Tightening the energy-based boson truncation bound in quantum simulation

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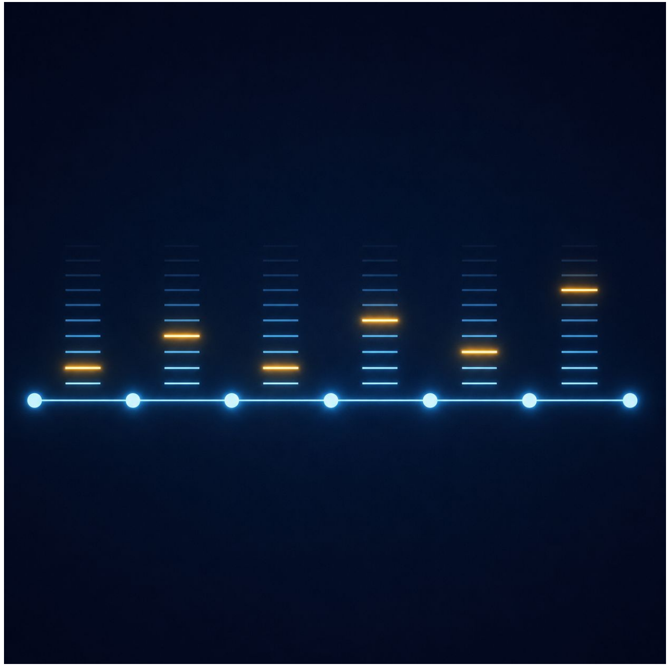
Overview

[arxiv:2604.24896] PRD

- Boson truncation error
- Jordan-Lee-Preskill (JLP) energy-based bound
- Our improved bound
  - Monte Carlo trick
  - *p*-norm trick
- Results



Lattice field theory: two approaches

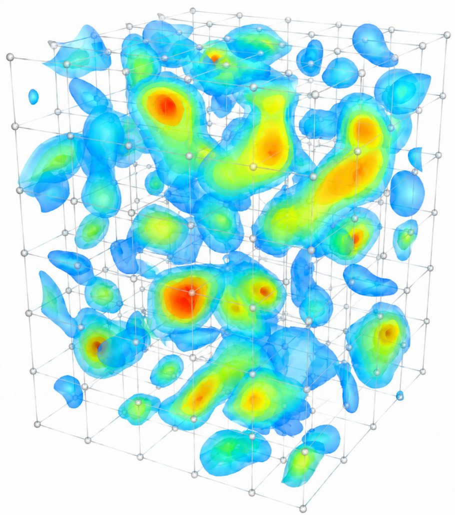
Lagrangian formulation

$$\langle O \rangle = \frac{\int d\phi\, O\, e^{-S[\phi]}}{\int d\phi\, e^{-S[\phi]}}$$

Classical Monte Carlo

Static property of
- ground state
- one-particle state
- Gibbs state

Low-energy scattering
(Lüscher formalism)



Hamiltonian formulation

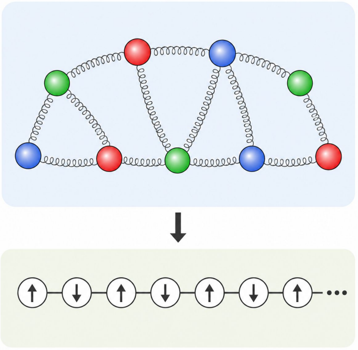
$$|\psi(0)\rangle \xrightarrow{e^{-iHt}} |\psi(t)\rangle$$

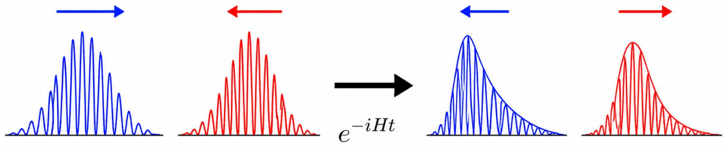
Quantum simulation

Real-time dynamics
- wave-packet scattering
- string-breaking

Finite-density physics

Heavy nuclei

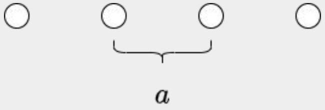




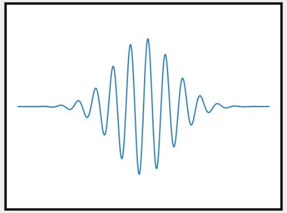
Uncertainties in quantum simulation

Physical systematic uncertainties

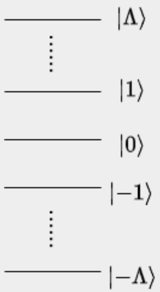
Discretization effects



Finite volume effects



Boson truncation



Obtaining physical results requires taking the limits:

