Finite temporal size effects on hadronic resonances using lattice QCD

Jakob Hoffmann, **Peter Lowdon, Owe Philipsen (Goethe University Frankfurt)**

based on **[2607.22346]**

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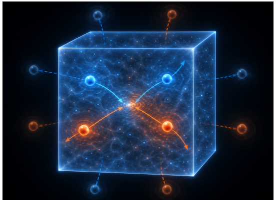
Strong-interaction matter
under extreme conditions



Hadronic resonances in finite volume

- In infinite volume resonance properties are determined by the scattering amplitude $\mathcal{M}(E)$
- Lattice QCD energies are computed in a finite 4d volume $L^3 \times L_\tau$ with periodic boundary conditions in Euclidean time
- **No** direct access to the real-time scattering amplitude on the lattice
- Quantization condition maps discrete two-particle energy spectrum $\{E_n\}$ computed in a finite spatial box for fixed momentum $\vec{P}$ to infinite-volume two-particle scattering amplitude $\mathcal{M}(E)$ [*M. Hansen, S.Sharpe* [1204.0826]]

$$\det\left[\mathcal{M}^{-1}(E_n^*) + F(E_n, \vec{P}; L)\right] = 0$$

- Derivation of quantization condition assumes $L_\tau \to \infty$

What about the finite temporal extent of the lattice?

Finite temporal extent

- Periodic Lattice with finite $L_\tau \equiv$ Lattice at finite temperature $T = 1/\beta = 1/L_\tau$

  $\rightarrow$ energies become discrete Matsubara frequencies $\omega_n = \frac{2\pi n_4}{\beta}$, $n_4 \in \mathbb{Z}$

- Spectral decomposition of thermal correlation function:

$$C_{L,\beta}(\tau) = \frac{1}{Z} \sum_{m,n} e^{-\beta E_n} \, e^{-(E_m - E_n)\,\tau} \langle n|\mathcal{O}|m\rangle \, \langle m|\mathcal{O}^\dagger|n\rangle$$

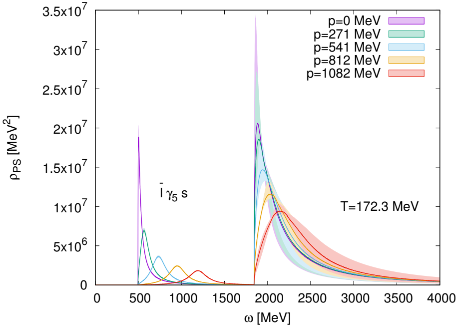
- Fourier transformed correlation function $C_{L,\beta}(E, \vec{P})$ has **poles** at finite-volume thermal energies $\mathcal{E}(\beta, \vec{P}, L) = E_m - E_n$

- Because L_τ is finite in a lattice simulation extracted energies **always** correspond to thermal energies

  $\rightarrow$ Need a finite-temperature scattering description

Thermoparticles as scattering states

- On-shell vacuum states are not good in-medium asymptotic states because interaction with medium is present also at large temporal and spatial separations
- Asymptotic spectrum is dominated by in-medium discrete modes (**Thermoparticles**) [hep-th/9606046]



K and K^* spectral functions from [2310.13476]

- Thermoparticle spectral function $\rho_{\mathrm{TP}}(\omega, \vec{p})$ fixed by interaction dependent damping factor $D_{m,\beta}(\vec{x})$
- Spectral functions have finite width, are asymmetric with branch-point at $\omega = m$

  $\rightarrow$ Thermoparticle does not decay into lighter modes
- Thermoparticle propagators have **complex** singularities

Thermal scattering amplitude and unitarity

- Thermoparticle asymptotic states can be used to define a thermal scattering amplitude:

$$_{\mathsf{TP}}\langle p_1', p_2' | T | p_1, p_2 \rangle_{\mathsf{TP}} = (2\pi)^4 \delta^{(4)}(p_1' + p_2' - p_1 - p_2)\, \mathcal{M}_{\mathsf{TP}}(p_1', p_2'; p_1, p_2)$$

- Scattering amplitude needs to fulfill S-matrix unitarity ($SS^\dagger = 1$), in the partial-wave basis:

