Gauge-invariant renormalization of $\Delta F = 1$ four-quark operators

M. Constantinou[1], M. Costa[2,3], H. Panagopoulos[4],
G. Spanoudes[4,*]

1. Temple University, 2. Cyprus University of Technology, 3. Rinnoco Ltd,
4. University of Cyprus, * Speaker

The 43rd International Symposium on Lattice Field Theory,
University of Maryland, College Park, USA, 26 July - 01 August 2026

Introduction - Motivation

- **Rare flavor-changing neutral-current decays are powerful probes of new physics** [M. Fedele, PoS (WIFAI2023) 022]
  - $b \to s(d)$ transitions: $B \to K^* \gamma$, $B \to K^{(*)} \ell^+ \ell^-$, $\bar{B}_s \to \gamma \ell^+ \ell^-$
- **Long-distance hadronic matrix elements involving four-quark operators**
  - Dominant theoretical uncertainty in B-decay phenomenology
- **First-principles nonperturbative determinations of complex Minkowski-space amplitudes via lattice QCD**
  - Spectral-density methods [R. Frezzotti et al., Phys. Rev. D 108 (2023) 074510]

$$i \int d^4x \, e^{ip \cdot x} \langle F | \, T\big[J_1(x) J_2(0) \big] \, | I \rangle$$

- **Spectral reconstruction makes the renormalization of four-quark operators even more important**
  - Complicated mixing pattern with operators of the same and/or lower dimension (UV power divergences).
  - Renormalization of contact (UV power) divergences in the time-ordered products of four-quark operators and electromagnetic currents

Introduction - Motivation

- **Gauge-invariant/coordinate-space renormalization scheme (GIRS):**
[V. Gimenez et al., PLB598 (2004) 227,
M. Costa et al., Phys. Rev. D 103 (2021) 094509]
• Consider vacuum expectation values of two or more gauge-invariant operators $\mathcal{O}_i$ at different spacetime points:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n)\rangle, \quad (x_1 \neq x_2 \neq \cdots \neq x_n)$$

• Integrate (sum, on the lattice) over time slices

$$\int d^3\vec{x}_1 \cdots \int d^3\vec{x}_{n-1} \langle \mathcal{O}_1(\vec{x}_1, t_1)\cdots\mathcal{O}_{n-1}(\vec{x}_{n-1}, t_{n-1})\mathcal{O}_n(\vec{0}, 0)\rangle,$$

$$(t_1 \neq t_2 \neq \cdots \neq t_{n-1} \neq 0)$$

• **Good features:**
1. No need for gauge-fixing, **2.** No contact terms, **3.** No need to consider mixing with gauge-non-invariant operators

• **Challenges:**
1. When mixing occurs, $n > 2$-point Green's functions are needed,
2. Calculation of Feynman diagrams with more loops at each perturbative order than Green's functions with external elementary fields,
3. Narrow renormalization window: $\Lambda_{\rm QCD} \ll \mu \ll a^{-1}, \quad \mu \sim |x_i - x_j|^{-1}$.

Operator Setup

- **Four-quark operators:** (Lorentz scalar and pseudoscalar)

$$(\mathcal{O}_{\Gamma\Gamma'})^{f_1,f_2}_{f_3,f_4} \equiv \Big(\sum_a \sum_{\alpha,\beta} \bar{\psi}^{a,f_1}_{\alpha}(x)\, \Gamma_{\alpha\beta}\, \psi^{a}_{\beta,f_3}(x) \Big) \Big(\sum_{a'} \sum_{\alpha',\beta'} \bar{\psi}^{a',f_2}_{\alpha'}(x)\, \Gamma'_{\alpha'\beta'}\, \psi^{a'}_{\beta',f_4}(x) \Big),$$

where $\Gamma \in \{\mathbf{1}, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}, \sigma_{\mu\nu}\gamma_5\} \equiv \{S, P, V, A, T, \tilde{T}\}$, $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$,
$\Gamma' \in \{\Gamma, \gamma_5\Gamma\}$

- We define operators with symmetric/antisymmetric flavor indices:

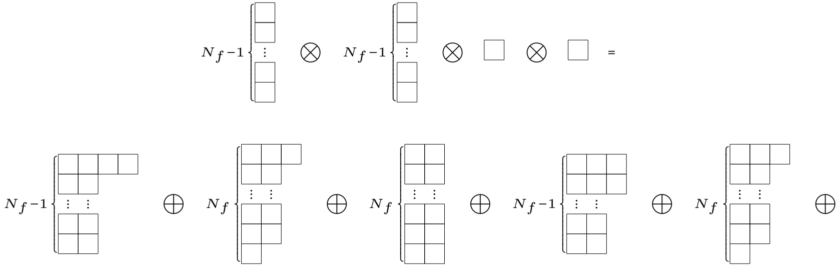
$$\begin{aligned}
(\mathcal{O}^{+}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4} &\equiv \frac{1}{2}\Big[(\mathcal{O}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4} + (\mathcal{O}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_4,f_3}\Big],\\
(\mathcal{O}^{-}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4} &\equiv \frac{1}{2}\Big[(\mathcal{O}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4} - (\mathcal{O}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_4,f_3}\Big],\\
(\mathcal{O}^{+}_{[\Gamma,\Gamma']})^{f_1,f_2}_{f_3,f_4} &\equiv \frac{1}{2}\Big[(\mathcal{O}_{[\Gamma,\Gamma']})^{f_1,f_2}_{f_3,f_4} + (\mathcal{O}_{[\Gamma,\Gamma']})^{f_1,f_2}_{f_4,f_3}\Big],\\
(\mathcal{O}^{-}_{[\Gamma,\Gamma']})^{f_1,f_2}_{f_3,f_4} &\equiv \frac{1}{2}\Big[(\mathcal{O}_{[\Gamma,\Gamma']})^{f_1,f_2}_{f_3,f_4} - (\mathcal{O}_{[\Gamma,\Gamma']})^{f_1,f_2}_{f_4,f_3}\Big],
\end{aligned}$$

where

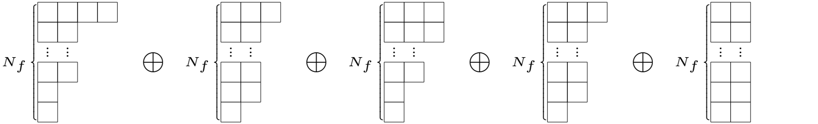
$$\begin{aligned}
(\mathcal{O}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4} &\equiv (\mathcal{O}_{\Gamma\Gamma'+\Gamma'\Gamma})^{f_1,f_2}_{f_3,f_4} \equiv (\mathcal{O}_{\Gamma\Gamma'})^{f_1,f_2}_{f_3,f_4} + (\mathcal{O}_{\Gamma'\Gamma})^{f_1,f_2}_{f_3,f_4},\\
(\mathcal{O}_{[\Gamma,\Gamma']})^{f_1,f_2}_{f_3,f_4} &\equiv (\mathcal{O}_{\Gamma\Gamma'-\Gamma'\Gamma})^{f_1,f_2}_{f_3,f_4} \equiv (\mathcal{O}_{\Gamma\Gamma'})^{f_1,f_2}_{f_3,f_4} - (\mathcal{O}_{\Gamma'\Gamma})^{f_1,f_2}_{f_3,f_4},
\end{aligned}$$