Lattice spacing
$$a \to 0$$

Box size
$$L \to \infty$$

Truncation cutoff
$$\Lambda \to \infty$$

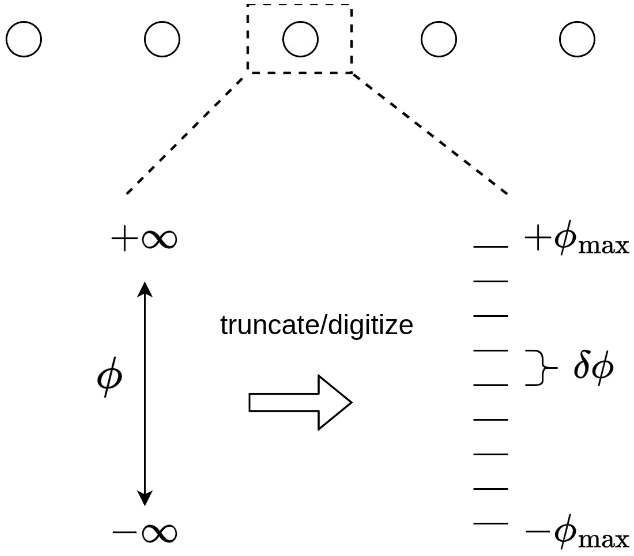
At finite values, we need to understand the approximation errors.

Boson truncation

Scalar field theory

$$[\hat{\phi}_i, \hat{\pi}_j] = i\delta_{ij}$$

$$\hat{H} = \frac{1}{a}\sum_{\mathbf{x}}\left(\frac{1}{2}\hat{\pi}(\mathbf{x})^2 + \frac{1}{2}(\nabla\hat{\phi})^2(\mathbf{x}) + \frac{1}{2}m_0^2\hat{\phi}(\mathbf{x})^2 + \frac{\lambda_0}{4!}\hat{\phi}(\mathbf{x})^4\right)$$



We need to choose $\phi_{\max}$ and $\delta\phi$ appropriately.

Or, equivalently $\phi_{\max}$ and $\pi_{\max}$

Jordan, Lee, and Preskill [arxiv:1111.3633](https://arxiv.org/abs/1111.3633) proposed an energy-based truncation error bound.

Truncation error

Define truncation projector

$$P_{\phi_{\max}} = \int_{-\phi_{\max}}^{\phi_{\max}} d\phi_1 \cdots d\phi_{\mathcal{V}} \, |\phi_1, \ldots, \phi_{\mathcal{V}}\rangle \langle \phi_1, \ldots, \phi_{\mathcal{V}}|$$

Given state $|\psi\rangle$, one can obtain a truncated state $P_{\phi_{\max}} |\psi\rangle$.

$$|\psi\rangle \overset{?}{\approx} P_{\phi_{\max}} |\psi\rangle$$

One can quantify the truncation error by

$$\epsilon = 1 - \langle\psi|P_{\phi_{\max}}|\psi\rangle$$

$$\Rightarrow \quad \epsilon = \Pr\left[\max_i |\phi_i| > \phi_{\max}\right] \qquad \epsilon \lesssim ?$$

JLP energy-based bound

Recall $\epsilon = \Pr\left[\max_i |\phi_i| > \phi_{\max}\right]$

For single site $\mathbf{x}_i$, by Chebyshev inequality

$$\Pr\left[|\phi_i| > \phi_{\max}\right] \leq \frac{\langle\psi|\,\hat{\phi}^2(\mathbf{x}_i)\,|\psi\rangle}{\left(\phi_{\max}-\sqrt{\langle\psi|\hat{\phi}^2(\mathbf{x}_i)|\psi\rangle}\right)^2} \sim \frac{\langle\psi|\,\hat{\phi}^2(\mathbf{x}_i)\,|\psi\rangle}{\left(\phi_{\max}\right)^2}$$