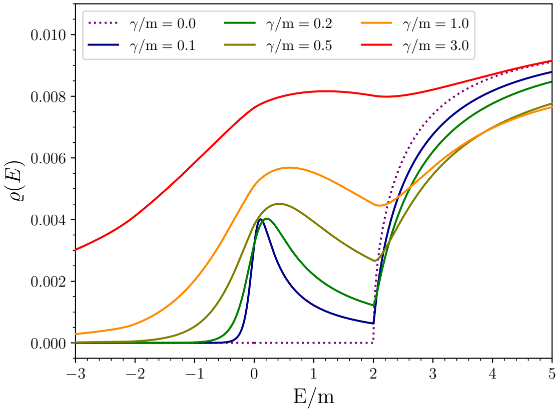
$$\mathrm{Im}(\mathcal{M}(E, \vec{P})) = \varrho_{\mathsf{TP}}(E, \vec{P}) |\mathcal{M}(E, \vec{P})|^2$$

- $\varrho_{\mathsf{TP}}(E, \vec{P})$ is the partial-wave projected frame-dependent thermal two-particle phase space

- Possible solution of unitarity condition:

$$\mathcal{M}_{\mathsf{TP}}(E, \vec{P}) = \frac{1}{\mathcal{K}^{-1}_{\mathsf{TP}}(E, \vec{P}) - i\varrho_{\mathsf{TP}}(E, \vec{P})}, \qquad \mathcal{K}^{-1}_{\mathsf{TP}}(E, \vec{P}) : \text{inverse K-matrix}$$

Thermal phase space for $\vec{P} = 0$



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- Damping factor: $D_{m,\beta}(\vec{x}) = \alpha\, e^{-\gamma|\vec{x}|}$ with $\alpha = 1$ and $\gamma = T$ (for illustration)
- Phase space is non-vanishing for $E < 2m$

  $\rightarrow$ no threshold at $E = 2m$
- Phase space is non-vanishing for $E < 0$

  $\rightarrow$ particles can absorb energy from the medium and create suppressed hole-like states
- Contribution for $E < 2m$ exclusively generated by interactions of thermoparticles with surrounding medium

Thermal finite-volume scattering

- Consider two identical, spinless thermoparticles in a finite spatial cubic box with periodic boundary conditions

  $\rightarrow$ spatial momenta are discrete, i.e. $\vec{k} \in \frac{2\pi}{L}\,\mathbb{Z}^3$

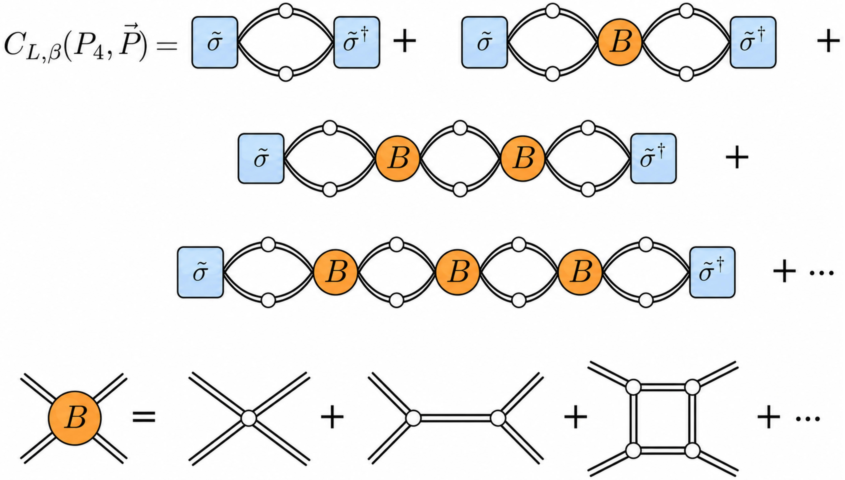
  $\rightarrow$ energies are discrete, i.e. $\omega \in \frac{2\pi}{\beta}\,\mathbb{Z}$

- Quantized energies and momenta imply replacements

$$\int \frac{d^3k}{(2\pi)^3} \longrightarrow \frac{1}{L^3} \sum_{\vec{k} \in \frac{2\pi}{L}\,\mathbb{Z}^3}, \qquad \int \frac{dk_0}{2\pi} \longrightarrow \frac{1}{\beta} \sum_{k_0 \in \frac{2\pi i}{\beta}\,\mathbb{Z}}$$

RFT skeleton expansion

- **Idea:** expand finite-volume correlation functions with **Thermoparticle** propagators:



