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From Lattice to Boundary: Symmetries and Anomalies of 3+1D Staggered Fermions

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Based on arXiv:2509.04906 w/ Tetsuya Onogi (U. Osaka)
arXiv:2604.02078 w/ Tatsuhiro Misumi (Kindai U) and Tetsuya Onogi (U. Osaka)

1. Introduction

Nielsen-Ninomiya's theorem

[Nielsen, Ninomiya (1981)] [D. Friedan (1982)] etc.

Lattice Hamiltonian has

- Locality properties
- **Chiral symmetry**
- Translation Invariance
- Hermiticity

$\Longleftrightarrow$ Doubling problem

(There are extra excited Dirac fermions.)

It is highly non-trivial to realize the chiral (flavor) symmetries on the lattice.

Long-standing Question

How can we encode **on the lattice**
chiral symmetries and their 't Hooft anomalies
of the target continuum theory?

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How can we encode **on the lattice**
chiral symmetries and their 't Hooft anomalies
of the target continuum theory?

★ [Chatterjee, D. Pace, Shao (2024)] realized the chiral symmetry and its ’t Hooft anomaly in 1+1 D staggered fermion system.

Key Idea : Onsager charges

[Chatterjee, D. Pace, Shao (2024)]

- 1+ 1 D staggered fermion Hamiltonian has the translational symmetry acting on each Majorana fermions,

$$c_j = \frac{1}{2}(a_j + ib_j) \quad \{a_j, a_{j'}\} = \{b_j, b_{j'}\} = 2\delta_{j,j'} \quad \{c_j, c^\dagger_{j'}\} = \delta_{j,j'} \quad \{a_j, b_{j'}\} = 0$$

$$T_b a_j (T_b)^{-1} = a_j \,, \quad T_b b_j (T_b)^{-1} = b_{j+1}$$

- The system has two conserved charges,

$$Q_V = \sum_{j=1}^{2N} \left(c_j^\dagger c_j - \frac{1}{2} \right) \qquad Q_A = T_b Q_V \, (T_b)^{-1}$$

- In the continuum limit, Q_V, Q_A generate **vector** U(1) and **axial** U(1) symmetries respectively.
- Q_V, Q_A do not commute each other but generate the **Onsager algebra** on the lattice.

  (We call these conserved charges Onsager charges in this talk.)

  ― This non-commutativity avoids the Nielsen-Ninomiya theorem.

- The presence of both conserved Q_V, Q_A forces the system to be gapless.

  ➡ **’t Hooft anomaly = Obstruction to a symmetric, trivially-gapped phase**

What is the anomaly?

['t Hooft (1980)]　[Seiberg, Shao, Zhang (2025)]

The **standard** 't Hooft anomaly in continuum QFT:

coupling to background gauge fields for internal symmetries makes the partition function ill-defined.

$$Z[A] \to Z[A]\underbrace{e^{2\pi i \int_X \alpha}}_{\text{Anomaly}}$$

Anomaly = Obstruction to gauging the symmetry

What is the anomaly?

[Chang, Lin, Shao, Wang, Ying (2018)] [Thorngren and Wang (2019)]
[Seiberg, Shao, Zhang (2025)]

However, when conserved charges are highly non-local,
it is not clear if there is a sensible prescription for gauging it.

(e.g.) Conserved charges which generate the Onsager algebra = Onsager charges

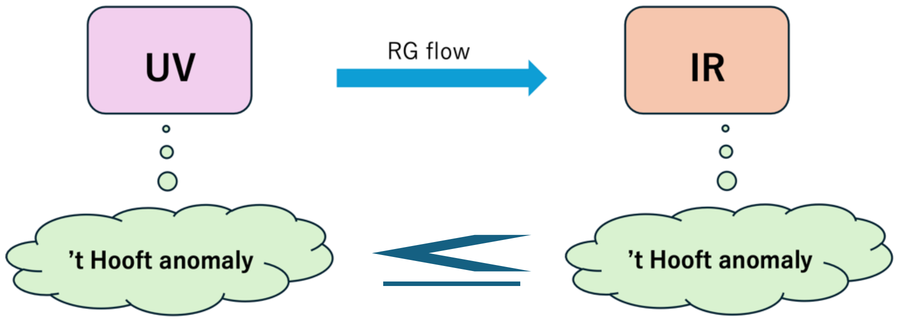
Another definition of anomaly :

Anomaly = **Obstruction to a symmetric, trivially-gapped phase**

∵ 't Hooft anomaly matching condition

(e.g.)Lieb-Schultz-Mattis (LSM)-like constraints

[Lieb, Schultz, Mattis (1961)]

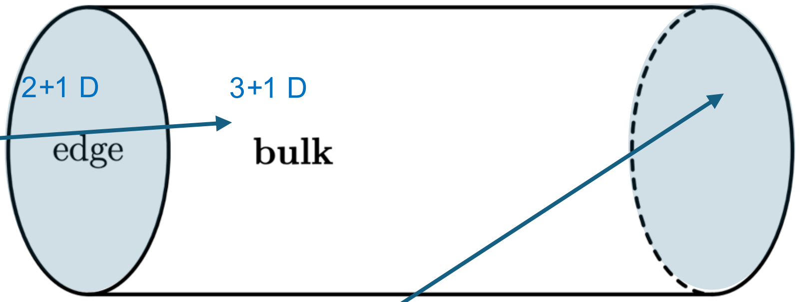


Our work = Applications of the Onsager charges in the other systems

Today's question :

・What about in **3+1 D staggered fermion system**?

[Onogi, TY (2025)]



・What about on **2+1 D boundary** of the 3+1 D staggered fermion system?

[Misumi, Onogi, TY (2026)]

2. 3+1 D staggered fermions

2-1. Set up

2-1. New conserved charge; Onsager charges

3+1 D staggered fermion Hamiltonian

[Kogut, Susskind (1975)]
[Catterall, Pradhan, Samlodia 2024]

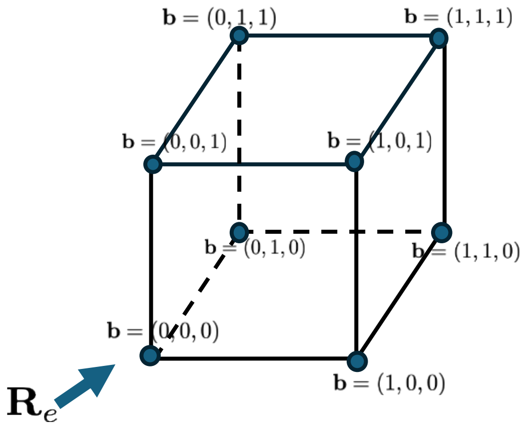
$$\boxed{H = \sum_{\boldsymbol{r}}^{L} \sum_{i=1}^{3} \frac{\mathrm{i}}{2} \eta_i(\boldsymbol{r}) \left(\chi(\boldsymbol{r})^\dagger \chi(\boldsymbol{r}+\hat{i}) + \chi(\boldsymbol{r})\chi^\dagger(\boldsymbol{r}+\hat{i}) \right) = \sum_{\boldsymbol{r}} \sum_{i=1}^{3} \frac{1}{4} \eta_i(\boldsymbol{r}) \left(a(\boldsymbol{r})a(\boldsymbol{r}+\hat{i}) + b(\boldsymbol{r})b(\boldsymbol{r}+\hat{i}) \right)}$$