Denote the total number of sites by $\mathcal{V}$. By union bound, we have

$$\epsilon \lesssim \frac{\mathcal{V}\langle\psi|\,\hat{\phi}^2(\mathbf{x})\,|\psi\rangle}{\left(\phi_{\max}\right)^2},$$

provided that we can bound $\langle\psi|\,\hat{\phi}^2(\mathbf{x})\,|\psi\rangle$ uniformly.

JLP energy-based bound

Assuming $\langle\psi|\,\hat{H}\,|\psi\rangle \leq E$, we have

$$
\begin{aligned}
E &\geq \langle\psi|\,\hat{H}\,|\psi\rangle \\
&= \langle\psi|\,\frac{1}{a}\sum_{\mathbf{x}}\left(\frac{1}{2}\hat{\pi}(\mathbf{x})^2 + \frac{1}{2}(\nabla\hat{\phi})^2(\mathbf{x}) + \frac{1}{2}m_0^2\hat{\phi}(\mathbf{x})^2 + \frac{\lambda_0}{4!}\hat{\phi}(\mathbf{x})^4\right)|\psi\rangle \\
&\geq \frac{1}{a}\langle\psi|\frac{1}{2}m_0^2\hat{\phi}(\mathbf{x}_i)^2\,|\psi\rangle,
\end{aligned}
$$

Hence, $\langle\psi|\,\hat{\phi}^2(\mathbf{x})\,|\psi\rangle$ can be upper-bounded.

To conclude, $\epsilon \lesssim \dfrac{\mathcal{V}2aE}{m_0^2\phi_{\max}^2}$. Or, given error budget ϵ, we can choose $\phi_{\max}$ as

$$
\phi_{\max} \sim \sqrt{\frac{\mathcal{V}2aE}{m_0^2\epsilon}}
$$

Volume scaling of JLP bound

One obtains the required truncation cutoff

$$\phi_{\max} \sim \sqrt{\frac{\mathcal{V} 2 a E}{m_0^2 \epsilon}}$$

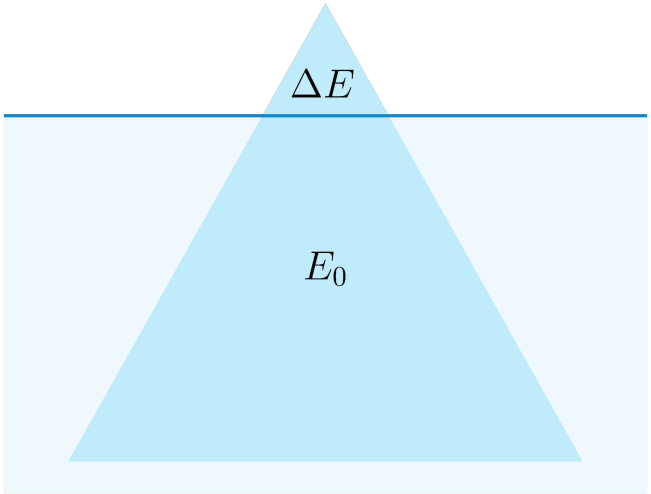
Here, $\sqrt{\mathcal{V}}$ contributes to an explicit factor of $\sqrt{\mathcal{V}}$.

