Fuzzy regularization of Hamiltonian CP^N models

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Based on:

AB, Karl Jansen, Stefan Kühn, Marco Panero and Antonio Smecca

arXiv:26MM.XXXXX

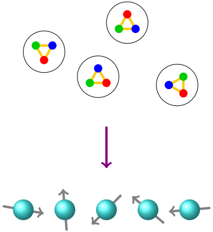
CP^N models in $1+1D$

Toy models for $3+1D$ gauge theories

- $\mathrm{CP}^N \sim \mathrm{SU}(N+1)/\mathrm{U}(N)$, global $\mathrm{SU}(N+1)$ symmetry
- Physics similar to $3+1D$ $\mathrm{SU}(N+1)$ gauge theories
  - Linear confinement
  - $1/N$ expansion
- MCMC: sign problem when studying
  - θ terms
  - chemical potentials
  - real-time dynamics

Encoding of a theory

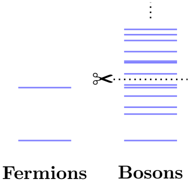
- Alternative: Hamiltonian simulations
- **This talk**: finite dimensional Hilbert space and continuum limit



Hilbert space truncation

Truncation schemes

- Tensor networks, digital QC (…) need a finite Hilbert space
- Different truncation schemes



Angular momentum truncation
Quantum link models
Finite subgroups
Orbifold
q-deformations
Fuzzy
Loop-string-hadron
Heisenberg comb
Renormalized dual basis
Large-N_c

… and many more!

Full Hilbert space limit

- Unphysical finite Hilbert space effects (cfrt finite-a and finite-V)
- Extrapolation $\dim \mathcal{H} \to \infty$ (in general costly + systematic errors)
- For some systems, this might not be needed after all …

[Plenary by S. Chandrasekharan, Lattice conference 2024]

Fuzzy manifolds

Continuous manifolds

- Geometry (CP^N) = Algebra of functions ($C^\infty(\mathrm{CP}^N)$)

$$f(\mathbf{x}) = f_0 + f_a x^a + \dots$$

$$f_1 * f_2 = f_3 \in C^\infty$$

Truncation

- Possibility: truncate the Taylor series (**angular momentum truncation**)

$$f(\mathbf{x}) = f_0 + f_a x^a \in C_{\text{trunc}}$$

- The product is not closed anymore

$$f_1 * f_2 = \dots + f_{ab} x^a x^b \notin C_{\text{trunc}}$$

Fuzzy truncation

- Coordinates $\to$ matrices

$$f(\lambda) = f_0 \mathbb{1} + f_a \lambda^a \in M$$

$$f_1 * f_2 = f_3 \in M$$

$$\partial_a f \to [\lambda_a, f]$$

- The continuous symmetry of the manifold is retained

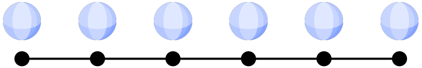
[Madore; Class.Quant.Grav. 9 (1992)]
[Balachandran et al., J.Geom.Phys. 43 (2002)]

Fuzzy sphere approach for the $O(3)$ sigma model (aka CP^1)

$O(3)$ sigma model

$$aH = \frac{1}{J}\sum_{i=1}^{L/a} L_i^2 + J\sum_{i=1}^{L/a-1} \mathbf{x}_i \cdot \mathbf{x}_{i+1}$$

$$\sum_{a=1}^{3} x^a x^a = 1, \quad [x^a, x^b] = 0$$

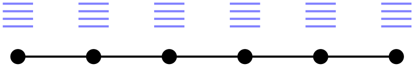


Asymptotic freedom: $am \sim \exp(-J)$

Fuzzy $O(3)$ sigma model

$$aH = \frac{1}{J}\sum_{i=1}^{L/a}\sum_{a=1}^{3}[\Sigma_i^a, [\Sigma_i^a, \cdot]] + J\sum_{i=1}^{L/a-1}\sum_{a=1}^{3}\Sigma_i^a\Sigma_{i+1}^a$$

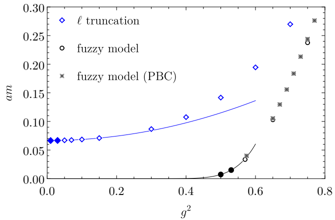
$$\Sigma\,|M\rangle = |\sigma M\rangle, \quad [\Sigma^a, \Sigma^b] = i\varepsilon^{abc}\Sigma^c$$

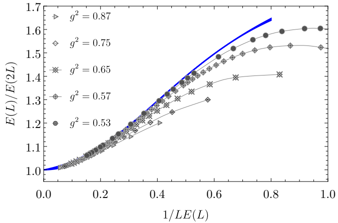


[Alexandru et al., PRL 123 (2019)]

Fuzzy sphere approach for the $O(3)$ sigma model (aka CP^1)

[Alexandru et al., PRD 107 (2023)]





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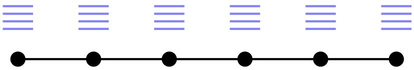