This Hamiltonian can be rewritten to **2-flavor Dirac fermion** Hamiltonian,

$$\boxed{H = i \sum_{\mathbf{R}_e} \Psi^\dagger(\mathbf{R}_e) \left[(\alpha_i \otimes I_{2\times 2}) \frac{\nabla_i}{2} + \underbrace{\beta(\gamma_5 \otimes \sigma_i^T) \frac{\nabla_i^2}{2}}_{\text{Twisted Wilson mass}} \right] \Psi(\mathbf{R}_e)}$$

where $\alpha_i = \gamma_0 \gamma_i \ (i = 1, 2, 3), \quad \beta = \gamma_0$

$$\left.\Psi(R_e) = \underbrace{\begin{pmatrix} \psi_{1+}(R_e) & \psi_{2+}(R_e) \\ \psi_{1-}(R_e) & \psi_{2-}(R_e) \end{pmatrix}}_{\text{Flavor 1,2}}\ \right\} \text{4-component spinor}$$



3+1 D staggered fermion Hamiltonian

[Kogut, Susskind (1975)]

[Catterall, Pradhan, Samlodia 2024]

In the continuum,

$$H = i\sum_{\mathbf{R}_e} \Psi^\dagger(\mathbf{R}_e)\left[(\alpha_i \otimes I_{2\times 2})\frac{\nabla_i}{2} + \cancel{\beta(\gamma_5 \otimes \sigma_i^T)\frac{\nabla_i^2}{2}}\right]\Psi(\mathbf{R}_e)$$

Twisted Wilson mass

Describing Two-flavor massless Dirac fermion system.

$$G^{(3+1D)}_{\text{classical}} = \mathrm{U}(1)_V \times \underline{\mathrm{U}(1)_A \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R}$$

These symmetries will be restored in the continuum limit.

3+1 D staggered fermion Hamiltonian

On the lattice,

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2. 3+1 D staggered fermions

2-1. Set up

2-1. New conserved charge; Onsager charges

Translational symmetry

[Kogut, Susskind (1975)]

[Catterall, Pradhan, Samlodia 2024]

$$H=\sum_{\boldsymbol{r}}^{L}\sum_{i=1}^{3}\frac{\mathrm{i}}{2}\eta_i(\boldsymbol{r})\left(\chi(\boldsymbol{r})^\dagger\chi(\boldsymbol{r}+\hat{i})+\chi(\boldsymbol{r})\chi^\dagger(\boldsymbol{r}+\hat{i})\right)=\sum_{\boldsymbol{r}}\sum_{i=1}^{3}\frac{1}{4}\eta_i(\boldsymbol{r})\left(a(\boldsymbol{r})a(\boldsymbol{r}+\hat{i})+b(\boldsymbol{r})b(\boldsymbol{r}+\hat{i})\right)$$

(We assume that the number of lattice sites on each direction is even $\boldsymbol{L}$ and the periodic boundary condition.)

$$\chi(\boldsymbol{r})=\frac{1}{2}\left(a(\boldsymbol{r})+\mathrm{i}b(\boldsymbol{r})\right)$$

- Majorana fermions a and b are decoupled from each other.

→ Translational symmetries acting on each Majorana fermion.

e.g. $T_{\hat{i}}^{(b)}a(\boldsymbol{r})\left(T_{\hat{i}}^{(b)}\right)^{-1}=a(\boldsymbol{r})\ ,\ T_{\hat{i}}^{(b)}b(\boldsymbol{r})\left(T_{\hat{i}}^{(b)}\right)^{-1}=\zeta_i(\boldsymbol{r})b(\boldsymbol{r}+\hat{i})\ ,\ [T_{\hat{i}}^{(b)},H]=0\,.$

$$\zeta_i(\boldsymbol{r}):=(-1)^{\sum_{k=i+1}^{3}r_k}$$

Conserved charges on the lattice

[Onogi, TY (2025)]

We can define two types of conserved charges :

$$Q_0 = \sum_{\boldsymbol{r}} \left(\chi^\dagger(\boldsymbol{r})\chi(\boldsymbol{r}) - \frac{1}{2} \right) = \sum_{\boldsymbol{r}} \frac{\mathrm{i}}{2} a(\boldsymbol{r}) b(\boldsymbol{r})$$

$$Q_i = T_{\hat{i}}^{(b)} Q_0 \left(T_{\hat{i}}^{(b)} \right)^{-1} = \sum_{\boldsymbol{r}} \frac{\mathrm{i}}{2} \zeta_i(\boldsymbol{r}) a(\boldsymbol{r}) b(\boldsymbol{r} + \hat{i}) \,, \qquad i = x, y, z$$

New Conserved Charges

$$\because \; [Q_0, H] = [T_{\hat{i}}^{(b)}, H] = 0$$

Onsager charges in the system

These conserved charges do **Not** commute with each other.

$$[Q_i, Q_j] =: G_{ij} \neq 0 \qquad [Q_0, Q_i] =: G_{0i} \neq 0$$

G_{ij} also commutes with the Hamiltonian.

$Q_0,\ Q_i$ and G_{ij} generate the Onsager algebras. [Aoki, Kikukawa, Takemoto (2025)]

Thus

$\boxed{Q_0, Q_i \text{ and } G_{ij} \text{ are } \textbf{Onsager charges} \text{ !!}}$

Physical interpretation

[Onogi, TY (2025)] [Misumi, Onogi, TY (2026)]

We consider following conserved charges, $Q_x, Q_y, Q_z, R_{xy}, R_{yz}, R_{zx}$

$$\Psi(R_e) = \begin{pmatrix} \psi_{1+}(R_e) & \psi_{2+}(R_e) \\ \psi_{1-}(R_e) & \psi_{2-}(R_e) \end{pmatrix}$$

Taking **continuum** limit, we find

$$\lim_{L\to\infty}[Q_i, \Psi(R_e)] = (\gamma_5 \otimes \sigma_i)\Psi(R_e)\,, \qquad \lim_{L\to\infty}[R_{ij}, \Psi(R_e)] = 2\mathrm{i}\epsilon_{ijk}(\mathbb{1}\otimes\sigma_k)\Psi(R_e)$$

The conserved charges generate

chiral flavor $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ **symmetry.**

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The conserved charges generate

chiral flavor $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ **symmetry.**

— thought to be absent on the lattice by the twisted-Wilson mass, yet realized exactly (**emanant**).