Meanwhile, $\sqrt{E}$ also implicitly contributes.

$$E = E_0 + \Delta E$$

ground-state energy a.k.a. vacuum energy $\leftarrow E_0$; $\Delta E \rightarrow$ physical energy

Typical scenario in scattering



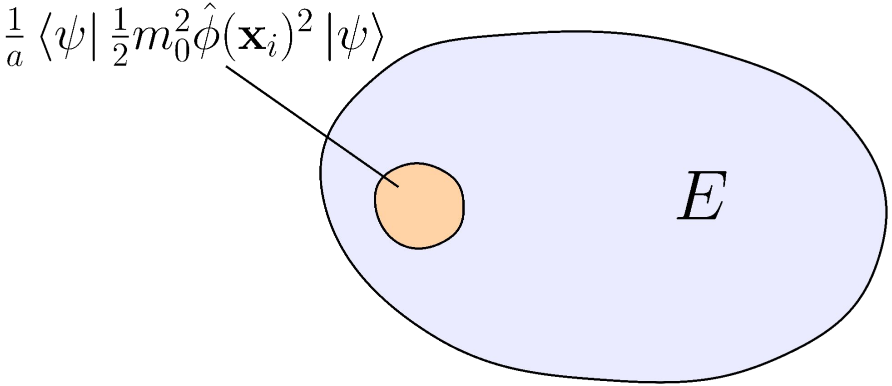
We have $E_0 \propto \mathcal{V}$

Hence, we have $\phi_{\max} \propto \mathcal{V}$ here. Can we do better?

Conservative energy bound

Many terms are thrown away in the energy-based bound.

$$
\begin{aligned}
E &\geq \langle\psi|\, \hat{H}\, |\psi\rangle \\
&= \langle\psi|\, \frac{1}{a} \sum_{\mathbf{x}} \left(\frac{1}{2}\hat{\pi}(\mathbf{x})^2 + \frac{1}{2}(\nabla\hat{\phi})^2(\mathbf{x}) + \frac{1}{2}m_0^2\hat{\phi}(\mathbf{x})^2 + \frac{\lambda_0}{4!}\hat{\phi}(\mathbf{x})^4 \right) |\psi\rangle \\
&\geq \frac{1}{a}\, \langle\psi|\, \frac{1}{2}m_0^2\hat{\phi}(\mathbf{x}_i)^2\, |\psi\rangle
\end{aligned}
$$



This results in a conservative bound.

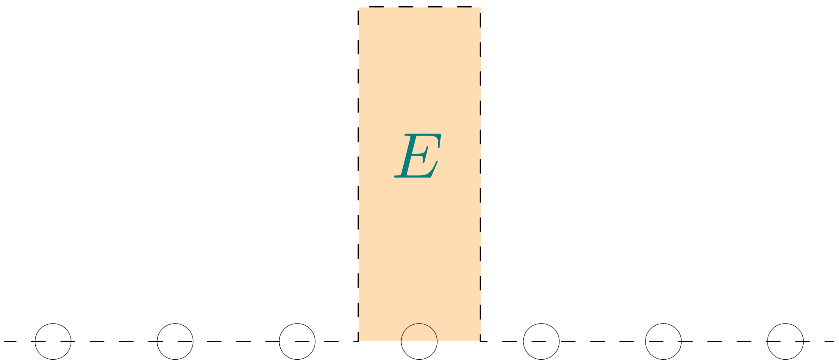
In fact, $\frac{1}{a}\, \langle\psi|\, \frac{1}{2}m_0^2\hat{\phi}(\mathbf{x}_i)^2\, |\psi\rangle \ll E$

Monte Carlo trick

Original approach uses E in the bound.

Note, $E_0 \propto \mathcal{V}$

$$\langle\psi|\, \frac{1}{2a} m_0^2 \hat{\phi}(\mathbf{x}_i)^2 \,|\psi\rangle \le E = E_0 + \Delta E$$

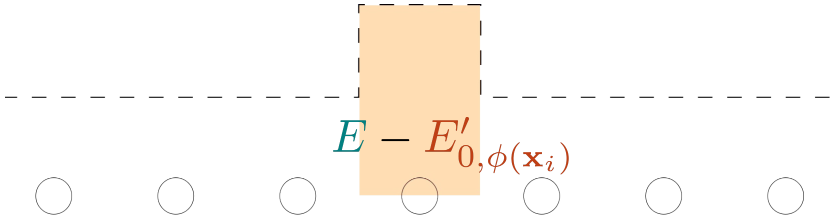


Original approach

By contrast, we use $E_0 + \Delta E - E'_{0,\phi(\mathbf{x}_i)}$ which is much smaller.