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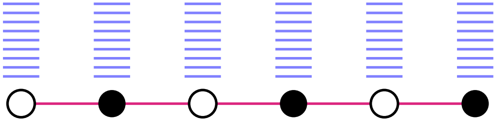
Fuzzy regularization for CP^N models

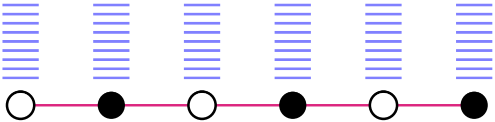
[**AB**, Jansen, Kühn, Panero, Smecca, arXiv:26MM.XXXXX]

$$aH = \frac{1}{J}\sum_{i=1}^{L-1}\sum_{a=1}^{(N+1)^2-1}[\Lambda_i^a,[\Lambda_i^a,\cdot]] + J\sum_{i\ \text{even}}\sum_{a=1}^{(N+1)^2-1}\Lambda_i^a\bar{\Lambda}_{i+1}^a + J\sum_{i\ \text{odd}}\sum_{a=1}^{(N+1)^2-1}\bar{\Lambda}_i^a\Lambda_{i+1}^a$$

$$\Lambda^a\,|M\rangle = |\lambda^a M\rangle \qquad \bar{\Lambda}^a = -(\Lambda^a)^*$$

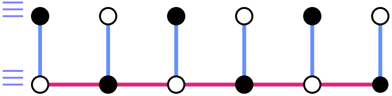


Fuzzy $\mathrm{CP}^N = \mathrm{SU}(N+1)$ comb



change of basis

$$aH = \sum_{i\text{ even}}^{L} \sum_{a=1}^{(N+1)^2-1} \left\{ \frac{1}{J} \lambda_i^a (-\lambda^a)^*_{i+1/2} + J \lambda_i^a (-\lambda^a)^*_{i+1} \right\} +$$
$$\sum_{i\text{ odd}}^{L} \sum_{a=1}^{(N+1)^2-1} \left\{ \frac{1}{J} (-\lambda^a)^*_i \lambda^a_{i+1/2} + J (-\lambda^a)^*_i \lambda^a_{i+1} \right\}$$



See also [Bhattacharya et al., PRL 126 (2021)]

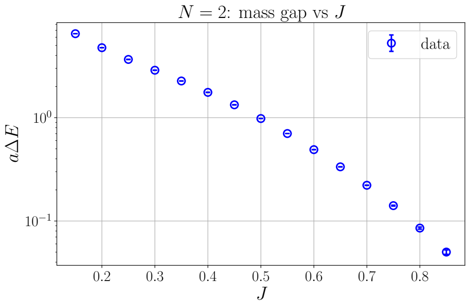
CP^2: mass gap

Expectation

- MPS simulations, $D \to \infty$ and $L \to \infty$ extrapolations
- $a\Delta E$ exponentially closing with $J \to \infty$

$$a\Delta E = AJ^{\alpha} e^{-bJ}$$

MPS simulation



CP^2: entanglement entropy

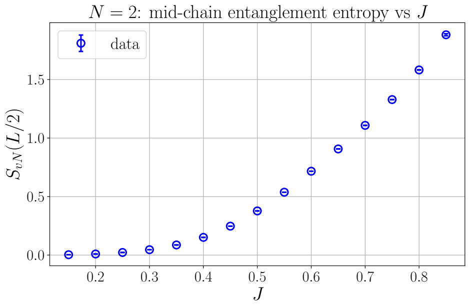
Expectation

- Asymptotically linear function of J

$$S = \frac{c}{6}(bJ - \alpha \log J) + k$$

- The central charge c can be computed

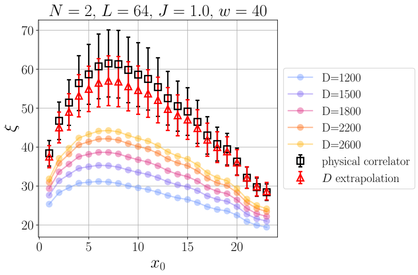
MPS simulation



CP^2: step scaling

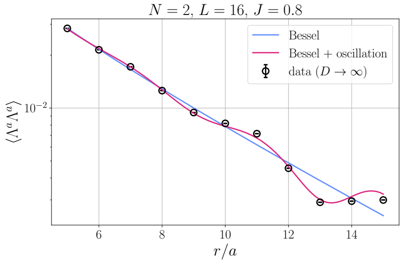
ξ from correlators: large L

- ξ extracted fitting $\langle \Lambda^a \Lambda^a \rangle$
- Large bond dimensions required



ξ from correlators: small L

- Non-trivial boundary effects at small volumes



Conclusions and Outlook

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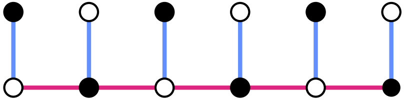
- Proposed fuzzy regularization for quantum simulation of CP^N models
  - Extension of fuzzy/comb constructions
  - Continuum limit at finite truncation
- Preliminary results on the mass gap and entanglement entropy
- Next steps: refine the step scaling analysis

Outlook

- Study of entanglement measures in asymptotically free theories
- How to extend this approach?