= These symmetries emanate from the underlying exact lattice symmetries.

$$H = i\sum_{\mathbf{R}_e}\Psi^\dagger(\mathbf{R}_e)\left[(\alpha_i\otimes I_{2\times 2})\frac{\nabla_i}{2} + \beta(\gamma_5\otimes\sigma_i^T)\frac{\nabla_i^2}{2}\right]\Psi(\mathbf{R}_e)$$

3. 2+1 D boundary of the 3+1 D staggered fermion system

From 3+1 D bulk to 2+1 D edge

■ 3+1 dim. Staggered fermion without the kink mass

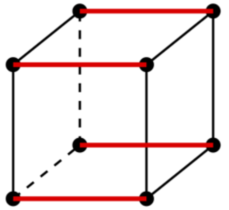
- describes a **two-flavor Dirac fermion system**.
- has Onsager charges; $Q_x, Q_y, Q_z, R_{xy}, R_{yz}, R_{zx}$

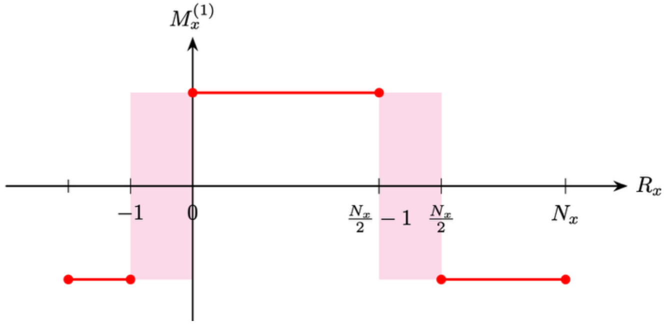
  generating chiral flavor $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry in the continuum limit.

One-link mass in the x-direction [Misumi, Onogi, TY (2026)]

Let us introduce a kink profile of the **one-link mass in the x-direction = Domain wall.**

$$\delta H_x^{(1)} = \mathrm{i} \sum_{\boldsymbol{R}} \frac{m}{2} \theta(R_x) \Psi^\dagger(\boldsymbol{R}_e)(\gamma_0\gamma_5 \otimes \sigma_1)\Psi(\boldsymbol{R}_e)\,,$$





	P	R_x	R_y	R_z	$\mathcal{T}$	$\mathcal{C}$	S_x	S_y	S_z	Q_0	Q_x	Q_y	Q_z
$\delta H_x^{(1)}$	×	×	○	○	×	○	×	○	○	○	×	○	○

➡ preserves $\cancel{Q_x}, Q_y, Q_z, \cancel{R_{xy}}, R_{yz}, \cancel{R_{zx}}$

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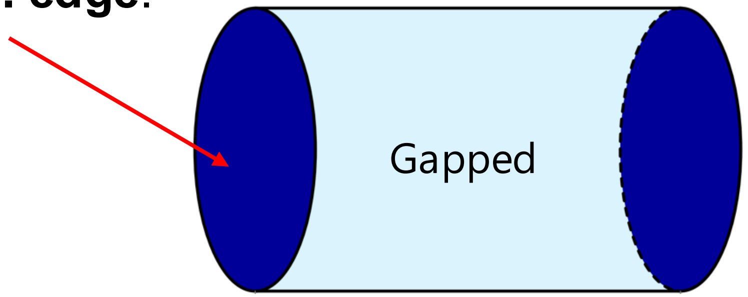
Introducing the kink mass $\delta H_x^{(1)} = \mathrm{i} \sum_{\boldsymbol{R}} \frac{m}{2} \theta(R_x) \Psi^\dagger(\boldsymbol{R}_e)(\gamma_0\gamma_5 \otimes \sigma_1)\Psi(\boldsymbol{R}_e)\,,$

	P	R_x	R_y	R_z	$\mathcal{T}$	$\mathcal{C}$	S_x	S_y	S_z	Q_0	Q_x	Q_y	Q_z
$\delta H_x^{(1)}$	×	×	○	○	×	○	×	○	○	○	×	○	○

- The fermions on the bulk are gapped.
- **Two-flavor two-component Dirac fermion is localized on the 2+1 Dim. edge**.
- This system has Onsager charges;

$$\cancel{Q_x}, Q_y, Q_z, \cancel{R_{xy}}, R_{yz}, \cancel{R_{zx}}$$



Physical interpretation of Onsager charges on the edge

two-flavor two-component Dirac fermion

In the continuum limit,

$$\lim_{L\to\infty}[Q_y,\hat{\Psi}] = (\mathbb{1}_{2\times 2}\otimes\sigma_2)\hat{\Psi}\,,\quad \lim_{L\to\infty}[Q_z,\hat{\Psi}] = -(\mathbb{1}_{2\times 2}\otimes\sigma_1)\hat{\Psi}\,,\quad \lim_{L\to\infty}[R_{yz},\hat{\Psi}] = 2\mathrm{i}(\mathbb{1}_{2\times 2}\otimes\sigma_3)\hat{\Psi}$$

Q_y, Q_z and G_{yz} generate a flavor $\mathrm{SU}(2)$ symmetry on the Domain wall.

Parity anomaly b/w $\mathrm{SU}(2)$ and $\mathbb{Z}_2^{\mathrm{R}_i}$ on the edge

Comment: It is known that 2+1 D two-flavor Dirac fermion (**continuum**) system has parity anomaly associated w/ flavor SU(2) symmetry and spacetime reflection symmetry.

[Seiberg and Witten (2016)] [Pace, Kim, Chatterjee, Shao (2025)]

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In our system,

	P	R_x	R_y	R_z	$\mathcal{T}$	$\mathcal{C}$	S_x	S_y	S_z	Q_0	Q_x	Q_y	Q_z
$\delta H_x^{(1)}$	×	×	○	○	×	○	×	○	○	○	×	○	○

Imposing
- Flavor $\mathrm{SU}(2)$ symmetry generated by Q_y, Q_z, R_{yz}
- Reflection symmetry $\mathbb{Z}_2^{\mathrm{R}_i}$, $i = y, z$

⇒ No mass term is allowed.(= Obstruction to a symmetric, trivially-gapped phase)

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- Reflection symmetry $\mathbb{Z}_2^{\mathrm{R}_i}$, $i = y, z$

⇒ No mass term is allowed.(= Obstruction to a symmetric, trivially-gapped phase)

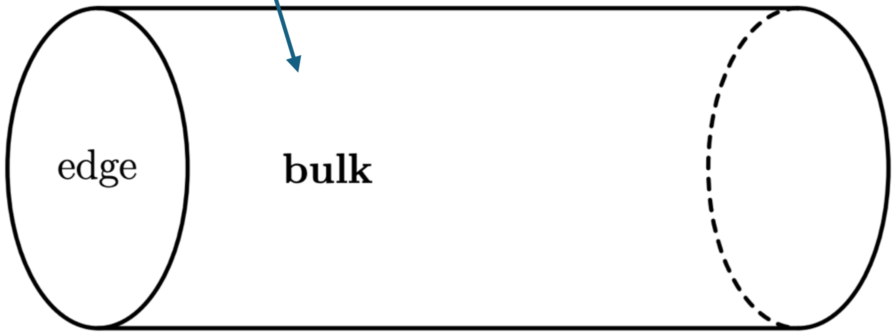
This establishes the parity anomaly associated with $\mathrm{SU}(2)$ and $\mathbb{Z}_2^{\mathrm{R}_i}$.