Operator Setup

- **Classification into irreps of** $SU(N_f)$ (for mass-degenerate quarks)



$(\breve{\mathcal{O}}^{+}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4}$ $(\overline{\mathcal{O}}^{+}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4}$ $(\widehat{\mathcal{O}}^{+}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4}$ $(\breve{\mathcal{O}}^{-}_{[\Gamma,\Gamma']})^{f_1,f_2}_{f_3,f_4}$ $(\overline{\mathcal{O}}^{-}_{[\Gamma,\Gamma']})^{f_1,f_2}_{f_3,f_4}$



$(\breve{\mathcal{O}}^{+}_{[\Gamma,\Gamma']})^{f_1,f_2}_{f_3,f_4}$ $(\overline{\mathcal{O}}^{+}_{[\Gamma,\Gamma']})^{f_1,f_2}_{f_3,f_4}$ $(\breve{\mathcal{O}}^{-}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4}$ $(\overline{\mathcal{O}}^{-}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4}$ $(\widehat{\mathcal{O}}^{-}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4}$

Operator Setup

- **Classification into irreps of** $SU(N_f)$ (for mass-degenerate quarks)

$$
\begin{aligned}
(\check{\mathcal{O}}^{\pm}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4} &\equiv (\mathcal{O}^{\pm}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4} - \frac{1}{(N_f \pm 2)}(\overline{\mathcal{O}}^{\pm}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4} - \frac{1}{N_f(N_f \pm 1)}(\hat{\mathcal{O}}^{\pm}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4}, \\
(\overline{\mathcal{O}}^{\pm}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4} &\equiv \sum_{f'}\Big(\delta^{f_1}{}_{f_3}(\mathcal{O}^{\pm}_{\{\Gamma,\Gamma'\}})^{f',f_2}_{f',f_4} + \delta^{f_2}{}_{f_3}(\mathcal{O}^{\pm}_{\{\Gamma,\Gamma'\}})^{f_1,f'}_{f',f_4} \\
&\quad +\delta^{f_1}{}_{f_4}(\mathcal{O}^{\pm}_{\{\Gamma,\Gamma'\}})^{f',f_2}_{f_3,f'} + \delta^{f_2}{}_{f_4}(\mathcal{O}^{\pm}_{\{\Gamma,\Gamma'\}})^{f_1,f'}_{f_3,f'}\Big) - \frac{2}{N_f}(\hat{\mathcal{O}}^{\pm}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4}, \\
(\hat{\mathcal{O}}^{\pm}_{\{\Gamma,\Gamma'\}})^{f_1,f_2}_{f_3,f_4} &\equiv (\delta^{f_1}{}_{f_3}\delta^{f_2}{}_{f_4} \pm \delta^{f_1}{}_{f_4}\delta^{f_2}{}_{f_3}) \sum_{f',f''}(\mathcal{O}^{\pm}_{\{\Gamma,\Gamma'\}})^{f',f''}_{f',f''}, \\
(\check{\mathcal{O}}^{\pm}_{[\Gamma,\Gamma']})^{f_1,f_2}_{f_3,f_4} &\equiv (\mathcal{O}^{\pm}_{[\Gamma,\Gamma']})^{f_1,f_2}_{f_3,f_4} - \frac{1}{N_f}(\overline{\mathcal{O}}^{\pm}_{[\Gamma,\Gamma']})^{f_1,f_2}_{f_3,f_4}, \\
(\overline{\mathcal{O}}^{\pm}_{[\Gamma,\Gamma']})^{f_1,f_2}_{f_3,f_4} &\equiv \sum_{f'}\Big(\delta^{f_1}{}_{f_3}(\mathcal{O}^{\pm}_{[\Gamma,\Gamma']})^{f',f_2}_{f',f_4} + \delta^{f_2}{}_{f_3}(\mathcal{O}^{\pm}_{[\Gamma,\Gamma']})^{f_1,f'}_{f',f_4} \\
&\quad +\delta^{f_1}{}_{f_4}(\mathcal{O}^{\pm}_{[\Gamma,\Gamma']})^{f',f_2}_{f_3,f'} + \delta^{f_2}{}_{f_4}(\mathcal{O}^{\pm}_{[\Gamma,\Gamma']})^{f_1,f'}_{f_3,f'}\Big),
\end{aligned}
$$

$\check{\mathcal{O}}^{\pm}$: completely traceless, $\hat{\mathcal{O}}^{\pm}$: pure trace (flavor singlets), $\overline{\mathcal{O}}^{\pm}$: support the adjoint representation of the flavor group.

Operator Setup

- **Operators for flavor-changing processes**

  - $\underline{\Delta F = 2\textbf{:}}$ $\boxed{(\mathrm{f}_1 = f_2 \text{ and } f_1 \notin \{f_3, f_4\}) \text{ or } (f_3 = f_4 \text{ and } f_3 \notin \{f_1, f_2\})}$

    $\Rightarrow$ Only traceless operators:

$$\breve{\mathcal{O}}^{\pm}_{\{\Gamma,\Gamma'\}},\ \breve{\mathcal{O}}^{\pm}_{[\Gamma,\Gamma']}$$

  - $\underline{\Delta F = 1\textbf{:}}$

    1. $\boxed{(\mathrm{f}_1, f_2, f_3, f_4) \text{ are all different}}$

    $\Rightarrow$ Only traceless operators:

$$\breve{\mathcal{O}}^{\pm}_{\{\Gamma,\Gamma'\}},\ \breve{\mathcal{O}}^{\pm}_{[\Gamma,\Gamma']}$$

    2. $\boxed{(\mathrm{f}_1 \in \{f_3, f_4\} \text{ and } f_2 \notin \{f_3, f_4\}) \text{ or } (f_1 \notin \{f_3, f_4\} \text{ and } f_2 \in \{f_3, f_4\})}$