Here, E_0 and $E'_{0,\phi(\mathbf{x}_i)}$ can be evaluated classically.

$$\langle\psi|\, \frac{1}{2a} m_0^2 \hat{\phi}(\mathbf{x}_i)^2 \,|\psi\rangle \le E_0 + \Delta E - E'_{0,\phi(\mathbf{x}_i)}$$



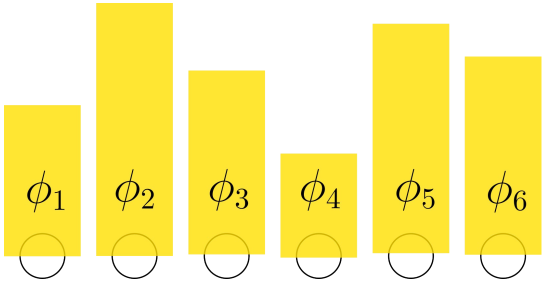
Improved approach

p-norm trick

Original approach

Bound error contribution at each site $\sim \langle\psi|\,\hat{\phi}(\mathbf{x})^2\,|\psi\rangle$

Bound total error of all sites $\sim \mathcal{V}\,\langle\psi|\,\hat{\phi}(\mathbf{x})^2\,|\psi\rangle$



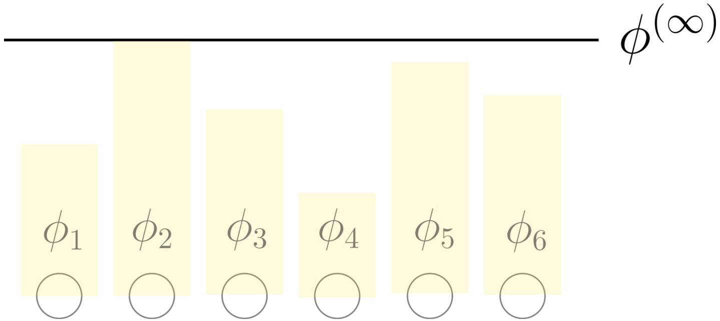
Improved approach

Define $\hat{\phi}^{(p)} = \left(\sum_{i=1}^{\mathcal{V}} |\hat{\phi}(\mathbf{x}_i)|^p\right)^{1/p}$

Directly bound total error of all sites $\sim \langle\psi|\,(\hat{\phi}^{(p)})^2\,|\psi\rangle$

Here, a good choice is $p = \infty$.

For context, $\phi^{(\infty)} = \max_i |\phi_i|$



Improved boson truncation bound

To estimate the truncation cutoff $\phi_{\max}$, we set $p = \infty$ and combine both Monte Carlo trick and *p*-norm trick.

$$\epsilon \lesssim \frac{\langle\psi|(\hat{\phi}^{(\infty)})^2|\psi\rangle}{(\phi_{\max})^2}$$

We have

$$\langle\psi|\,(\hat{\phi}^{(\infty)})^2\,|\psi\rangle \leq \left\langle\Omega_{\phi^{(\infty)},\eta}\right|\,(\hat{\phi}^{(\infty)})^2\,\left|\Omega_{\phi^{(\infty)},\eta}\right\rangle + \frac{2}{\eta m_0^2}a\Delta E$$

To be evaluated in a path integral

$$\frac{\int \mathcal{D}\phi\,\left(\phi^{(\infty)}\right)^2 e^{-S'_{\phi^{(\infty)},\eta}[\phi]}}{\int \mathcal{D}\phi\, e^{-S'_{\phi^{(\infty)},\eta}[\phi]}}$$

Physical energy of particles, e.g. 5 GeV

Improved boson truncation bound

To estimate the truncation cutoff $\phi_{\max}$, we set $p = \infty$ and combine both Monte Carlo trick and *p*-norm trick.