∵ ’t Hooft anomaly matching condition

Emanant symmetries

In the continuum limit, Q_y, Q_z, R_{yz} conserved **on the lattice** generate

$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ on the bulk



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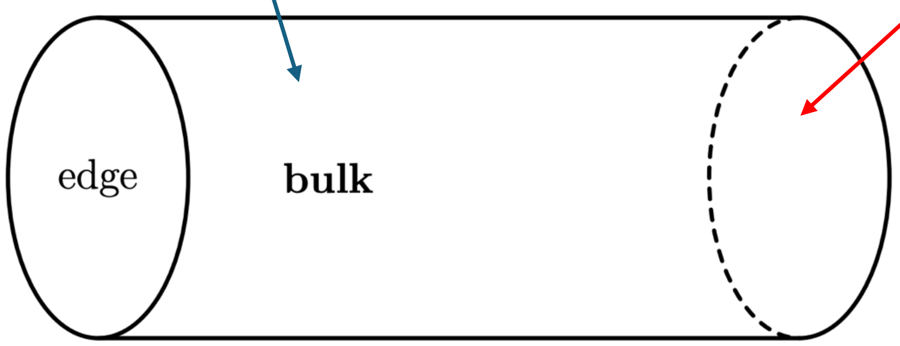
$\boxed{\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \text{ on the bulk}} \Rightarrow \boxed{\mathrm{SU}(2) \text{ on the edge}}$



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$\mathbb{Z}_2^{\mathrm{R}_y}$, $\mathbb{Z}_2^{\mathrm{R}_z}$: exact on bulk and edge

Parity anomaly

Emanant symmetries

In the continuum limit, Q_y, Q_z, R_{yz} conserved **on the lattice** generate

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Parity anomaly

This flavor $\mathrm{SU}(2)$ symmetry and the parity anomaly are not emergent but **Emanant**.

i.e., These originate from conserved charges defined in the bulk lattice theory.

Summary

3+1 dim. Staggered fermion (without the kink mass)

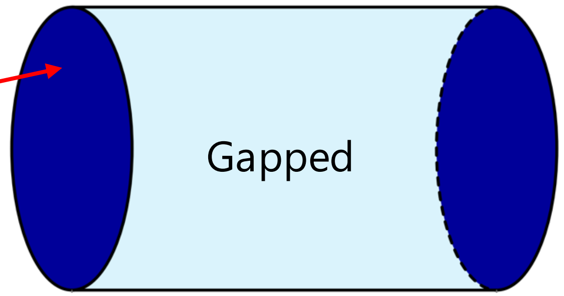
・describes a **two-flavor Dirac fermion system**.

・has Onsager charges; $Q_x, Q_y, Q_z, R_{xy}, R_{yz}, R_{zx}$

generating chiral flavor $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry in the continuum limit.

On the 2+1 dim. edge

・**Two-flavor two-component Dirac fermion is localized on the 2+1 Dim. edge**.



・This system has Onsager charges; $\cancel{Q_x}, Q_y, Q_z, \cancel{R_{xy}}, R_{yz}, \cancel{R_{zx}}$

・The charges generate the flavor SU(2) symmetry and realize the Parity anomaly in the continuum limit.

Future directions

・What about the 4+1 dim. Theory ?

・Application of the Onsager charges to the soft pion theorem.

Work in progress with Misumi and Onogi.

Main message

In both the bulk and the edge,

the **Onsager charges** are **the lattice avatars** of symmetries

previously thought to emerge only in the IR

— and, when those symmetries are anomalous, **avatars of** their anomalies as well.

→ Such symmetries and anomalies are not emergent but "**emanant.**"
= These emanate from the underlying exact lattice symmetries and anomalies.

→ The IR QFT information can be extracted from the Onsager charges on the lattice.

Back Up

Onsager algebra (Ons3)

[Aoki, Kikukawa, Takemoto (2025)]

$$T_g^{(b)} := (T_x^{(b)})^{n_x}(T_y^{(b)})^{n_y}(T_z^{(b)})^{n_z} \qquad g(\boldsymbol{r}) = \boldsymbol{r} + n_x\hat{x} + n_y\hat{y} + n_z\hat{z}$$

$$Q_g = T_g^{(b)} Q_0 (T_g^{(b)})^{-1} = \frac{\mathrm{i}}{2}\sum \zeta_g(\boldsymbol{r}) a_{\boldsymbol{r}} b_{g(\boldsymbol{r})}\,,$$

$$G_{g,h} = \frac{\mathrm{i}}{2}\sum \left(\zeta_g a_{\boldsymbol{r}} a_{g(\boldsymbol{r})} + \zeta_h b_{\boldsymbol{r}} b_{g(\boldsymbol{r})}\right)$$

$$\begin{aligned}
[Q_g, Q_h] &= \sqrt{-1}\,G_{g^{-1}h,hg^{-1}}\,,\\
[Q_g, G_{h,k}] &= \sqrt{-1}(Q_{gh^{-1}} + Q_{k^{-1}g} - Q_{gh} - Q_{kg}),\\
[G_{g_1,h_1}, G_{g_2,h_2}] &= \sqrt{-1}(G_{g_2g_1,h_1h_2} - G_{g_2^{-1}g_1,h_1h_2^{-1}} - G_{g_1g_2,h_2h_1} + G_{g_1g_2^{-1},h_2^{-1}h_1}).
\end{aligned}$$

What is the anomaly?

['t Hooft (1980)] [Seiberg, Shao, Zhang (2025)]

- The standard definition of 't Hooft anomaly in **continuum QFT** involves placing the system on a general Euclidean spacetime with background gauge fields for the **internal** symmetries **and finding that the partition function is not well-defined**.

$$Z[A] \to Z[A]\underbrace{e^{2\pi i \int_X \alpha}}_{\text{Anomaly}}$$

Anomaly = Obstruction to gauging the symmetry

What is the anomaly?

[Chang, Lin, Shao, Wang, Ying (2018)] [Thorngren and Wang (2019)]
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However, when conserved charges are highly non-local,
it is not clear if there is a sensible prescription for gauging it.

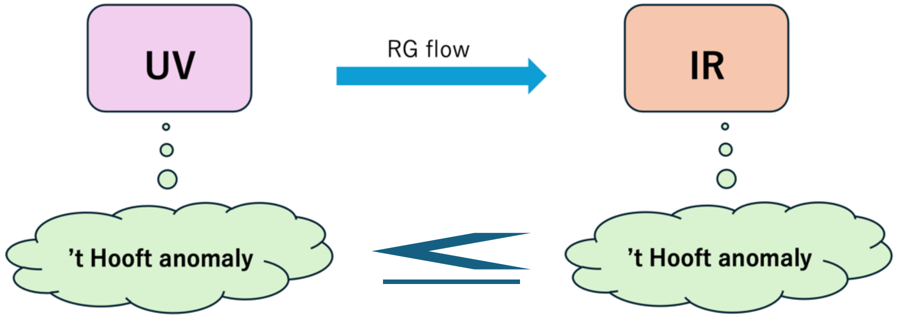
(e.g.) Conserved charges which generate the Onsager algebra.

