perturbation theory and identify those with reduced mixing effects.
- ▶ For promising variants, we have extracted the next-to-leading-order conversion matrices between GIRS and $\overline{\text{MS}}$ scheme.
- ▶ A renormalization study of $\Delta F = 1$ operators that mix with lower-dimensional operators on the lattice is currently in progress.
- ▶ A renormalization study of contact terms appearing in long-distance amplitudes calculated by spectral reconstruction methods is currently in progress (together with Prof. Martinelli)
- ▶ Future plans: Extend our study to $\Delta F = 0$ operators

Relevant Papers:

1. M. Constantinou, M. Costa, H. Herodotou, H. Panagopoulos and G. Spanoudes, “Gauge-invariant renormalization of four- quark operators,” Phys. Rev. D 110 (2024) 074506 [arXiv:2406.08065 [hep-lat]].
2. G. Spanoudes, M. Costa, K. Mitsidi and H. Panagopoulos, “Classification of four-quark operators with $\Delta F \leq 2$ under flavor symmetry and their renormalization in a gauge-invariant scheme,” Phys. Rev. D 113 (2026) 034504 [arXiv:2511.04305 [hep-lat]]

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BACKUP SLIDES

$\Delta F = 1$ operators in the adjoint representations

- We employ Osterwalder-Seiler (OS) fermions with a clover term [ETMC, Phys. Rev. D 107 (2023) 074506]

$$S_{\text{OS/cl}}(m_f, r_f) = \sum_x \bar{\psi}_f(x) \left[\gamma \cdot \tilde{\nabla} - i\gamma_5 W_{\text{cr}}^{\text{cl}}(r_f) + m_f \right] \psi_f(x),$$

$$\gamma \cdot \tilde{\nabla} = \frac{1}{2} \sum_{\mu} \gamma_{\mu} (\nabla_{\mu} + \nabla_{\mu}^*), \quad W_{\text{cr}}^{\text{cl}}(r_f) = -\frac{r_f}{2} \sum_{\mu} \nabla_{\mu}^* \nabla_{\mu} + M_{\text{cr}}(r_f) + \frac{r_f}{32} c_{\text{SW}} \sum_{\mu, \nu} \sigma_{\mu\nu} [Q_{\mu\nu}(x) - Q_{\nu\mu}(x)]$$

- Spurionic symmetries of the OS action:

$$\mathcal{CS}^{SP} \equiv \mathcal{C} \times \mathcal{S} \times (r_s \leftrightarrow r_b) \times (m_s \leftrightarrow m_b)$$

$$\mathcal{P}^{SP} \equiv \mathcal{P} \times (r_f \rightarrow -r_f), \text{ for all } f, \quad \mathcal{C} : \begin{cases} \psi_f(x) \rightarrow C^{-1} \bar{\psi}_f(x)^T, \\ \bar{\psi}_f(x) \rightarrow -\psi_f(x)^T C, \\ U_{\mu}(x) \rightarrow (U_{\mu}^{\dagger}(x))^T \end{cases}, \quad \mathcal{S} : \begin{cases} \psi_s \leftrightarrow \psi_b, \\ \bar{\psi}_s \leftrightarrow \bar{\psi}_b, \end{cases}$$

$$\mathcal{P}_5^{SP} \equiv \mathcal{P}_5 \times (m_f \rightarrow -m_f), \text{ for all } f$$

$$\mathcal{P} : \begin{cases} \psi_f(x) \rightarrow \gamma_0 \psi_f(x_{\mathcal{P}}), \quad x_{\mathcal{P}} = (x_0, -\vec{x}) \\ \bar{\psi}_f(x) \rightarrow \bar{\psi}_f(x_{\mathcal{P}}) \gamma_0, \\ U_0(x) \rightarrow U_0(x_{\mathcal{P}}), \\ U_k(x) \rightarrow U_k^{\dagger}(x_{\mathcal{P}} - a\hat{k}), \quad k = 1, 2, 3 \end{cases}, \quad \mathcal{P}_5 : \begin{cases} \psi_f(x) \rightarrow \gamma_5 \gamma_0 \psi_f(x_{\mathcal{P}}), \quad x_{\mathcal{P}} = (x_0, -\vec{x}) \\ \bar{\psi}_f(x) \rightarrow -\bar{\psi}_f(x_{\mathcal{P}}) \gamma_0 \gamma_5, \\ U_0(x) \rightarrow U_0(x_{\mathcal{P}}), \\ U_k(x) \rightarrow U_k^{\dagger}(x_{\mathcal{P}} - a\hat{k}), \quad k = 1, 2, 3 \end{cases}$$

