Spatial Densities without Form Factors

- Problems of obtaining hadron spatial densities with Fourier transforms from form factors
- Define spatial densities with the help of Ward identity
- Applications

T. Draper, C. Zimmermann, KFL

Lattice 2026, U. Maryland, July 28, 2026

Wave Packet and Fourier Transform

- The conventional approach to defining a density is the FT of the form factor in the Breit frame

$$\rho(r) = \frac{1}{(2\pi)^3}\int d^3q\, e^{-i\vec{q}\cdot\vec{r}} F(q^2)|_{q^0=0}$$

  - Form factor is from states with different momenta (*N.B.* Breit frame).
  - Not clear what r is refereced to.
- Use wavepacket to localize the hadron → relativistic corrections

M. Burkardt, PRD 62 (2000) 071503

$$F_\Psi(\vec{q}) = 1 - \frac{R^2}{6}\vec{q}^2 - \frac{R^2}{6}\int d^3p\,|\Psi(\vec{p})|^2 \frac{(\vec{q}\cdot\vec{p})^2}{E_p^2} + \int d^3p\,|\vec{q}\cdot\vec{\nabla}\Psi(\vec{p})|^2 + \ldots$$

R^2 is slope of $F(q^2)$

  - Relativistic corrections include Lorentz contraction and size of wave packet..
  - Non-relativistic case is fine.
  - Solution is IMF with $P_z \gg M, |\vec{q}|$
  - transverse distribution from DVCS at $\zeta = 0$

Density and Fourier Transforms

- In non-relativistic bound state of two scalar particles

$$\Psi(\vec{r}_1, \vec{r}_2) = e^{i\vec{P}\cdot\vec{R}}\phi(\vec{r}).$$

$$F_{NR}(|\vec{q}|) = \int d^3r|\phi(\vec{r})|^2 e^{i\vec{q}\cdot\vec{r}/2}$$

- In relativistic system, the Bethe-Salpeter wavefuntion depends on the total momentum P.
- The EM form factor differs markedly from the NR case.
- The form factor from the overlap of different incoming and outgoing wavefunction defies the notion of density.
(G. Miller – arXiv:1002.0355, 2507.14388)

Wave Packet and Fourier Transform

- The criteria for a good approximation of the density from the FT of a form factor (R.L. Jaffe – arXiv:2010.15887)

$$\frac{1}{M} \ll \Delta \ll R$$

- The wave packet size should be << the hadron size to be localized, but >> the Compton wavelength to avoid large relativistic corrections.

- Atoms: $R \sim a_0 = 5 \times 10^{-11}$ m, $1/M \sim 2.1 \times 10^{-16}$ m
- Nuclei: Carbon (A = 12) R ~ 3 fm, 1/M ~ 0.018 fm
- Nucleon: R ~ 0.84 fm, 1/M ~ 0.2 fm: RM ~ 4, too small
- Pion: R ~ 0.6 fm, 1/M ~ 1.44 fm, RM ~ 0.4

Challenge of defining density on lattice

- Consider a forward matrix element of a local operator O in a hadron. Due to translational invariance

$$\langle P|O(x)|P\rangle = e^{i(\vec{p}-\vec{p})\cdot\vec{x}}\langle P|O(0)|P\rangle$$

- There is no information on the location of the operator O. One needs the location of the hadron, e.g. the center of mass to define densities.

Challenge of defining density on lattice

- Here comes the Ward identity

$$[H, Q] = 0, \qquad Q\psi = \psi$$

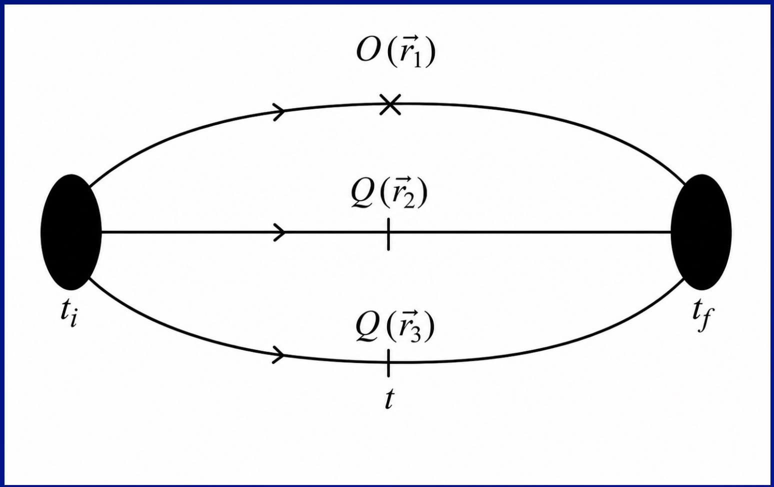
$$\sum_{\vec{z}} \langle \psi(x) J_0(\vec{z}, t) \bar{\psi}(y) \rangle = \langle \psi(x) \bar{\psi}(y) \rangle$$