Another definition of anomaly :

Anomaly = **Obstruction to a symmetric, trivially-gapped phase**

$\because$ 't Hooft anomaly matching condition

(e.g.)Lieb-Schultz-Mattis (LSM)-like constraints
[Lieb, Schultz, Mattis (1961)]



Conserved charges in 1+1 D Staggered fermion

[Chatterjee, Pace, Shao (2024)]

- The Hamiltonian has a manifest conserved charge : $Q_V = \sum_{j=1}^{2N} \left(c_j^\dagger c_j - \frac{1}{2} \right)$
  - It is quantized, i.e. $Q_V \in \mathbb{Z}$
  - It generate a vector U(1) symmetry

$$c_j = \frac{1}{2}(a_j + i b_j) \quad \{a_j, a_{j'}\} = \{b_j, b_{j'}\} = 2\delta_{j,j'}$$

$$\{c_j, c_{j'}^\dagger\} = \delta_{j,j'} \quad \{a_j, b_{j'}\} = 0$$

$$T_b a_j (T_b)^{-1} = a_j \,, \quad T_b b_j (T_b)^{-1} = b_{j+1}$$

- There is an extra conserved charge : $Q_A = T_b Q_V \left(T_b\right)^{-1} = \frac{1}{2} \sum_{j=1}^{2N} \left(c_j + c_j^\dagger\right) \left(c_{j+1} - c_{j+1}^\dagger\right) = \frac{\mathrm{i}}{2} \sum_{j=1}^{2N} a_j b_{j+1}$
  - It is quantized since Q_V and Q_A are unitary equivalent. $Q_A \in \mathbb{Z}$
  - It generates axial U(1) symmetry in the continuum limit.

Important properties of the conserved charges

1) Q_V, Q_A are part of the **Onsager algebra**. $\rightarrow$ We call the charges **Onsager charges** in this talk.

2) If we impose both symmetries, possible local terms δH that we can add to the Hamiltonian are strongly restricted.

a) Only bilinear terms are allowed.

b) Moreover, mass terms are prohibited.

➔The presence of conserved Q_V, Q_A forces the system to be gapless. → **'t Hooft anomaly**

➔ Onsager charges are Lattice avatars of the symmetries and their anomalies in the IR continuum theory.
= These symmetry and anomaly in IR emanate from the underlying exact lattice symmetries and anomalies.

➔ **Strong constraints on the vacuum structure**

A hint to consider possible symmetric mass generation

Applications of the conserved Onsager charges

- 2+1D Honeycomb lattice system (describing 2-flavor Dirac fermion system in IR)
  - Onsager charges Q_n generate the flavor (valley) SU(2) that had been regarded as emergent, and its parity anomaly (with spacetime reflection) appears as the LSM anomaly of Q_n on the lattice.

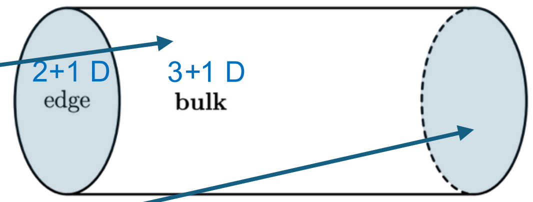
  $\Rightarrow$ This SU(2) and its parity anomaly are **emanant**, not emergent.
  = These emanate from the underlying exact lattice symmetries and anomalies.

  [Pace, Kim, Chatterjee, Shao (2025)]

- 3+1D lattice model with the minimal fermion content (a single Weyl doublet) [Gioia, Thorngren (2025)] [see also Gioia's talk]
- Lattice model having Fermi surface [Pace, Kim, Chatterjee, Shao (2025)]

Today's Topic : Our work

- What about in **3+1 D staggered fermion system**? [Onogi, TY (2025)]
- What about **on 2+1 D boundary** of the 3+1 D staggered fermion system? [Misumi, Onogi, TY (2026)]



Hamiltonian

[Kogut, Susskind (1975)]
[Catterall, Pradhan, Samlodia 2024]

$$H=\sum_{\boldsymbol{r}}^{L}\sum_{i=1}^{3}\frac{\mathrm{i}}{2}\eta_i(\boldsymbol{r})\left(\chi(\boldsymbol{r})^{\dagger}\chi(\boldsymbol{r}+\hat{i})+\chi(\boldsymbol{r})\chi^{\dagger}(\boldsymbol{r}+\hat{i})\right)=\sum_{\boldsymbol{r}}\sum_{i=1}^{3}\frac{1}{4}\eta_i(\boldsymbol{r})\left(a(\boldsymbol{r})a(\boldsymbol{r}+\hat{i})+b(\boldsymbol{r})b(\boldsymbol{r}+\hat{i})\right)$$

(We assume that the number of lattice sites on each direction is even $\boldsymbol{L}$ and the periodic boundary condition.)

$$\chi(\boldsymbol{r})=\frac{1}{2}\left(a(\boldsymbol{r})+ib(\boldsymbol{r})\right)$$

- staggered phase $\eta_i(\boldsymbol{r})=(-1)^{r_1+\cdots+r_{i-1}}$ corresponds to inserting a π flux in the unit cell that gives the **eight** zeromodes at two momentum points in the continuum limit.

➡ Two-flavor Dirac fermions (We will see this in the next slides. (3pages))