Quantization condition

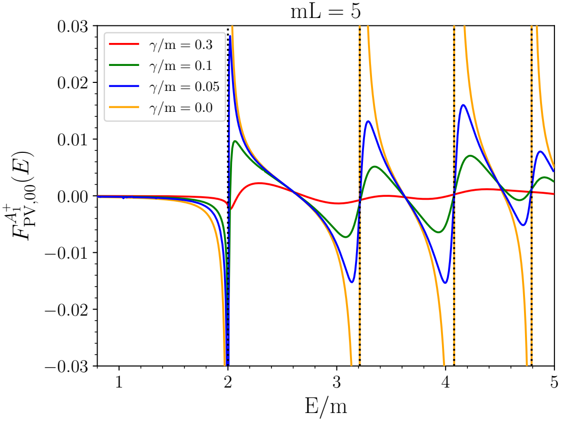
- Separate leading-order finite-volume effects due to two-particle loop from subleading effects due to Bethe-Salpether kernels

- Isolating poles of $C_{L,\beta}(E, \vec{P})$ yields frame-dependent thermal two-particle quantization condition (**QC2TP**)

$$\det\left[\mathcal{M}_{\mathsf{TP}}^{-1}(E, \vec{P}) + F(E, \vec{P}; L, \beta)\right] = 0$$

$$F_{\ell' m'; \ell m}(E, \vec{P}; L, \beta) = \frac{1}{4}\left(\frac{1}{L^3}\sum_{\vec{k}\in\frac{2\pi}{L}\mathbb{Z}^3} - \int\frac{d^3k}{(2\pi)^3}\right)\int_{-\infty}^{\infty}\frac{dq_0}{2\pi}\int_{-\infty}^{\infty}\frac{dr_0}{2\pi}\,\frac{4\pi\,Y_{\ell' m'}(\hat{k})\,Y_{\ell m}^{*}(\hat{k})|\vec{k}|^{\ell+\ell'}}{E - q_0 - r_0 + i\epsilon}$$
$$\times\left[\rho_{\mathsf{TP}}(q_0, \vec{P}-\vec{k})\rho_{\mathsf{TP}}(r_0, \vec{k}) + \rho_{\mathsf{TP}}(q_0, \vec{k})\rho_{\mathsf{TP}}(r_0, \vec{P}-\vec{k})\right]\coth\left(\frac{\beta q_0}{2}\right)$$

Finite-volume effects for $\vec{P} = 0$ and $\gamma = T$



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- For nonzero temperatures poles at non-interacting energies turn into **finite** peaks

  → Regularization by nonzero width of the thermoparticle spectral function

- Finite-volume effects are exponentially suppressed for $E < 2m$

  → Subthreshold energy levels have exponentially suppressed finite-volume corrections

- Peak structure diminishes for sufficiently large temperatures

  → Other thermal effects become relevant

Connection to vacuum scattering amplitude

- Replace infinite-volume part of $F(E, \vec{P}; L, \beta)$:

$$\frac{1}{\beta} \sum_{k_0 \in \frac{2\pi i}{\beta}\mathbb{Z}} \int \frac{d^3k}{(2\pi)^3}\, \rho_{\mathsf{TP}}(q_0, \vec{k})\, \rho_{\mathsf{TP}}(r_0, \vec{P}-\vec{k}) \longrightarrow \int \frac{dk_0}{2\pi} \int \frac{d^3k}{(2\pi)^3}\, \rho_{\mathsf{vac}}(q_0, \vec{k})\, \rho_{\mathsf{vac}}(r_0, \vec{P}-\vec{k})$$

- Resulting quantization condition allows combined study of leading-order finite-volume and finite-temporal size effects on vacuum scattering amplitudes for a fixed lattice ensemble pair (L, L_τ)

- Sum part is identical to that in **QC2TP**

  $\rightarrow$ Leading order finite-volume corrections are finite for all E, have poles only for $L_\tau \rightarrow \infty$

Conclusion and Outlook

- Thermoparticles can be used to formulate finite- and infinite-volume scattering for finite Euclidean-time extent
- Thermoparticle asymptotic states depend on an interaction dependent damping factor $D_{m,\beta}(\vec{x})$
- Scattering with the medium induces non-vanishing phase space for $E < 2m$
- Finite temporal size L_τ **regularizes** the poles of the $L_\tau \to \infty$ leading-order finite-volume corrections
- Generalizations of the formalism to non-identical particles and particles with spin are underway