    $\Rightarrow$ Operators transforming under traceless or adjoint reps:

$$\breve{\mathcal{O}}^{\pm}_{\{\Gamma,\Gamma'\}},\ \breve{\mathcal{O}}^{\pm}_{[\Gamma,\Gamma']},\ \overline{\mathcal{O}}^{\pm}_{\{\Gamma,\Gamma'\}},\ \overline{\mathcal{O}}^{\pm}_{[\Gamma,\Gamma']}$$

  - $\underline{\Delta F = 0\textbf{:}}$ $\boxed{(\mathrm{f}_1, f_2) \text{ are pairwise equal to } (f_3, f_4)}$

    $\Rightarrow$ All classes of operators:

$$\breve{\mathcal{O}}^{\pm}_{\{\Gamma,\Gamma'\}},\ \breve{\mathcal{O}}^{\pm}_{[\Gamma,\Gamma']},\overline{\mathcal{O}}^{\pm}_{\{\Gamma,\Gamma'\}},\ \overline{\mathcal{O}}^{\pm}_{[\Gamma,\Gamma']},\ \hat{\mathcal{O}}^{\pm}_{\{\Gamma,\Gamma'\}}$$

$\Delta F = 1$ traceless operators

- **Operator basis** [Frezzotti and Rossi, JHEP 10 (2004) 070]

Parity even	**Parity odd**
$Q_1^{\pm} \equiv \breve{\mathcal{O}}^{\pm}_{VV+AA}$,	$\mathcal{Q}_1^{\pm} \equiv \breve{\mathcal{O}}^{\pm}_{VA+AV}$,
$Q_2^{\pm} \equiv \breve{\mathcal{O}}^{\pm}_{VV-AA}$,	$\mathcal{Q}_2^{\pm} \equiv \breve{\mathcal{O}}^{\pm}_{VA-AV}$,
$Q_3^{\pm} \equiv \breve{\mathcal{O}}^{\pm}_{SS-PP}$,	$\mathcal{Q}_3^{\pm} \equiv \breve{\mathcal{O}}^{\pm}_{PS-SP}$,
$Q_4^{\pm} \equiv \breve{\mathcal{O}}^{\pm}_{SS+PP}$,	$\mathcal{Q}_4^{\pm} \equiv \breve{\mathcal{O}}^{\pm}_{PS+SP}$,
$Q_5^{\pm} \equiv \breve{\mathcal{O}}^{\pm}_{TT}$,	$\mathcal{Q}_5^{\pm} \equiv \breve{\mathcal{O}}^{\pm}_{T\tilde{T}}$.

- **Mixing sets:** Consider also CP symmetries [Spanoudes et al., Phys.Rev.D 113 (2026) 034504]

$$
\begin{aligned}
S_1^{\pm} &: \left\{Q_1^{\pm},\ Q_2^{\pm},\ Q_3^{\pm},\ Q_4^{\pm},\ Q_5^{\pm}\right\},\\
S_{2a}^{\pm} &: \left\{\mathcal{Q}_1^{\pm}\right\},\\
S_{2b}^{\pm} &: \left\{\mathcal{Q}_4^{\pm}, \mathcal{Q}_5^{\pm}\right\},\\
S_3^{\pm} &: \left\{\mathcal{Q}_2^{\pm},\ \mathcal{Q}_3^{\pm}\right\},
\end{aligned}
$$

Renormalization Setup

- **Mixing matrices**

$$Q_l^{\pm,R} = \sum_{m=1}^{5} Z_{lm}^{\pm} \cdot Q_m^{\pm}, \qquad \mathcal{Q}_l^{\pm,R} = \sum_{m=1}^{5} \mathcal{Z}_{lm}^{\pm} \cdot \mathcal{Q}_m^{\pm}, \qquad l = 1, \ldots, 5$$

$$Z^{\pm} = \begin{pmatrix} Z_{11}^{\pm} & Z_{12}^{\pm} & Z_{13}^{\pm} & Z_{14}^{\pm} & Z_{15}^{\pm} \\ Z_{21}^{\pm} & Z_{22}^{\pm} & Z_{23}^{\pm} & Z_{24}^{\pm} & Z_{25}^{\pm} \\ Z_{31}^{\pm} & Z_{32}^{\pm} & Z_{33}^{\pm} & Z_{34}^{\pm} & Z_{35}^{\pm} \\ Z_{41}^{\pm} & Z_{42}^{\pm} & Z_{43}^{\pm} & Z_{44}^{\pm} & Z_{45}^{\pm} \\ Z_{51}^{\pm} & Z_{52}^{\pm} & Z_{53}^{\pm} & Z_{54}^{\pm} & Z_{55}^{\pm} \end{pmatrix}, \quad \mathcal{Z}^{\pm} = \begin{pmatrix} \mathcal{Z}_{11}^{\pm} & 0 & 0 & 0 & 0 \\ 0 & \mathcal{Z}_{22}^{\pm} & \mathcal{Z}_{23}^{\pm} & 0 & 0 \\ 0 & \mathcal{Z}_{32}^{\pm} & \mathcal{Z}_{33}^{\pm} & 0 & 0 \\ 0 & 0 & 0 & \mathcal{Z}_{44}^{\pm} & \mathcal{Z}_{45}^{\pm} \\ 0 & 0 & 0 & \mathcal{Z}_{54}^{\pm} & \mathcal{Z}_{55}^{\pm} \end{pmatrix}.$$

• To determine all mixing coefficients (diagonal+nondiagonal), we need to define 25 conditions for Z^+ and 9 conditions for $\mathcal{Z}^+$. Similarly, for Z^- and $\mathcal{Z}^-$.

Renormalization Setup

- **Green's functions in GIRS:**

• The determination of the mixing matrices requires the calculation of both two-point and three-point functions.
• For the parity-even $Q^{\pm}$, we consider:

$$
\begin{aligned}
G^{2\text{pt}}_{Q_i^{\pm};\,Q_j^{\pm}}(t) &\equiv \int d^3\vec{x}\,\left\langle Q_i^{\pm}(\vec{x},t)\left[Q_j^{\pm}(\vec{0},0)\right]^{\dagger}\right\rangle,\\
G^{3\text{pt}}_{\mathcal{O}_\Gamma;\,Q_i^{\pm};\,\mathcal{O}_\Gamma}(t,t') &\equiv \int d^3\vec{x}\int d^3\vec{x}'\,\left\langle \mathcal{O}_\Gamma(\vec{x},t)\;Q_i^{\pm}(\vec{0},0)\;\mathcal{O}_\Gamma(-\vec{x}',-t')\right\rangle,
\end{aligned}
$$

where $\left(\mathcal{O}_\Gamma\right)^{f_1}{}_{f_2}(x) \equiv \bar{\psi}^{f_1}(x)\Gamma\psi_{f_2}(x)$ is a quark bilinear operator.