From staggered fermions to Dirac fermions

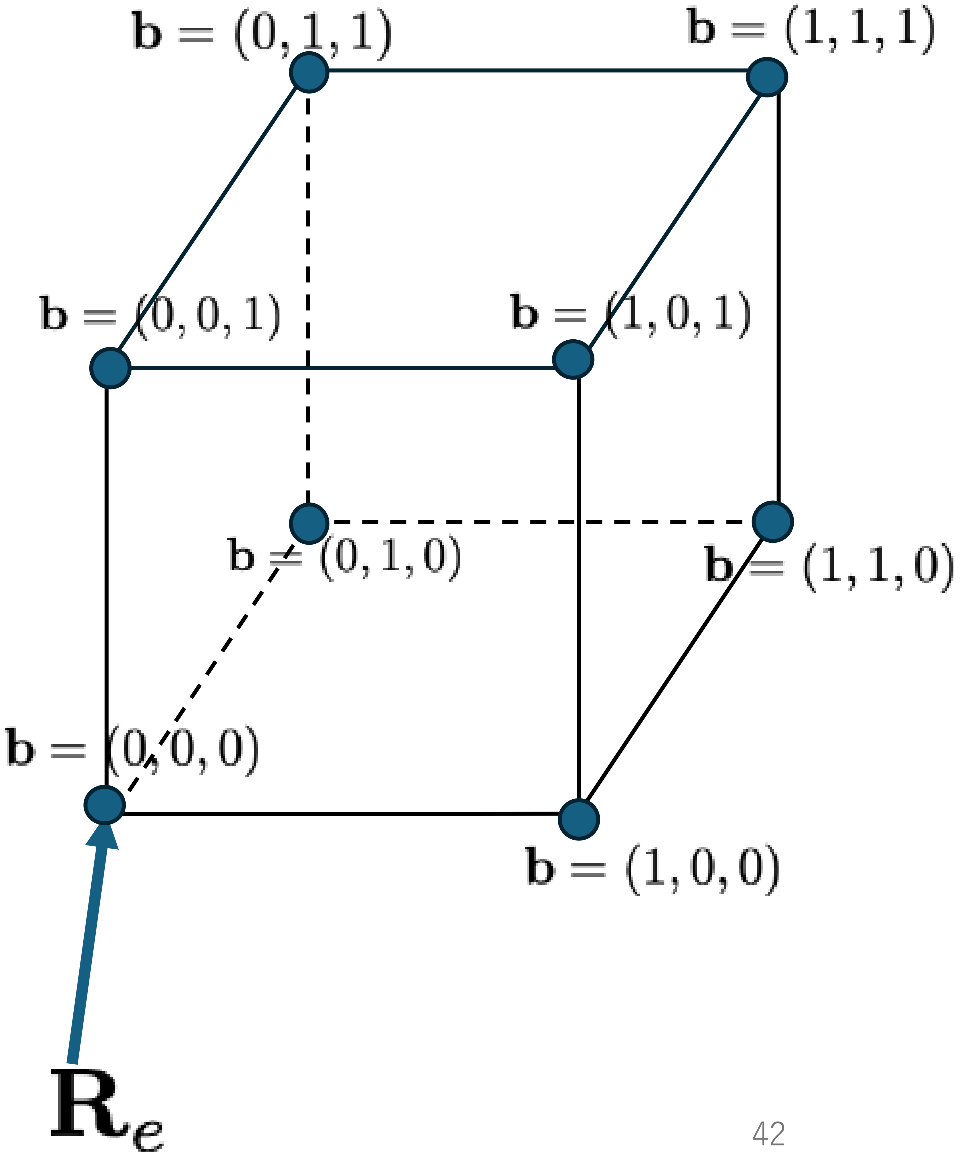
We consider a unit cell with 8 points

$$\mathbf{r} = \mathbf{R}_e + \mathbf{b}$$

Vectors labelling different cells

Vectors within the unit cell

We interpret $\mathbf{R}_e$ as spatial points,
and $\mathbf{b}$ as spin-flavor degrees of freedom



From staggered fermions to Dirac fermions

In chiral basis,

$$\alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$$

Using 4x4 matrices $\alpha_i = \gamma_0\gamma_i \ (i = 1, 2, 3), \quad \beta = \gamma_0$

$$\alpha^{\mathbf{b}} = \alpha_1^{b_1}\alpha_2^{b_2}\alpha_3^{b_3}$$

We define matrix valued fermion from staggered fermion χ as

$$\Phi(\boldsymbol{R}_e) = N_0 \sum_{\{\boldsymbol{b}\}} \chi(\boldsymbol{R}_e + \boldsymbol{b})\, \alpha^b = \begin{pmatrix} \psi_{1+}(\boldsymbol{R}_e) & \psi_{2+}(\boldsymbol{R}_e) & 0 & 0 \\ 0 & 0 & \psi_{3-}(\boldsymbol{R}_e) & \psi_{4-}(\boldsymbol{R}_e) \end{pmatrix}$$

Combining Weyl fermions as

$$\Psi(\mathbf{R}_e) = \underbrace{\begin{pmatrix} \psi_{1+}(\mathbf{R}_e) & \psi_{2+}(\mathbf{R}_e) \\ \psi_{3-}(\mathbf{R}_e) & \psi_{4-}(\mathbf{R}_e) \end{pmatrix}}_{\text{Flavor 1,2}} \Bigg\} \text{4-component spinor}$$

we obtain 2-flavor Dirac fermions.

From staggered fermions to Dirac fermions

[Kogut, Susskind (1975)]
[Catterall, Pradhan, Samlodia 2024]

This Hamiltonian can be rewritten to **2-flavor Dirac fermion** Hamiltonian,

$$H = i\sum_{\mathbf{R}_e} \Psi^\dagger(\mathbf{R}_e)\left[(\alpha_i \otimes I_{2\times 2})\frac{\nabla_i}{2} + \underbrace{\beta(\gamma_5 \otimes \sigma_i^T)\frac{\nabla_i^2}{2}}_{\text{Twisted Wilson mass}}\right]\Psi(\mathbf{R}_e)$$

$$\alpha_i = \gamma_0\gamma_i \ (i=1,2,3), \quad \beta = \gamma_0$$

where $$\Psi(\mathbf{R}_e) = \underbrace{\begin{pmatrix} \psi_{1+}(\mathbf{R}_e) & \psi_{2+}(\mathbf{R}_e) \\ \psi_{3-}(\mathbf{R}_e) & \psi_{4-}(\mathbf{R}_e) \end{pmatrix}}_{\text{Flavor 1,2}} \Big\} \text{ 4-component spinor}$$

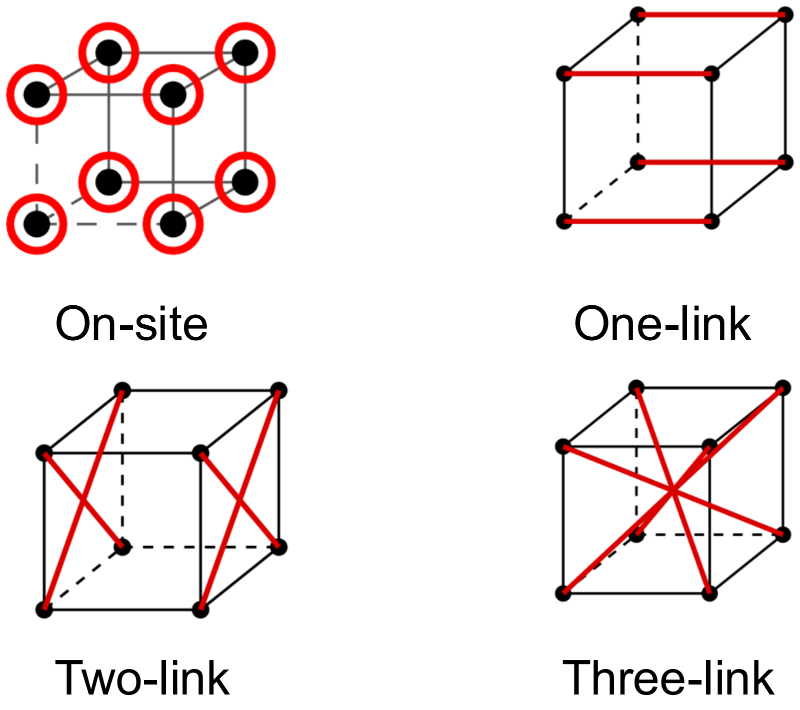
Classification of Dirac mass terms

A Dirac fermion is reconstructed from 8 one-component staggered fermions in a unit cube.

→ mass terms need not be on-site
— they may arise from one-, two-, or three-link hoppings within the cube.