Spurionic symmetry transformations

Operator	$\mathcal{CS}^{sp}$	$\mathcal{P}^{sp}$	$\mathcal{P}_5^{sp}$
$\mathcal{O}_1^{2F;d=3}$	+	+	-
$\mathcal{O}_2^{2F;d=3}$	+	-	+
$\mathcal{O}_1^{2F;d=5}$	+	+	-
$\mathcal{O}_2^{2F;d=5}$	+	-	+
$\mathcal{O}_3^{2F;d=5}$	+	+	-
$\mathcal{O}_4^{2F;d=5}$	+	-	+
$\mathcal{O}_1^{2F;d=6}$	-	+	+
$\mathcal{O}_2^{2F;d=6}$	+	-	-
$\mathcal{O}_3^{2F;d=6}$	+	+	+
$\mathcal{O}_4^{2F;d=6}$	-	-	-
$\mathcal{O}_5^{2F;d=6}$	-	+	+
$\mathcal{O}_6^{2F;d=6}$	+	-	-
$\mathcal{O}_7^{2F;d=6}$	+	+	+
$\mathcal{O}_8^{2F;d=6}$	-	-	-
$\mathcal{O}_9^{2F;d=6}$	-	+	+
$\mathcal{O}_{10}^{2F;d=6}$	+	-	-
$\mathcal{O}_{11}^{2F;d=6}$	+	+	+
$\mathcal{O}_{12}^{2F;d=6}$	-	-	-

Spurionic symmetry transformations

Operator	CS^{SP}	$\mathcal{P}^{SP}$	$\mathcal{P}_5^{SP}$
$\mathcal{O}_{1f}^{4F;d=6}$	+	+	+
$\mathcal{O}_{2f}^{4F;d=6}$	+	+	+
$\mathcal{O}_{3f}^{4F;d=6}$	+	+	+
$\mathcal{O}_{4f}^{4F;d=6}$	+	+	+
$\mathcal{O}_{5f}^{4F;d=6}$	+	+	+
$\mathcal{O}_{6f}^{4F;d=6}$	+	+	+
$\mathcal{O}_{7f}^{4F;d=6}$	+	+	+
$\mathcal{O}_{8f}^{4F;d=6}$	+	+	+
$\mathcal{O}_{9f}^{4F;d=6}$	+	+	+
$\mathcal{O}_{10f}^{4F;d=6}$	+	+	+
$\mathcal{O}_{11f}^{4F;d=6}$	-	-	-
$\mathcal{O}_{12f}^{4F;d=6} + \mathcal{O}_{17f}^{4F;d=6}$	+	-	-
$\mathcal{O}_{12f}^{4F;d=6} - \mathcal{O}_{17f}^{4F;d=6}$	-	-	-
$\mathcal{O}_{13f}^{4F;d=6} + \mathcal{O}_{18f}^{4F;d=6}$	-	-	-
$\mathcal{O}_{13f}^{4F;d=6} - \mathcal{O}_{18f}^{4F;d=6}$	+	-	-
$\mathcal{O}_{14f}^{4F;d=6}$	+	-	-
$\mathcal{O}_{15f}^{4F;d=6}$	+	-	-
$\mathcal{O}_{16f}^{4F;d=6}$	-	-	-
$\mathcal{O}_{19f}^{4F;d=6}$	+	-	-
$\mathcal{O}_{20f}^{4F;d=6}$	+	-	-