• Similarly, for the parity-odd $\mathcal{Q}^{\pm}$, we consider:

$$
\begin{aligned}
G^{2\text{pt}}_{\mathcal{Q}_i^{\pm};\,\mathcal{Q}_j^{\pm}}(t) &\equiv \int d^3\vec{x}\,\left\langle \mathcal{Q}_i^{\pm}(\vec{x},t)\left[\mathcal{Q}_j^{\pm}(\vec{0},0)\right]^{\dagger}\right\rangle,\\
G^{3\text{pt}}_{\mathcal{O}_\Gamma;\,\mathcal{Q}_i^{\pm};\,\mathcal{O}_{\Gamma\gamma_5}}(t,t') &\equiv \int d^3\vec{x}\int d^3\vec{x}'\,\left\langle \mathcal{O}_\Gamma(\vec{x},t)\;\mathcal{Q}_i^{\pm}(\vec{0},0)\;\mathcal{O}_{\Gamma\gamma_5}(-\vec{x}',-t')\right\rangle,
\end{aligned}
$$

Renormalization Setup

- **Renormalization conditions in GIRS:**

  • For the parity-even operators $Q^{\pm}$, the **25** conditions can be obtained:

  ∗ **15** conditions from two-point functions:

$$[G^{2\text{pt}}_{Q^{\pm}_i;Q^{\pm}_j}(t)]^{\text{GIRS}} \equiv \sum_{k,l=1}^{5} (Z^{\pm}_{ik})^{\text{GIRS}} (Z^{\pm}_{jl})^{\text{GIRS}}\, G^{2\text{pt}}_{Q^{\pm}_k;Q^{\pm}_l}(t) = [G^{2\text{pt}}_{Q^{\pm}_i;Q^{\pm}_j}(t)]^{\text{free}},\ i \leq j,$$

  ∗ **10** conditions from three-point functions: 30!/(10! 20!) possible choices!

$$[G^{3\text{pt}}_{\mathcal{O}_\Gamma;Q^{\pm}_i;\mathcal{O}_\Gamma}(t,t)]^{\text{GIRS}} \equiv (Z^{\text{GIRS}}_{\mathcal{O}_\Gamma})^2 \sum_{k=1}^{5} (Z^{\pm}_{ik})^{\text{GIRS}}\, G^{3\text{pt}}_{\mathcal{O}_\Gamma;Q^{\pm}_k;\mathcal{O}_\Gamma}(t,t) = [G^{3\text{pt}}_{\mathcal{O}_\Gamma;Q^{\pm}_i;\mathcal{O}_\Gamma}(t,t)]^{\text{free}},$$

  where $i \in [1,5]$, $\Gamma \in \{\mathbf{1},\, \gamma_5,\, \gamma_m,\, \gamma_m\gamma_5,\, \sigma_{mn},\, \sigma_{m4}\}$ $(m, n$ are spatial directions).

Renormalization Setup

- **Renormalization conditions in GIRS:**

  • For the parity-odd operators $\mathcal{Q}^{\pm}$, the **9** conditions can be obtained:
  $*$ **7** conditions from two-point functions: (1 + 3 + 3 for each block, respectively)

$$[G^{2\mathrm{pt}}_{\mathcal{Q}^{\pm}_1;\mathcal{Q}^{\pm}_1}(t)]^{\mathrm{GIRS}} \equiv [(\mathcal{Z}^{\pm}_{11})^{\mathrm{GIRS}}]^2 \; G^{2\mathrm{pt}}_{\mathcal{Q}^{\pm}_1;\mathcal{Q}^{\pm}_1}(t) = [G^{2\mathrm{pt}}_{\mathcal{Q}^{\pm}_1;\mathcal{Q}^{\pm}_1}(t)]^{\mathrm{free}},$$

$$[G^{2\mathrm{pt}}_{\mathcal{Q}^{\pm}_i;\mathcal{Q}^{\pm}_j}(t)]^{\mathrm{GIRS}} \equiv \sum_{k,l=2}^{3} (\mathcal{Z}^{\pm}_{ik})^{\mathrm{GIRS}} (\mathcal{Z}^{\pm}_{jl})^{\mathrm{GIRS}} \; G^{2\mathrm{pt}}_{\mathcal{Q}^{\pm}_k;\mathcal{Q}^{\pm}_l}(t) = [G^{2\mathrm{pt}}_{\mathcal{Q}^{\pm}_i;\mathcal{Q}^{\pm}_j}(t)]^{\mathrm{free}}, \;\; (i,j = 2,3),$$

$$[G^{2\mathrm{pt}}_{\mathcal{Q}^{\pm}_i;\mathcal{Q}^{\pm}_j}(t)]^{\mathrm{GIRS}} \equiv \sum_{k,l=4}^{5} (\mathcal{Z}^{\pm}_{ik})^{\mathrm{GIRS}} (\mathcal{Z}^{\pm}_{jl})^{\mathrm{GIRS}} \; G^{2\mathrm{pt}}_{\mathcal{Q}^{\pm}_k;\mathcal{Q}^{\pm}_l}(t) = [G^{2\mathrm{pt}}_{\mathcal{Q}^{\pm}_i;\mathcal{Q}^{\pm}_j}(t)]^{\mathrm{free}}, \;\; (i,j = 4,5),$$

  where $i \leq j$.
  $*$ **2** conditions from three-point functions: 4 possible choices!