For $$t_x < t < t_y$$

- Ward identity for lattice point-split vector current with Wilson fermion --- F. Guerin, Nucl. Phys B282, 495 (1987)

Challenge of defining density on lattice

- For a matrix element $\langle P|O(x)|P\rangle$, insert total charge Q on the quark propagators other than the propagator with insertion O on the same time slice



- Define spatial density

$$\langle P|O(x)|P\rangle = \lim_{t_f \gg t, t \gg t_i} \frac{C_3(t_i, t, t_f)}{C_2(t_i, t_f)},$$
$$= \sum_{\vec{r}_1, \vec{r}_2, \vec{r}_3} f(\vec{r}_1, \vec{r}_2, \vec{r}_3) = \sum_{\vec{r}} \rho(\vec{r})$$

where,

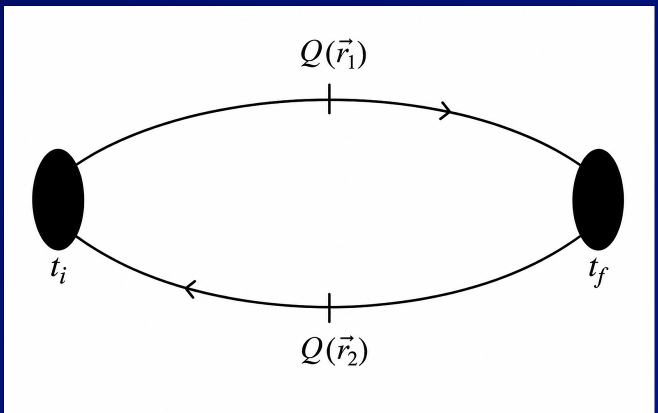
$$\rho(\vec{r}) = \sum_{\vec{r}_1, \vec{r}_2, \vec{r}_3} f(\vec{r}_1, \vec{r}_2, \vec{r}_3)\, \delta(\vec{r} - (\vec{r}_1 - \vec{r}_{c.m}))$$