We classify **all bilinear Dirac mass terms** that are

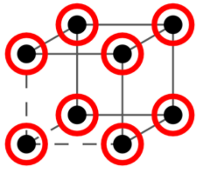
- Hermitian
- particle-number-preserving
- localized within a unit cube
- gap out the continuum Dirac fermions



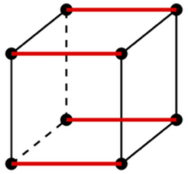
Mass terms in the unit cell

[Misumi, Onogi, TY (2026)]

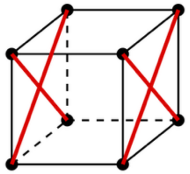
On- site mass : $$\delta H^{(0)} = \sum_{\boldsymbol{R}_e} m\Psi^\dagger(\boldsymbol{R}_e)(\gamma_0 \otimes \mathbb{1})\Psi(\boldsymbol{R}_e)$$



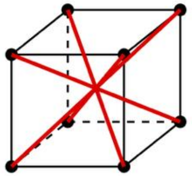
One-link mass : $$\delta H_i^{(1)} = \sum_{\boldsymbol{R}_e} m\Psi^\dagger(\boldsymbol{R}_e)(\mathrm{i}\gamma_0\gamma_5 \otimes \sigma_i)\Psi(\boldsymbol{R}_e)$$



Two-link mass : $$\delta H_i^{(2)} = \sum_{\boldsymbol{R}_e} m\Psi^\dagger(\boldsymbol{R}_e)(\gamma_0 \otimes \sigma_i)\Psi(\boldsymbol{R}_e)$$

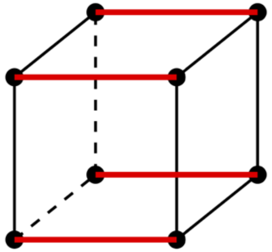


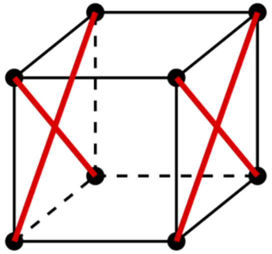
Three-link mass : $$\delta H^{(3)} = \sum_{\boldsymbol{R}_e} m\Psi^\dagger(\boldsymbol{R}_e)(\mathrm{i}\gamma_0\gamma_5 \otimes \mathbb{1})\Psi(\boldsymbol{R}_e)$$

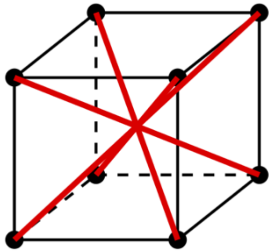


Masses and symmetries

[Misumi, Onogi, TY (2026)]







	P	R_x	R_y	R_z	T	C	$\mathcal{T}$	$\mathcal{C}$	S_x	S_y	S_z	Q_0	Q_x	Q_y	Q_z
$\delta H^{(0)}$	○	○	○	○	○	×	○	○	×	×	×	○	×	×	×
$\delta H_x^{(1)}$	×	×	○	○	○	○	×	○	×	○	○	○	×	○	○
$\delta H_y^{(1)}$	×	○	×	○	○	○	×	×	×	×	○	○	×	×	○
$\delta H_z^{(1)}$	×	○	○	×	○	○	×	×	×	×	×	○	×	×	×
$\delta H_x^{(2)}$	○	○	×	×	×	○	○	×	×	×	×	○	×	×	×
$\delta H_y^{(2)}$	○	×	○	×	×	○	○	×	×	×	×	○	×	×	×
$\delta H_z^{(2)}$	○	×	×	○	×	○	○	×	×	×	×	○	×	×	×
$\delta H^{(3)}$	×	×	×	○	○	×	○	×	×	×	×	○	×	×	×

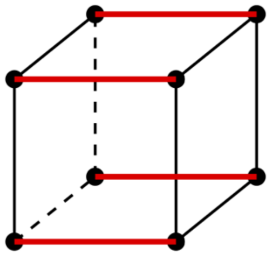
Table 1. Commutation relation between masses and symmetries

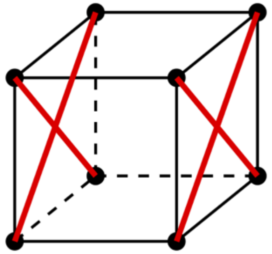
○ : commutative , × : non-commutative.

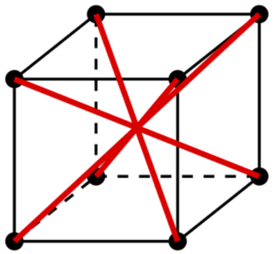
Masses and symmetries

[Misumi, Onogi, TY (2026)]

$$\delta H^{(0)} = \sum_{\boldsymbol{R}_e} m \Psi^\dagger(\boldsymbol{R}_e)(\gamma_0 \otimes \mathbb{1})\Psi(\boldsymbol{R}_e)$$







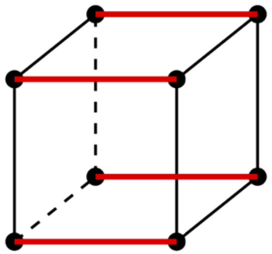
	P	R_x	R_y	R_z	T	C	$\mathcal{T}$	$\mathcal{C}$	S_x	S_y	S_z	Q_0	Q_x	Q_y	Q_z
$\delta H^{(0)}$	○	○	○	○	○	×	○	○	×	×	×	○	×	×	×
$\delta H_x^{(1)}$	×	×	○	○	○	○	×	○	×	○	○	○	×	○	○
$\delta H_y^{(1)}$	×	○	×	○	○	○	×	×	×	×	○	○	×	×	○
$\delta H_z^{(1)}$	×	○	○	×	○	○	×	×	×	×	×	○	×	×	×
$\delta H_x^{(2)}$	○	○	×	×	×	○	○	×	×	×	×	○	×	×	×
$\delta H_y^{(2)}$	○	×	○	×	×	○	○	×	×	×	×	○	×	×	×
$\delta H_z^{(2)}$	○	×	×	○	×	○	○	×	×	×	×	○	×	×	×
$\delta H^{(3)}$	×	×	×	○	○	×	○	×	×	×	×	○	×	×	×

Table 1. Commutation relation between masses and symmetries
○ : commutative , × : non-commutative.

Masses and symmetries

[Misumi, Onogi, TY (2026)]

$$\delta H_i^{(1)} = \sum_{\boldsymbol{R}_e} m\Psi^\dagger(\boldsymbol{R}_e)(\mathrm{i}\gamma_0\gamma_5 \otimes \sigma_i)\Psi(\boldsymbol{R}_e)$$



	P	R_x	R_y	R_z	T	C	$\mathcal{T}$	$\mathcal{C}$	S_x	S_y	S_z	Q_0	Q_x	Q_y	Q_z
$\delta H^{(0)}$	○	○	○	○	○	×	○	○	×	×	×	○	×	×	×
$\delta H_x^{(1)}$	×	×	○	○	○	○	×	○	×	○	○	○	×	○	○
$\delta H_y^{(1)}$	×	○	×	○	○	○	×	×	×	×	○	○	×	×	○
$\delta H_z^{(1)}$	×	○	○	×	○	○	×	×	×	×	×	○	×	×	×

- $\delta H_x^{(1)}$ respects **many** of the symmetries of the staggered fermion Hamiltonian, and implies the **absence** of a mixed 't Hooft anomaly between $\mathrm{U}(1)_V$ and $\mathrm{U}(1)_{F_{i\neq x}}$
- This anisotropy originates from the staggered phase $\eta_i(\boldsymbol{r}) = (-1)^{r_1+\cdots+r_{i-1}}$, which singles out the x-direction.