$$\begin{aligned}[G^{3\mathrm{pt}}_{\mathcal{O}_\Gamma;\mathcal{Q}^{\pm}_i;\mathcal{O}_{\Gamma\gamma_5}}(t,t')]^{\mathrm{GIRS}} &\equiv Z^{\mathrm{GIRS}}_{\mathcal{O}_\Gamma} \; Z^{\mathrm{GIRS}}_{\mathcal{O}_{\Gamma\gamma_5}} \sum_{k=2}^{3} (\mathcal{Z}^{\pm}_{ik})^{\mathrm{GIRS}} \; G^{3\mathrm{pt}}_{\mathcal{O}_\Gamma;\mathcal{Q}^{\pm}_k;\mathcal{O}_{\Gamma\gamma_5}}(t,t) \\ &= [G^{3\mathrm{pt}}_{\mathcal{O}_\Gamma;\mathcal{Q}^{\pm}_i;\mathcal{O}_{\Gamma\gamma_5}}(t,t)]^{\mathrm{free}}, \;\; (i = 2 \text{ or } 3),\end{aligned}$$

$$\begin{aligned}[G^{3\mathrm{pt}}_{\mathcal{O}_\Gamma;\mathcal{Q}^{\pm}_i;\mathcal{O}_{\Gamma\gamma_5}}(t,t)]^{\mathrm{GIRS}} &\equiv Z^{\mathrm{GIRS}}_{\mathcal{O}_\Gamma} \; Z^{\mathrm{GIRS}}_{\mathcal{O}_{\Gamma\gamma_5}} \sum_{k=4}^{5} (\mathcal{Z}^{\pm}_{ik})^{\mathrm{GIRS}} \; G^{3\mathrm{pt}}_{\mathcal{O}_\Gamma;\mathcal{Q}^{\pm}_k;\mathcal{O}_{\Gamma\gamma_5}}(t,t) \\ &= [G^{3\mathrm{pt}}_{\mathcal{O}_\Gamma;\mathcal{Q}^{\pm}_i;\mathcal{O}_{\Gamma\gamma_5}}(t,t)]^{\mathrm{free}}, \;\; (i = 4 \text{ or } 5),\end{aligned}$$

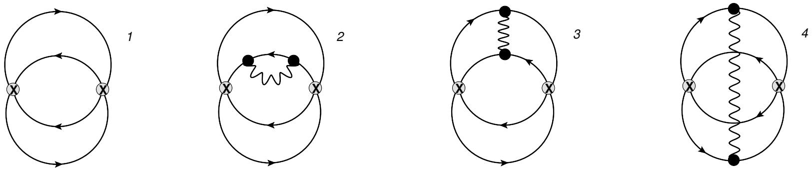
  where $\Gamma \in \{, \gamma_m, \sigma_{mn}\}$ (m, n are spatial directions).

Calculation

- **NLO calculation in dimensional regularization:** [M. Constantinou et al., Phys.Rev.D 110 (2024) 074506, G. Spanoudes et al., Phys.Rev.D 113 (2026) 034504]

  - **Two-point functions:**

$$G^{2\mathrm{pt}}_{Q^\pm_i;Q^\pm_j}(t) = \frac{N_c}{\pi^6|t|^9}\left(\delta^{f_1}{}_{f_3'}\,\delta^{f_2}{}_{f_4'} \pm \delta^{f_1}{}_{f_4'}\,\delta^{f_2}{}_{f_3'}\right)\times$$
$$\left(\delta_{f_3}{}^{f_1'}\,\delta_{f_4}{}^{f_2'} \pm \delta_{f_3}{}^{f_2'}\,\delta_{f_4}{}^{f_1'}\right)\left(a^\pm_{ij;0} + a^\pm_{ij;1}N_c\right),$$
$$G^{2\mathrm{pt}}_{\mathcal{Q}^\pm_i;\mathcal{Q}^\pm_j}(t) = \frac{N_c}{\pi^6|t|^9}\left(\delta^{f_1}{}_{f_3'}\,\delta^{f_2}{}_{f_4'} \pm \delta^{f_1}{}_{f_4'}\,\delta^{f_2}{}_{f_3'}\right)\times$$
$$\left((-1)^{\delta_{i2}+\delta_{i3}}\,\delta_{f_3}{}^{f_1'}\,\delta_{f_4}{}^{f_2'} \pm \delta_{f_3}{}^{f_2'}\,\delta_{f_4}{}^{f_1'}\right)\left(\tilde{a}^\pm_{ij;0} + \tilde{a}^\pm_{ij;1}N_c\right)$$



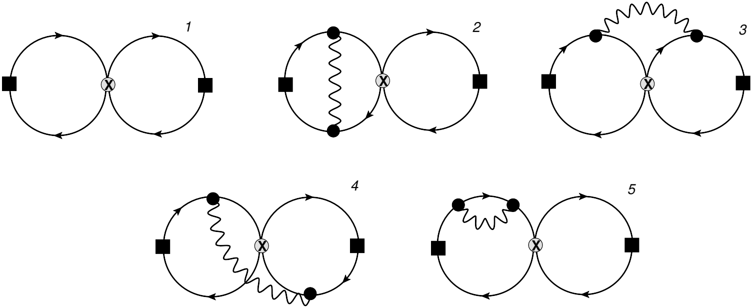
$\otimes = Q^\pm_i$ or $\mathcal{Q}^\pm_i$. Wavy (solid) lines represent gluons (quarks). Diagrams 2 and 4 have also mirror variants.

Calling

- **Three-point functions:**

$$G^{3\mathrm{pt}}_{\mathcal{O}_\Gamma;Q_i^\pm;\mathcal{O}_\Gamma}(t,t) = \frac{N_c}{\pi^4 t^6}\left(\delta^{f_1}{}_{f_3'}\,\delta^{f_2}{}_{f_4'} \pm \delta^{f_1}{}_{f_4'}\,\delta^{f_2}{}_{f_3'}\right)\times \left(\delta_{f_3}{}^{f_1'}\,\delta_{f_4}{}^{f_2'} \pm \delta_{f_3}{}^{f_2'}\,\delta_{f_4}{}^{f_1'}\right)\left(d^\pm_{i\Gamma;0} + d^\pm_{i\Gamma;1} N_c\right),$$

$$G^{3\mathrm{pt}}_{\mathcal{O}_\Gamma;\mathcal{Q}_i^\pm;\mathcal{O}_{\Gamma\gamma_5}}(t,t) = \frac{N_c}{\pi^4 t^6}\left(\delta^{f_1}{}_{f_3'}\,\delta^{f_2}{}_{f_4'} \pm \delta^{f_1}{}_{f_4'}\,\delta^{f_2}{}_{f_3'}\right)\times \left((-1)^{\delta_{i2}+\delta_{i3}}\,\delta_{f_3}{}^{f_1'}\,\delta_{f_4}{}^{f_2'} \pm \delta_{f_3}{}^{f_2'}\,\delta_{f_4}{}^{f_1'}\right)\left(\tilde{d}^\pm_{i\Gamma;0} + \tilde{d}^\pm_{i\Gamma;1} N_c\right)$$



$\otimes = Q_i^\pm$ or $\mathcal{Q}_i^\pm$, $\blacksquare = \mathcal{O}_\Gamma$ or $\mathcal{O}_{\Gamma\gamma_5}$. Wavy (solid) lines represent gluons (quarks). Diagrams 2-5 have also mirror variants.