Original JLP bound

$$\phi_{\max} = \left(\sqrt{\frac{\mathcal{V}}{\epsilon}} + 1\right)\sqrt{\langle\psi|\,\hat{\phi}^2(\mathbf{x})\,|\psi\rangle}$$

$$\langle\psi|\,\hat{\phi}^2(\mathbf{x})\,|\psi\rangle \le \frac{2}{m_0^2} a E_0 + \frac{2}{m_0^2} a \Delta E$$

$$\phi_{\max} \propto \mathcal{V}$$

Our improved bound

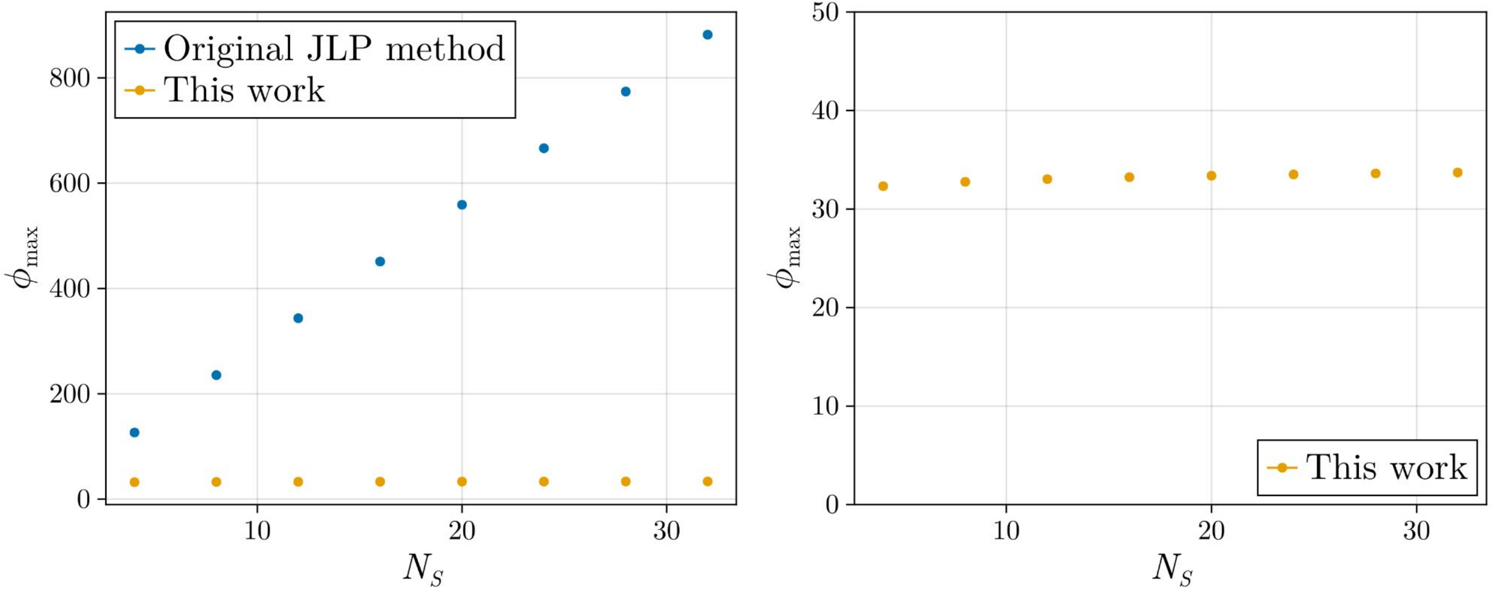
$$\phi_{\max} = \left(\frac{1}{\sqrt{\epsilon}} + 1\right)\sqrt{\langle\psi|\,(\hat{\phi}^{(\infty)})^2\,|\psi\rangle}$$

$$\langle\psi|\,(\hat{\phi}^{(\infty)})^2\,|\psi\rangle \le \frac{\int \mathcal{D}\phi\,\left(\phi^{(\infty)}\right)^2 e^{-S'_{\phi^{(\infty)}}[\phi]}}{\int \mathcal{D}\phi\, e^{-S'_{\phi^{(\infty)}}[\phi]}} + \frac{2}{m_0^2} a \Delta E$$

Mild $\mathcal{V}$ - dependence

Original bound vs improved bound

1+1D ϕ^4 theory $\quad \epsilon = 0.01 \quad a\Delta E = 1.0$



(a) (b)