Under a cyclic $\mathbb{Z}_3$ relabeling of {x,y,z},
a different one-link mass would preserve the largest residual symmetry.

Discrete symmetries

[Li, Wang, You (2024)] [Aoki, Kikukawa, Takemoto (2025)]

3+1 D staggered fermion system possesses the following discrete symmetries :

- Parity symmetry $\mathbb{Z}_2^{\mathrm{P}}$
- Reflection symmetry $\mathbb{Z}_2^{\mathrm{R}_i}$
- Time-reversal symmetry $\mathbb{Z}_2^{\mathrm{T}}$
- Charge conjugation symmetry $\mathbb{Z}_2^{\mathrm{C}}$
- Shift symmetry $\mathbb{Z}_{L_i}^{S_i}$

Lattice size along the i th direction

Action on the staggered fermions

$$
\begin{aligned}
\mathrm{P}: &\quad \mathrm{P}\chi(\boldsymbol{r})\mathrm{P}^{-1} = \epsilon(\boldsymbol{r})\,\chi(-\boldsymbol{r})\,, \qquad \text{with} \quad \mathrm{P} = \mathrm{R}_x\mathrm{R}_y\mathrm{R}_z\,,\\
\mathrm{R}_i: &\quad \mathrm{R}_i\chi(\boldsymbol{r})\mathrm{R}_i^{-1} = (-1)^{r_i}\,\chi(\tilde{R}_i(\boldsymbol{r})) = \epsilon(\boldsymbol{r})\eta_i(\boldsymbol{r})\zeta_i(\boldsymbol{r})\chi(\tilde{R}_i(\boldsymbol{r}))\,,\\
\mathrm{T}: &\quad \mathrm{T}\chi(\boldsymbol{r})\mathrm{T}^{-1} = \epsilon(\boldsymbol{r})\,\chi(\boldsymbol{r})\,, \qquad \mathrm{T}\,\mathrm{i}\,\mathrm{T}^{-1} = -\mathrm{i}\,,\\
\mathrm{C}: &\quad \mathrm{C}\chi(\boldsymbol{r})\mathrm{C}^{-1} = \chi^\dagger(\boldsymbol{r})\,,\\
S_i: &\quad S_i\chi(\boldsymbol{r})S_i^{-1} = \zeta_i(\boldsymbol{r})\,\chi(\boldsymbol{r}+\hat{i})\,, \qquad S_i = T_{\hat{i}}^{(a)}T_{\hat{i}}^{(b)}\,,
\end{aligned}
$$

$$
\begin{aligned}
&\tilde{R}_i(r_1,\dots,r_i,\dots,r_3) = (r_1,\dots,-r_i,\dots,r_3)\\
&\epsilon(\boldsymbol{r}) = (-1)^{x+y+z}\\
&\zeta_i(\boldsymbol{r}) := (-1)^{\sum_{k=i+1}^{3} r_k}
\end{aligned}
$$

Discrete symmetries

In the continuum limit,

- Parity symmetry $\mathbb{Z}_2^{\mathrm{P}}$
- Reflection symmetry $\mathbb{Z}_2^{\mathrm{R}_i}$
- Time-reversal symmetry $\mathbb{Z}_4^{\mathcal{T}}$
- Charge conjugation symmetry $\mathbb{Z}_2^{\mathcal{C}}$
- Shift symmetry $\mathbb{Z}_{L_i}^{S_i}$

where $\mathcal{T} := -S_z S_x \mathrm{T}: \quad \mathcal{T}\chi(\boldsymbol{r})\mathcal{T}^{-1} = -(-1)^x \chi(\boldsymbol{r} + \hat{x} + \hat{z}), \qquad \mathcal{T}\,\mathrm{i}\,\mathcal{T}^{-1} = -\mathrm{i}\,,$

$\mathcal{C} := S_y \mathrm{C}: \quad \mathcal{C}\chi(\boldsymbol{r})\mathcal{C}^{-1} = (-1)^z \chi^\dagger(\boldsymbol{r} + \hat{y})\,.$

$$
\begin{aligned}
\mathrm{P}:&\quad \Psi(\boldsymbol{R}_e) \to (\beta \otimes \mathbb{1})\,\Psi(-\boldsymbol{R}_e)\,, \qquad \text{with} \quad \mathrm{P} = \mathrm{R}_x\mathrm{R}_y\mathrm{R}_z\,,\\
\mathrm{R}_x:&\quad \Psi(\boldsymbol{R}_e) \to (-\beta\gamma_5\alpha_1 \otimes \sigma_1^T)\,\Psi(\tilde{R}_x(\boldsymbol{R}_e)) = (-\mathrm{i}\beta\alpha_2\alpha_3 \otimes \sigma_1^T)\,\Psi(\tilde{R}_x(\boldsymbol{R}_e))\,,\\
\mathrm{R}_y:&\quad \Psi(\boldsymbol{R}_e) \to (\beta\gamma_5\alpha_2 \otimes \sigma_2^T)\,\Psi(\tilde{R}_y(\boldsymbol{R}_e)) = (\mathrm{i}\beta\alpha_3\alpha_1 \otimes \sigma_2^T)\,\Psi(\tilde{R}_y(\boldsymbol{R}_e))\,,\\
\mathrm{R}_z:&\quad \Psi(\boldsymbol{R}_e) \to (-\beta\gamma_5\alpha_3 \otimes \sigma_3^T)\,\Psi(\tilde{R}_z(\boldsymbol{R}_e)) = (-\mathrm{i}\beta\alpha_1\alpha_2 \otimes \sigma_3^T)\,\Psi(\tilde{R}_z(\boldsymbol{R}_e))\,,
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{T}:&\quad \Psi(\boldsymbol{R}_e) \to (-\alpha_1\alpha_3 \otimes \sigma_2^T)\,\Psi(\boldsymbol{R}_e)\,,\\
\mathrm{C}:&\quad \Psi(\boldsymbol{R}_e) \to -(\beta\alpha_3\alpha_1 \otimes \mathrm{i}\sigma_2^T)\,\Psi^*(\boldsymbol{R}_e)\,,\\
\mathcal{T}:&\quad \Psi(\boldsymbol{R}_e) \to (\alpha_1\alpha_3 \otimes \mathbb{1})\,\Psi(\boldsymbol{R}_e)\,,\\
\mathcal{C}:&\quad \Psi(\boldsymbol{R}_e) \to \mathrm{i}(\beta\alpha_2 \otimes \mathbb{1})\,\Psi^*(\boldsymbol{R}_e)\,,\\
\mathcal{S}_i:&\quad \Psi(\boldsymbol{R}_e) \to (\gamma_5 \otimes \sigma_i^T)\,\Psi(\boldsymbol{R}_e)\,, \qquad \text{with rotation angle } \frac{\pi}{2}\,.
\end{aligned}
$$