Conversion matrices

- **Conversion matrices from GIRS to $\overline{\text{MS}}$:**

$$(Z^{\pm})^{\overline{\text{MS}}} = (C^{\pm})^{\overline{\text{MS}},\text{GIRS}} (Z^{\pm})^{\text{GIRS}}, \qquad (\mathcal{Z}^{\pm})^{\overline{\text{MS}}} = (\tilde{C}^{\pm})^{\overline{\text{MS}},\text{GIRS}} (\mathcal{Z}^{\pm})^{\text{GIRS}}.$$

- We explore a wide range of GIRS variants and identify those with reduced mixing effects. For a promising variant, we have extracted the NLO conversion matrices to $\overline{\text{MS}}$ (see next slide).
  - **Selected 3pt functions for $Q^{\pm}$:**

$$G^{3\text{pt}}_{S;Q_1^{\pm};S}(t,t),\ G^{3\text{pt}}_{P;Q_1^{\pm};P}(t,t),\ G^{3\text{pt}}_{V_i;Q_1^{\pm};V_i}(t,t),\ G^{3\text{pt}}_{S;Q_2^{\pm};S}(t,t),\ G^{3\text{pt}}_{P;Q_2^{\pm};P}(t,t),$$
$$G^{3\text{pt}}_{S;Q_3^{\pm};S}(t,t),\ G^{3\text{pt}}_{S;Q_5^{\pm};S}(t,t),\ G^{3\text{pt}}_{P;Q_5^{\pm};P}(t,t),\ G^{3\text{pt}}_{V_i;Q_5^{\pm};V_i}(t,t),\ G^{3\text{pt}}_{A_i;Q_5^{\pm};A_i}(t,t),$$

  - **Selected 3pt functions for $\mathcal{Q}^{\pm}$:**

$$G^{3\text{pt}}_{S;\mathcal{Q}_2^{\pm};P}(t,t),\ G^{3\text{pt}}_{S;\mathcal{Q}_5^{\pm};P}(t,t).$$

- Alternative variants: a democratic version that treats all mixing operators uniformly (see [Spanoudes et al., Phys.Rev.D 113 (2026) 034504])

Conversion matrices

- **Conversion matrices from GIRS to $\overline{\mathrm{MS}}$:**

$$(C^{\pm}_{ij})^{\overline{\mathrm{MS}},\mathrm{GIRS}} = \delta_{ij} + \frac{g^2_{\overline{\mathrm{MS}}}(\bar{\mu})}{16\pi^2} \sum_{k=-1}^{+1} \left[g^{\pm}_{ij;k} + \left(\ln(\bar{\mu}^2 t^2) + 2\gamma_E \right) \, h^{\pm}_{ij;k} \right] N_c^k + \mathcal{O}(g^4_{\overline{\mathrm{MS}}}(\bar{\mu})),$$

$$(\tilde{C}^{\pm}_{ij})^{\overline{\mathrm{MS}},\mathrm{GIRS}} = \delta_{ij} + \frac{g^2_{\overline{\mathrm{MS}}}(\bar{\mu})}{16\pi^2} \sum_{k=-1}^{+1} \left[\tilde{g}^{\pm}_{ij;k} + \left(\ln(\bar{\mu}^2 t^2) + 2\gamma_E \right) \, \tilde{h}^{\pm}_{ij;k} \right] N_c^k + \mathcal{O}(g^4_{\overline{\mathrm{MS}}}(\bar{\mu})),$$

- More results on [Spanoudes et al., Phys.Rev.D 113 (2026) 034504]

i	j	$\tilde{g}^{\pm}_{ij;-1}$	$\tilde{g}^{\pm}_{ij;0}$	$\tilde{g}^{\pm}_{ij;+1}$	$\tilde{h}^{\pm}_{ij;-1}$	$\tilde{h}^{\pm}_{ij;0}$	$\tilde{h}^{\pm}_{ij;+1}$
1	1	$-869/140$	$\pm 379/140$	$7/2$	3	∓ 3	0
2	2	$-9/2$	0	$7/2$	-3	0	0
2	3	0	∓ 2	0	0	∓ 6	0
3	2	0	$\pm 99/280$	0	0	0	0
3	3	$-38/35$	0	$251/140$	-3	0	3
4	4	$-307/112 + 3\ln(2)$	$\pm 169/140$	$251/140$	-3	∓ 3	3
4	5	$-269/480 + 1/2\ \ln(2)$	$\pm(869/1680 - \ln(2))$	0	1	$\mp 1/2$	0
5	4	$-269/40 + 6\ln(2)$	$\mp(29/140 - 12\ln(2))$	0	12	± 6	0
5	5	$-1229/240 - 3\ln(2)$	$\pm 309/140$	$1709/420$	1	∓ 3	-1

$\Delta F = 1$ operators in the adjoint representations

- **Operator basis**

Parity even	**Parity odd**
$\overline{\mathcal{O}}^{\pm}_{VV+AA}$,	$\overline{\mathcal{O}}^{\pm}_{VA+AV}$,
$\overline{\mathcal{O}}^{\pm}_{VV-AA}$,	$\overline{\mathcal{O}}^{\pm}_{VA-AV}$,
$\overline{\mathcal{O}}^{\pm}_{SS-PP}$,	$\overline{\mathcal{O}}^{\pm}_{PS-SP}$,
$\overline{\mathcal{O}}^{\pm}_{SS+PP}$,	$\overline{\mathcal{O}}^{\pm}_{PS+SP}$,
$\overline{\mathcal{O}}^{\pm}_{TT}$,	$\overline{\mathcal{O}}^{\pm}_{T\tilde{T}}$.

- We focus on the weak current-current operators entering the effective Hamiltonian for $b \to s$ decays: [Frezzotti et al., Phys.Rev.D 113 (2026) 034509]

$$\left(\bar{\psi}_s^i \gamma_\rho P_L \psi_b^i\right)\left(\bar{\psi}_c^j \gamma_\rho P_L \psi_c^j\right), \quad \left(\bar{\psi}_s^i \gamma_\rho P_L \psi_c^i\right)\left(\bar{\psi}_c^j \gamma_\rho P_L \psi_b^j\right), \qquad P_L = (1-\gamma_5)/2.$$

- Mix with lower-dimensional operators (power-divergent mixing on the lattice) Nonperturbative renormalization strategy by Frezzotti et al., [Phys.Rev.D 113 (2026) 034509]
- To determine the full operator mixing set, we further investigate relevant spurionic symmetries on the lattice for mass-nongenerate quarks.
- A preliminary study in lattice perturbation theory, using Osterwalder-Seiler (OS) fermions, is ongoing.