Backup

Kinematic variables

- Kinematics of two-thermoparticle scattering can be described by the incoming/outgoing four momenta p_1, q_1, the total medium energy E and total momentum $\vec{P}$
- Due to loss of boost invariance, mandelstam variables s, t and u are not useful variables to describe scattering kinematics at finite temperature
- Thermoparticles do not have sharp dispersion relations

  $\rightarrow$ p_1, q_1 are not functions of the total energy E

  $\rightarrow$ In general 8 variables $(p_1^0, p_2^0, |\vec{p}_1|, |\vec{q}_1|, E, \vec{P})$ are needed to describe the scattering process

NRT: a thermal no-go theorem

Narnhofer, Requardt, Thirring (1983): "For a KMS state with **real** dispersion relation the initial and final asymptotic fields coincide and therefore the S matrix is **trivial**"

Narnhofer, H., Requardt, M. Thirring, Commun.Math. Phys. 92, 247–268 (1983)

$\rightarrow$ On-shell vacuum states that satisfy the KMS condition always lead to a trivial S matrix

Spectral function

- Decomposition of $\tilde{D}_\beta(\vec{u}, s)$ into discrete and continous part:

$$\tilde{D}_\beta(\vec{u}, s) = \sum_i D_\beta(|\vec{u}|)\, \delta(s - m_i^2) + \tilde{D}_{\beta,\mathsf{cont}}(\vec{u}, s)$$

- Discrete part is thermal particle-like contribution $\rightarrow$ **Thermoparticle**
- **Damping factor** $\tilde{D}_\beta(|\vec{u}|)$ describes the interaction of the thermoparticle with the surrounding medium
- $\tilde{D}_\beta(|\vec{u}|)$ is an interaction dependent non-perturbative object

  $\rightarrow$ Varies for different particles/theories

Asymptotic propagators

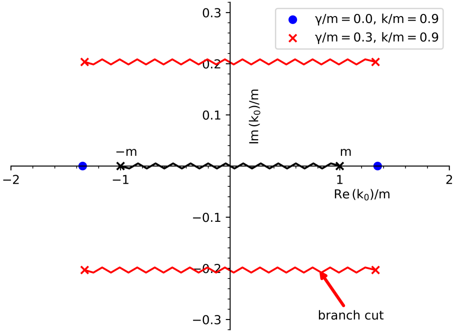
- Thermoparticle spectral function can be related to associated propagator as

$$G_\beta(k_0,\vec{k}) = -\int_{-\infty}^{\infty} dq_0\, \frac{\rho(k_0,\vec{k})}{k_0-q_0}$$

- Propagator for exponential position-space damping factor $D_\beta(x) = \alpha\, e^{-\gamma|\vec{x}|}$:

$$G_\beta(k_0,\vec{k}) = -\frac{\alpha}{k_0^2-|\vec{k}|^2-m^2-\gamma^2-2\gamma\sqrt{m^2-k_0^2}}$$

Asymptotic propagators for exponential damping



- Propagator has branch points at $k_0 = \pm m$
- Simple vacuum poles turn into **complex** singularities

  $\rightarrow$ particle goes off shell
- Complex singularities are branch points

  $\rightarrow$ A total of 3 branch cuts

Are thermoparticles resonances?

- Resonance propagators in vacuum can be expressed through self-energy $\Sigma(k)$:

$$G(k) = \frac{1}{k^2 - m^2 - \Sigma(k)}$$

- Nonzero imaginary part of $\Sigma(k)$ allows decay into lighter states
- Branch point at $k_0 = m$ in propagator **prevents** decay of thermoparticles to lighter states