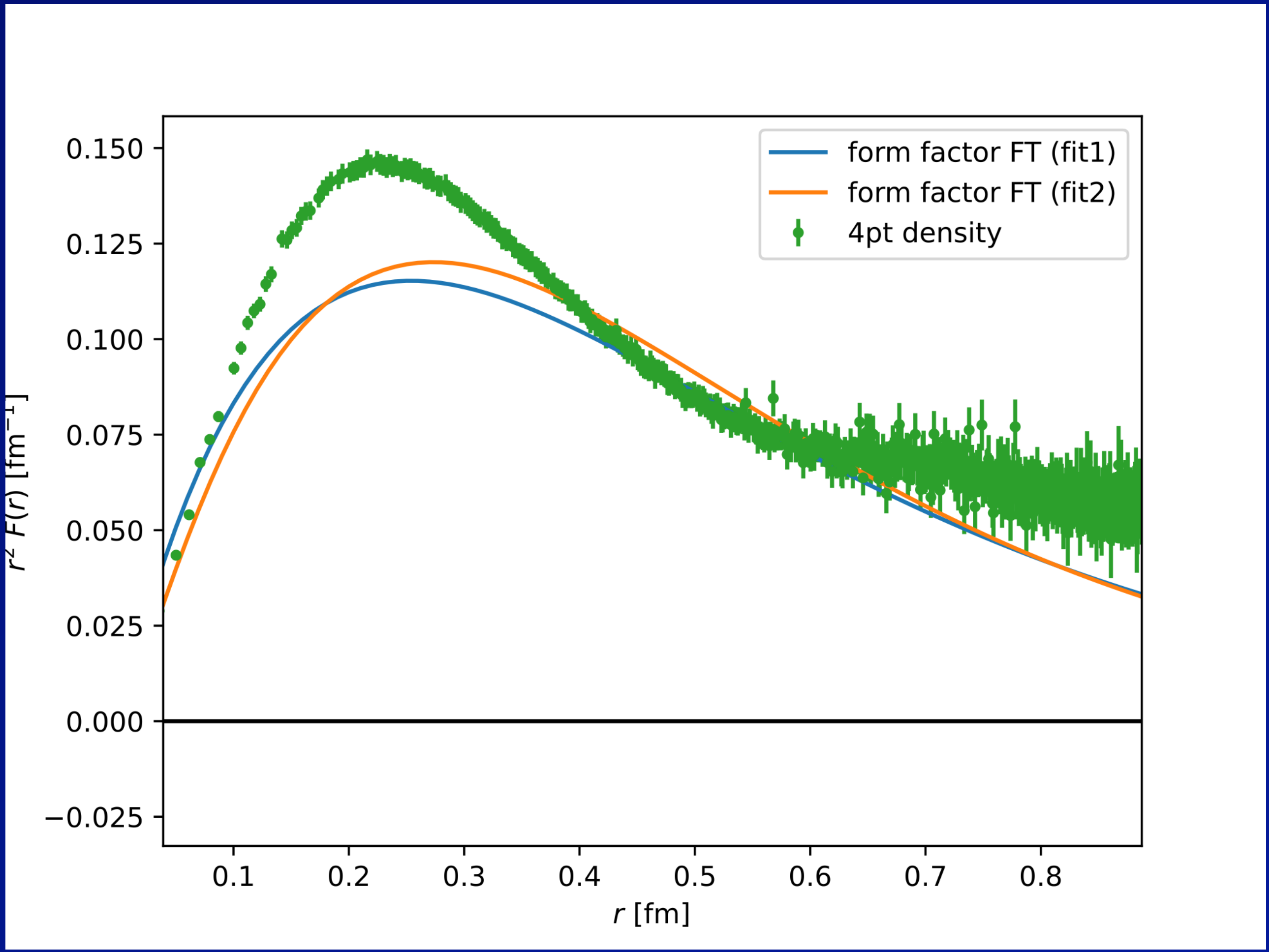
Previous attempts

$$\langle \pi | J_0(r) J_0(0) | \pi \rangle$$

- W. Wilcox and K.F. Liu, *Phys.Lett.B* 172 (1986) 62
  - Identifying the F.T. of the charge density as the form factor
- M.C. Chu, et al., *Nucl.Phys.B* 360 (1991) 31-66
- M. Burkardt et al., Annals Phys. 238 (1995) 441
  - Identify with $F\pi^2$
- C. Alexandrou et al., Phys. Rev. D68 (2003) 07450
- G. Bali et al., *JHEP* 12 (2018) 061
  - Identify with $F\pi^2$

Pion charge density





$N_f = 2$, $m_\pi \sim 300$ MeV
$a = 0.071$ fm,
$32^3 \times 64$, $L = 2.27$ fm
$40^3 \times 64$, $L = 2.84$ fm

G. Bali et al., *JHEP* 12 (2018) 061

Applications

- Charge and magnetization distribution, axial charge density, momentum and angular momentum density, pressure density, trace anomaly and quark condensate densities, topological charge distribution, …
- Longitudinal distribution as a function of hadron momentum

$$\langle P|O(0)|P\rangle = \sum_z \rho_{\|}(z,P)$$

Pion from rest frame to 1 GeV (v/c = 99 %)

- Orbital angular momentum

$$\mathrm{OAM} \equiv (\vec{r}-\vec{r}_{\mathrm{c.m}}) \times (E \times B)(\vec{r}-\vec{r}_{\mathrm{c.m}})$$

Conclusion

On the Impossibility of Obtaining Time-Independent, Three-Dimensional, Spherically-Symmetric Densities of Confined Systems of Relativistically Moving Constituents

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- It refers to F. T. of form factors

- Taking advantage of the Ward Identigy, local densities can be defined directly and calculated on the lattice without form factors.