$\Delta F = 1$ operators in the adjoint representations

- Candidate operators of dimensions ≤ 6 that can mix with the four-quark operators under study (in addition to the four-quark operators):

$$
\begin{aligned}
\mathcal{O}_1^{2F;d=3} &= \overline{\psi}_s(x)\psi_b(x) \\
\mathcal{O}_2^{2F;d=3} &= \overline{\psi}_s(x)\gamma_5\psi_b(x) \\
\mathcal{O}_1^{2F;d=5} &= g_0\,\overline{\psi}_s(x)\,\sigma_{\mu\nu}G_{\mu\nu}(x)\psi_b(x) \\
\mathcal{O}_2^{2F;d=5} &= g_0\,\overline{\psi}_s(x)\,\gamma_5\,\sigma_{\mu\nu}G_{\mu\nu}(x)\psi_b(x) \\
\mathcal{O}_3^{2F;d=5} &= \square(\bar{\psi}_s(x)\psi_b(x)) \\
\mathcal{O}_4^{2F;d=5} &= \square(\bar{\psi}_s(x)\gamma_5\psi_b(x)) \\
\mathcal{O}_1^{2F;d=6} &= g_0(\bar{\psi}_s(x)\gamma_\nu G_{\mu\nu}\overrightarrow{D}_\mu\psi_b(x) - \bar{\psi}_s(x)\overleftarrow{D}_\mu\gamma_\nu G_{\mu\nu}(x)\psi_b(x)) \\
\mathcal{O}_2^{2F;d=6} &= g_0(\bar{\psi}_s(x)\gamma_5\gamma_\nu G_{\mu\nu}\overrightarrow{D}_\mu\psi_b(x) - \bar{\psi}_s(x)\overleftarrow{D}_\mu\gamma_5\gamma_\nu G_{\mu\nu}(x)\psi_b(x)) \\
\mathcal{O}_3^{2F;d=6} &= g_0\,\partial_\mu(\bar{\psi}_s\gamma_\nu G_{\mu\nu}\psi_b) \\
\mathcal{O}_4^{2F;d=6} &= g_0\,\partial_\mu(\bar{\psi}_s\gamma_5\gamma_\nu G_{\mu\nu}\psi_b) \\
\mathcal{O}_5^{2F;d=6} &= \partial_\mu\partial_\mu\partial_\mu(\bar{\psi}_s(x)\gamma_\mu\psi_b(x)) \\
\mathcal{O}_6^{2F;d=6} &= \partial_\mu\partial_\mu\partial_\mu(\bar{\psi}_s(x)\gamma_5\gamma_\mu\psi_b(x))
\end{aligned}
$$

$\Delta F = 1$ operators in the adjoint representations

$$
\begin{aligned}
\mathcal{O}_7^{2F;d=6} &= \partial_\mu \partial_\mu (\bar{\psi}_s(x) \gamma_\mu \overleftrightarrow{D}_\mu \psi_b(x)) \\
\mathcal{O}_8^{2F;d=6} &= \partial_\mu \partial_\mu (\bar{\psi}_s(x) \gamma_5 \gamma_\mu \overleftrightarrow{D}_\mu \psi_b(x)) \\
\mathcal{O}_9^{2F;d=6} &= \partial_\mu (\bar{\psi}_s(x) \overleftarrow{D}_\mu \gamma_\mu \overrightarrow{D}_\mu \psi_b(x)) \\
\mathcal{O}_{10}^{2F;d=6} &= \partial_\mu (\bar{\psi}_s(x) \overleftarrow{D}_\mu \gamma_5 \gamma_\mu \overrightarrow{D}_\mu \psi_b(x)) \\
\mathcal{O}_{11}^{2F;d=6} &= \bar{\psi}_s(x) \overleftarrow{D}_\mu \gamma_\mu \overleftrightarrow{D}_\mu \overrightarrow{D}_\mu \psi_b(x) \\
\mathcal{O}_{12}^{2F;d=6} &= \bar{\psi}_s(x) \overleftarrow{D}_\mu \gamma_5 \gamma_\mu \overleftrightarrow{D}_\mu \overrightarrow{D}_\mu \psi_b(x)
\end{aligned}
$$

- The list has been substantially reduced when considering GIRS. Gauge-noninvariant operators and gauge-invariant operators that vanish by the equations of motion (~ 30) have been excluded.

Summary and outlook

- We classify the four-quark operators into traceless, singlets and operators that support the adjoint representation of the flavor group.
- We study the renormalization of traceless operators using GIRS
- We explore a wide range of GIRS variants in continuum perturbation theory and identify those with reduced mixing effects.
- For promising variants, we have extracted the next-to-leading-order conversion matrices between GIRS and $\overline{\text{MS}}$ scheme.
- A renormalization study of $\Delta F = 1$ operators that mix with lower-dimensional operators on the lattice is currently in progress.
- A renormalization study of contact terms appearing in long-distance amplitudes calculated by spectral reconstruction methods is currently in progress (together with Prof. Martinelli)
- Future plans: Extend our study to $\Delta F = 0$ operators

Relevant Papers:

1. M. Constantinou, M. Costa, H. Herodotou, H. Panagopoulos and G. Spanoudes, "Gauge-invariant renormalization of four- quark operators," Phys. Rev. D 110 (2024) 074506 [arXiv:2406.08065 [hep-lat]].
2. G. Spanoudes, M. Costa, K. Mitsidi and H. Panagopoulos, "Classification of four-quark operators with $\Delta F \leq 2$ under flavor symmetry and their renormalization in a gauge-invariant scheme," Phys. Rev. D 113 (2026) 034504 [arXiv:2511.04305 [hep-lat]]

THANK YOU FOR YOUR ATTENTION!

Acknowledgements: It is a pleasure to thank V. Lubicz and G. Martinelli for useful communication. The project EXCELLENCE/0524/0001 (StrongILA) is implemented under the programme of social cohesion "THALIA 2021-2027", co-funded by the European Union through the Cyprus Research and Innovation Foundation (RIF).