Observed saving factor is nearly proportional to the volume factor $\mathcal{V}$

Original bound vs improved bound

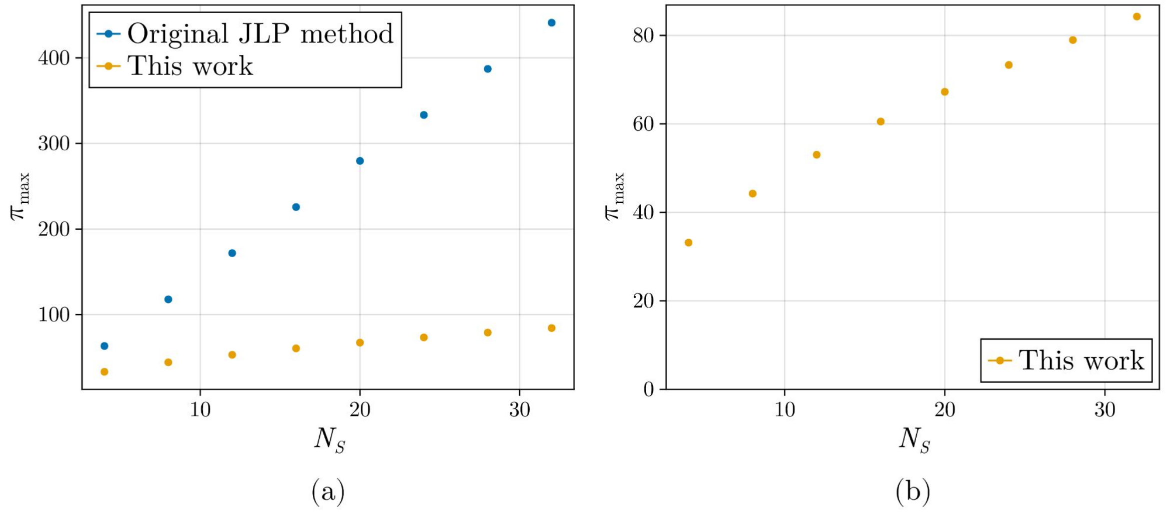
Bound for conjugate momentum field π

Original JLP bound

$$\pi_{\max} \lesssim \sqrt{\frac{\mathcal{V}\cdot 2aE}{\epsilon}}$$

Our improved bound

$$\pi_{\max} \lesssim \sqrt{\frac{2aE}{\epsilon}}$$

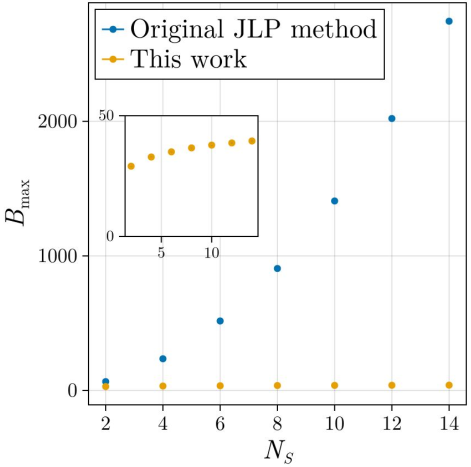


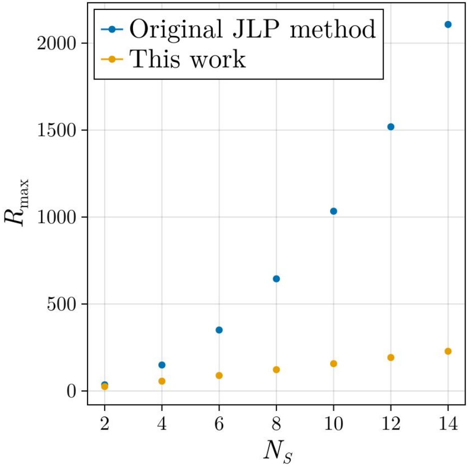
Saving factor: $\sqrt{\mathcal{V}}$

Extension to other theories

We also demonstrate the improved bound on 2+1D non-compact U(1) gauge theory in the dual formalism:

$$\hat{H}_{\mathrm{U}(1)} = \frac{1}{a}\frac{g^2}{2}\sum_{p,q}\hat{R}_p M_{p,q}\hat{R}_q + \frac{1}{a}\frac{1}{2g^2}\sum_p \left(\hat{B}_p\right)^2$$