  $\rightarrow$ Thermoparticles are not resonances

Extracting (α, γ) sets for one-pion correlator

- Fit spatial correlator with multi-exponential fit for large z

$$C(z) = A\, e^{-m_\pi^{\text{scr}} z} + B\, e^{-m_{\pi^*}^{\text{scr}} z}$$

$N_s^3 \times N_\tau$	a [fm]	$a\, m_{ud}$	T [MeV]	Fit range	A	m_π^{scr} [MeV]	B	$m_{\pi^*}^{\text{scr}}$ [MeV]	χ^2/d.o.f.
$32^3 \times 12$	0.075	0.001	220	$[2x_3^*, L/2]$	0.114(7)	776(17)	2.24(17)	3361(115)	0.581
$32^3 \times 8$	0.075	0.001	320	$[2x_3^*, L/2]$	0.347(7)	1781(5)	3.45(14)	4230(62)	0.686
$32^3 \times 8$	0.065	0.001	380	$[2x_3^*, L/2]$	0.325(9)	2153(9)	3.49(15)	4998(78)	0.444
$32^3 \times 8$	0.051	0.001	480	$[2x_3^*, L/2]$	0.343(11)	2860(11)	3.61(16)	6430(108)	0.314
$32^3 \times 4$	0.075	0.001	660	$[x_3^*, L/2]$	1.65(2)	3816(4)	5.96(6)	6012(39)	0.353
$32^3 \times 4$	0.051	0.001	960	$[x_3^*, L/2]$	1.52(4)	5559(10)	6.04(8)	8604(77)	1.243

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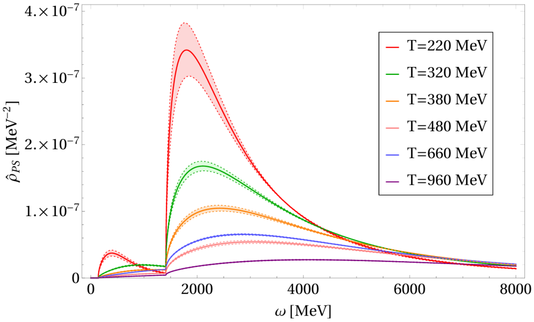
- m_{scr} and A can be related to ground state parameters (α_π, γ_π)

Evidence in Lattice QCD data?

Thermoparticle dominated pion spectral function

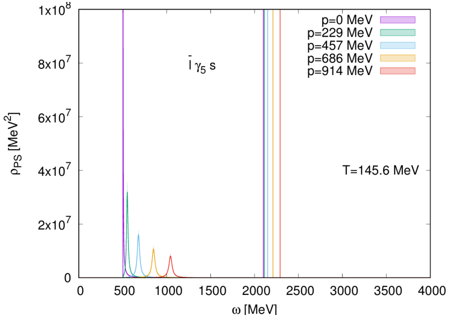


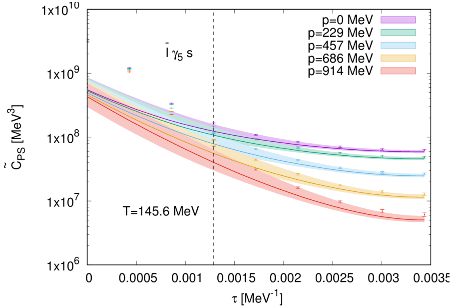
P.Lowdon, O. Philipsen, JHEP 10, 161 (2022)

- Damping factor $D_\beta(x)$ can be extracted from spatial correlators $C(z)$
- Spectral function is broadened and asymmetric for nonzero temperatures (not a Breit-Wigner!)

Evidence in lattice QCD data?

- Non-trivial thermoparticle hypothesis test with lattice QCD Kaon temporal correlators $C(\tau, \vec{p})$





D. Bala, O. Kaczmarek, P. Lowdon, O. Philipsen, and T. Ueding, JHEP 05, 332 (2024)

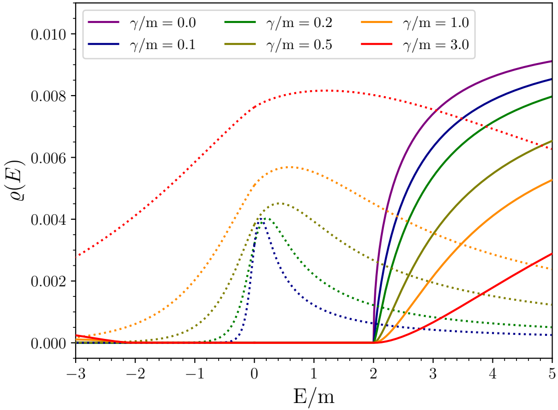
$\rightarrow$ Thermoparticle hypothesis for temporal correlators is **consistent** with lattice data

Thermal phase space for general $\vec{P}$

- Two-thermoparticle phase space defined as

$$\varrho_{\ell m;\ell' m',\mathsf{TP}}(E,\vec{P}) = \frac{\pi^2}{1-e^{-\beta E}} \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi}\, \rho_{\mathsf{TP}}(\omega_1,\vec{k})\rho_{\mathsf{TP}}(\omega_2,\vec{P}-\vec{k})$$
$$\times\, \delta(E-\omega_1-\omega_2) \left(\coth\left(\tfrac{\beta\omega_1}{2}\right)+\coth\left(\tfrac{\beta\omega_2}{2}\right)\right) Y_{\ell m}(\hat{k}) Y^*_{\ell' m'}(\hat{k}),$$