Co-funded by the European Union

Republic of Cyprus

RESEARCH & INNOVATION FOUNDATION

BACKUP SLIDES

$\Delta F = 1$ operators in the adjoint representations

- We employ Osterwalder-Seiler (OS) fermions with a clover term [ETMC, Phys. Rev. D 107 (2023) 074506]

$$S_{\mathrm{OS/cl}}(m_f, r_f) = \sum_x \bar{\psi}_f(x)\Big[\gamma\cdot\tilde{\nabla} - i\gamma_5\, W_{\mathrm{cr}}^{\mathrm{cl}}(r_f) + m_f\Big]\psi_f(x)\,,$$

$$\gamma\cdot\tilde{\nabla} = \frac{1}{2}\sum_\mu \gamma_\mu\left(\nabla_\mu + \nabla_\mu^*\right), \qquad W_{\mathrm{cr}}^{\mathrm{cl}}(r_f) = -\frac{r_f}{2}\sum_\mu \nabla_\mu^*\nabla_\mu + M_{\mathrm{cr}}(r_f) + \frac{r_f}{32}c_{\mathrm{SW}}\sum_{\mu,\nu}\sigma_{\mu\nu}\left[Q_{\mu\nu}(x) - Q_{\nu\mu}(x)\right]$$

- Spurionic symmetries of the OS action:

$$\begin{aligned}
\mathcal{CS}^{sp} &\equiv \mathcal{C}\times\mathcal{S}\times(r_s\leftrightarrow r_b)\times(m_s\leftrightarrow m_b)\\
\mathcal{P}^{sp} &\equiv \mathcal{P}\times(r_f\to -r_f),\ \text{for all } f\\
\mathcal{P}_5^{sp} &\equiv \mathcal{P}_5\times(m_f\to -m_f),\ \text{for all } f
\end{aligned}
\quad,\ \mathcal{C}:\begin{cases}\psi_f(x)\to C^{-1}\,\bar{\psi}_f(x)^T\,,\\ \bar{\psi}_f(x)\to -\psi_f(x)^T\,C,\\ U_\mu(x)\to (U_\mu^\dagger(x))^T\end{cases}
\quad \mathcal{S}:\begin{cases}\psi_s\leftrightarrow\psi_b\,,\\ \bar{\psi}_s\leftrightarrow\bar{\psi}_b\end{cases},$$

$$\mathcal{P}:\begin{cases}\psi_f(x)\to\gamma_0\,\psi_f(x_{\mathcal{P}}),\ x_{\mathcal{P}}=(x_0,-\vec{x})\\ \bar{\psi}_f(x)\to\bar{\psi}_f(x_{\mathcal{P}})\gamma_0\,,\\ U_0(x)\to U_0(x_{\mathcal{P}})\,,\\ U_k(x)\to U_k^\dagger(x_{\mathcal{P}}-a\hat{k}),\ k=1,2,3\end{cases}
\quad,\ \mathcal{P}_5:\begin{cases}\psi_f(x)\to\gamma_5\gamma_0\,\psi_f(x_{\mathcal{P}}),\ x_{\mathcal{P}}=(x_0,-\vec{x})\\ \bar{\psi}_f(x)\to -\bar{\psi}_f(x_{\mathcal{P}})\gamma_0\gamma_5\,,\\ U_0(x)\to U_0(x_{\mathcal{P}})\,,\\ U_k(x)\to U_k^\dagger(x_{\mathcal{P}}-a\hat{k}),\ k=1,2,3\end{cases}$$

Spurionic symmetry transformations

Operator	$\mathcal{CS}^{sp}$	$\mathcal{P}^{sp}$	$\mathcal{P}_5^{sp}$
$\mathcal{O}_1^{2F;d=3}$	+	+	−
$\mathcal{O}_2^{2F;d=3}$	+	−	+
$\mathcal{O}_1^{2F;d=5}$	+	+	−
$\mathcal{O}_2^{2F;d=5}$	+	−	+
$\mathcal{O}_3^{2F;d=5}$	+	+	−
$\mathcal{O}_4^{2F;d=5}$	+	−	+
$\mathcal{O}_1^{2F;d=6}$	−	+	+
$\mathcal{O}_2^{2F;d=6}$	+	−	−
$\mathcal{O}_3^{2F;d=6}$	+	+	+
$\mathcal{O}_4^{2F;d=6}$	−	−	−
$\mathcal{O}_5^{2F;d=6}$	−	+	+
$\mathcal{O}_6^{2F;d=6}$	+	−	−
$\mathcal{O}_7^{2F;d=6}$	+	+	+
$\mathcal{O}_8^{2F;d=6}$	−	−	−
$\mathcal{O}_9^{2F;d=6}$	−	+	+
$\mathcal{O}_{10}^{2F;d=6}$	+	−	−
$\mathcal{O}_{11}^{2F;d=6}$	+	+	+
$\mathcal{O}_{12}^{2F;d=6}$	−	−	−

Spurionic symmetry transformations

Operator	$\mathcal{CS}^{sp}$	$\mathcal{P}^{sp}$	$\mathcal{P}_5^{sp}$
${\mathcal{O}_1}_f^{4F;d=6}$	+	+	+
${\mathcal{O}_2}_f^{4F;d=6}$	+	+	+
${\mathcal{O}_3}_f^{4F;d=6}$	+	+	+
${\mathcal{O}_4}_f^{4F;d=6}$	+	+	+
${\mathcal{O}_5}_f^{4F;d=6}$	+	+	+
${\mathcal{O}_6}_f^{4F;d=6}$	+	+	+
${\mathcal{O}_7}_f^{4F;d=6}$	+	+	+
${\mathcal{O}_8}_f^{4F;d=6}$	+	+	+
${\mathcal{O}_9}_f^{4F;d=6}$	+	+	+
${\mathcal{O}_{10}}_f^{4F;d=6}$	+	+	+
${\mathcal{O}_{11}}_f^{4F;d=6}$	−	−	−
${\mathcal{O}_{12}}_f^{4F;d=6} + {\mathcal{O}_{17}}_f^{4F;d=6}$	+	−	−
${\mathcal{O}_{12}}_f^{4F;d=6} - {\mathcal{O}_{17}}_f^{4F;d=6}$	−	−	−
${\mathcal{O}_{13}}_f^{4F;d=6} + {\mathcal{O}_{18}}_f^{4F;d=6}$	−	−	−
${\mathcal{O}_{13}}_f^{4F;d=6} - {\mathcal{O}_{18}}_f^{4F;d=6}$	+	−	−
${\mathcal{O}_{14}}_f^{4F;d=6}$	+	−	−
${\mathcal{O}_{15}}_f^{4F;d=6}$	+	−	−
${\mathcal{O}_{16}}_f^{4F;d=6}$	−	−	−
${\mathcal{O}_{19}}_f^{4F;d=6}$	+	−	−
${\mathcal{O}_{20}}_f^{4F;d=6}$	+	−	−