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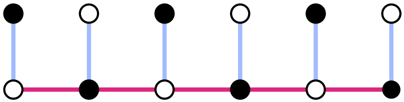

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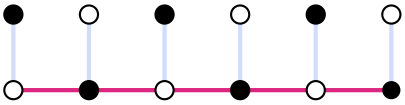

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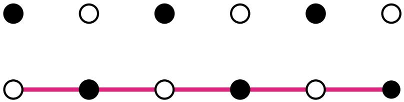


Backup slides

The fuzzy sphere

Quantizing the geometry

[Madore; Class.Quant.Grav. 9 (1992)]

- Geometry (S^2) = Algebra of functions ($C^\infty(S^2)$)
- Truncate the algebra of functions but preserve the algebraic structure

$$f(\mathbf{x}) = f_0 + f_a x_a + \dots \quad \longrightarrow \quad \hat{f} = f_0 \mathbb{1} + f_a \sigma_a$$

$$-\nabla^2 f \quad \longrightarrow \quad [\sigma_a, [\sigma_a, \hat{f}]]$$

- Matrix algebras are closed under multiplication

Numerical applications

- Radial quantization in 3D CFTs

  [Zhu et al., PRX 13 (2023)]
  [He, Zhu, arXiv:2607.01310]

- Hilbert space truncation

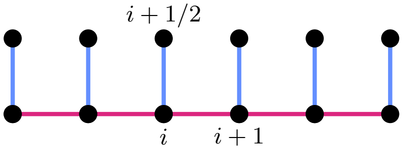
  [Alexandru et al., PRL 123 (2019)]
  [Alexandru et al., PLB 832 (2022)]
  [Alexandru et al., PRD 107 (2023)]
  [Alexandru et al., PRD 109 (2024)]
  [Bedaque et al., arXiv:2512.21282]

The Heisenberg comb

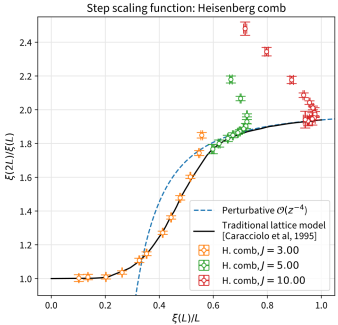
The Hamiltonian

[Bhattacharya et al., PRL 126 (2021)]

$$aH = \frac{1}{J}\sum_{i=1}^{L}\sum_{a=1}^{3}\sigma_i^a\sigma_{i+1/2}^a + J\sum_{i=1}^{L}\sum_{a=1}^{3}\sigma_i^a\sigma_{i+1}^a$$



Step scaling



[Bhattacharya et al., PRL 126 (2021)]

Fuzzy sphere = Heisenberg comb

A specific basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$$

$$(\Sigma^a)_{nm} = \langle b_n | \Sigma^a | b_m \rangle = \mathrm{Tr}\left(b_n^\dagger \sigma^a b_m \right)$$

$$(\Sigma^a)_{\mathcal{B}} = \sigma^a \otimes \mathbb{1}$$

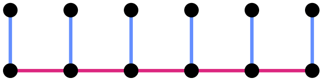
$$(L^a)_{\mathcal{B}} = \frac{1}{2}(\sigma^a \otimes \mathbb{1} + \mathbb{1} \otimes \sigma^a)$$

Equivalence of the two models

$$(L_i^a)^2_{\mathcal{B}} \sim \sigma_i^a \otimes \sigma_{i+1/2}^a$$

$$(\Sigma_i^a \Sigma_{i+1}^a)_{\mathcal{B}} \sim \sigma_i^a \otimes \mathbb{1}_{i+1/2} \otimes \sigma_{i+1}^a \otimes \mathbb{1}_{i+3/2}$$

$$(a H_{\mathrm{Fuzzy}})_{\mathcal{B}} =$$



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$$aH = \frac{1}{J}\sum_{i=1}^{L-1}\sum_{a=1}^{(N+1)^2-1}(L_i^a)^2 + J\sum_{i\ \mathrm{even}}\sum_{a=1}^{(N+1)^2-1}\Lambda_i^a\bar{\Lambda}_{i+1}^a + J\sum_{i\ \mathrm{odd}}\sum_{a=1}^{(N+1)^2-1}\bar{\Lambda}_i^a\Lambda_{i+1}^a$$

$$\Lambda^a\,|M\rangle = |\lambda^a M\rangle \qquad \bar{\Lambda}^a = -(\Lambda^a)^*$$

Computational basis

$$\mathcal{B} = \left\{\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}, \dots\right\}$$

$$(\Lambda^a)_{\mathcal{B}} = \lambda^a\otimes\mathbb{1}$$

$$(L^a)_{\mathcal{B}} = \frac{1}{2}[\lambda^a\otimes\mathbb{1} + \mathbb{1}\otimes(-\lambda^a)^*]$$

Comb-like structure