Threshold and purely thermal contributions to phase space



- Contributions at energies $E < 2m$ generated by thermal interactions of the two-particle system with the medium
- Purely thermal contribution compensates threshold at $E = 2m$ in the full phase space

Zero Temperature quantization condition

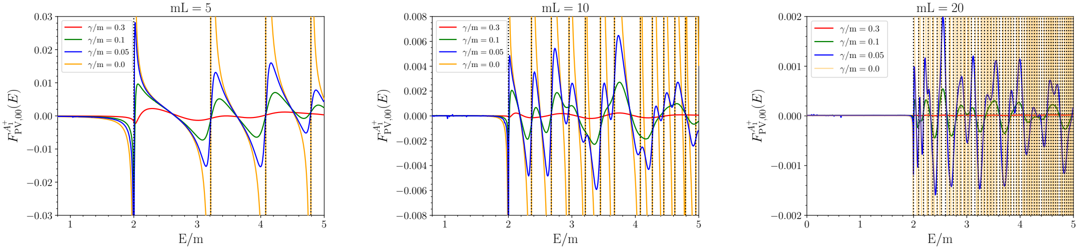
Two-particle quantization condition (QC2)

$$\det\left[\mathcal{M}^{-1}(E,\vec{P}) + F(E,\vec{P})\right] = 0,$$

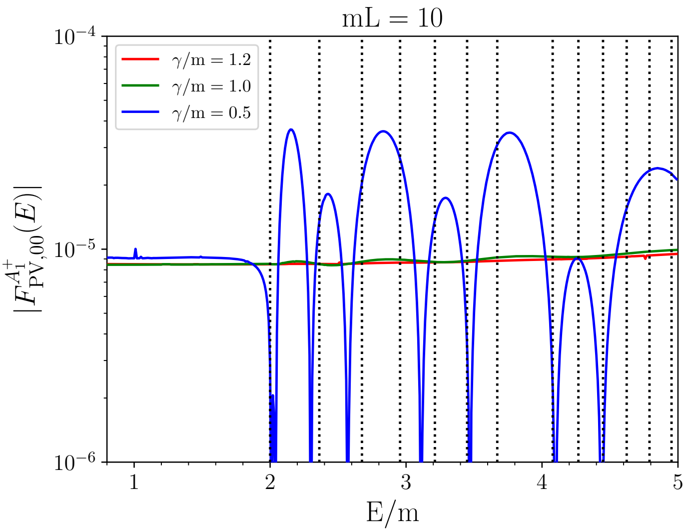
$$F_{\ell' m';\ell m}(E,\vec{P}) = \frac{1}{2}\left[\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right] \frac{\mathcal{Y}_{\ell' m'}(\vec{k})\,\mathcal{Y}^*_{\ell m}(\vec{k})}{2\omega_k\, 2\omega_{P-k}\,(E-\omega_k-\omega_{P-k}+i\epsilon)},$$

$$\mathcal{Y}_{\ell m}(\vec{k}) = \sqrt{4\pi}\left(\frac{k}{q}\right)^{\ell} Y_{\ell m}(\hat{k}), \quad \omega_k = \sqrt{k^2+m^2}, \quad q = \sqrt{E^2/4 - m^2}$$

Finite-volume effects for multiple volumes

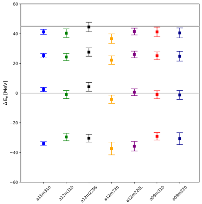


Finite-volume effects for high temperatures

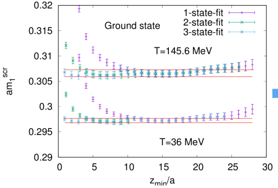


Possible application: $\rho \to (\pi\pi)_{\ell=1}$ at finite temperature

$\pi\pi(T_1^-)$ Levels



single π correlator



QC2TP/ Fit

$$\det\left[\mathcal{K}^{-1}_{\text{TP}}(E,\vec{P}) + F(E,\vec{P};L,\beta)\right] = 0$$

$D_\beta(\vec{u})$

Unitarity

$\mathcal{M}_{\text{TP}}(E,\vec{P})$

(no) pole

(no) resonance