	Variable	Saving factor
1+1D ϕ^4	ϕ	Nearly $\propto \mathcal{V}$
	π	$\propto \sqrt{\mathcal{V}}$
2+1D U(1)	B	Nearly $\propto \mathcal{V}$
	R	$\propto \sqrt{\mathcal{V}}$

Conclusion

JLP energy bound can be used to bound boson truncation error.

By combining **Monte Carlo** trick and ***p*-norm** trick, we tighten the bound.

[arxiv:2604.24896] PRD

By reducing truncation cutoffs, we achieve

Truncation cutoff $\downarrow$

smaller Hilbert space → fewer qubits $n_q \sim \log(\phi_{\max}\pi_{\max})$

smaller Hamiltonian norm/commutators → lower simulation cost bounds $\text{cost} \sim \text{poly}(\phi_{\max}, \pi_{\max})$

Thanks for listening!

Comments and suggestions appreciated.

Backup slides

Different approaches

- Numerical tests and extrapolation
  - Using Classical Monte Carlo
    - e.g. arxiv:1811.03629, arxiv:1906.11213, arxiv:2212.08546
  - Others
    - e.g. arxiv:2312.12506, arxiv:2507.22984
- Energy based bound (arxiv:1111.3633)
- Quantum speed limit (arxiv:2110.06942)
- Hilbert space fragmentation (arxiv:2508.00061)
- Nyquist-Shannon sampling theorem (arxiv:1802.07347)

Works using JLP energy-based bound

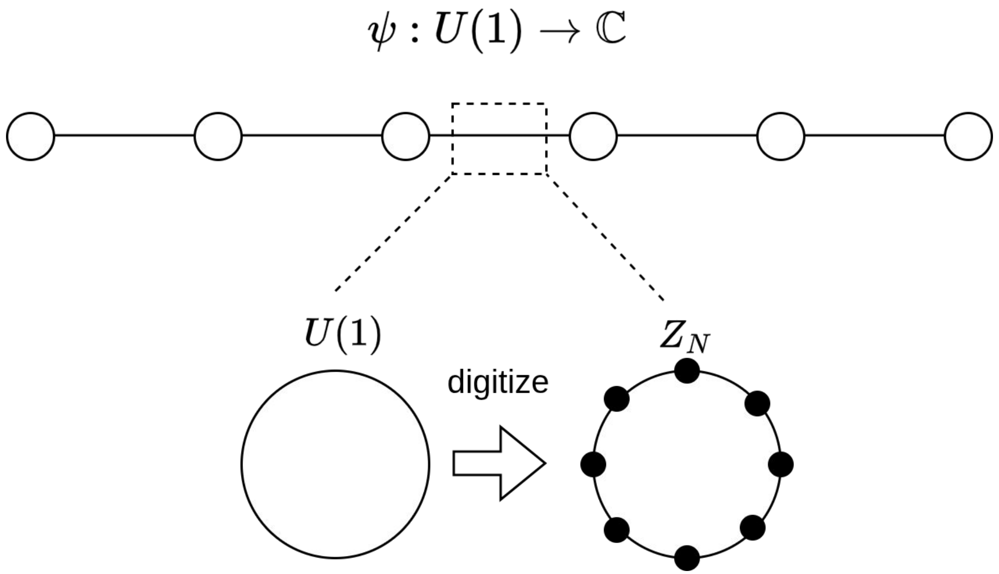
- On Quantum Simulation Of Cosmic Inflation [arxiv:2009.10921]
- Quantum simulation of gauge theory via orbifold lattice [arxiv:2011.06576]
- Quantum Algorithms for Simulating Nuclear Effective Field Theories [arxiv:2312.05344]
- Quantum Simulation of QED in Coulomb Gauge [arxiv:2507.01089]

Truncation of Kogut-Susskind Hamiltonian

For U(1) gauge theory, we have

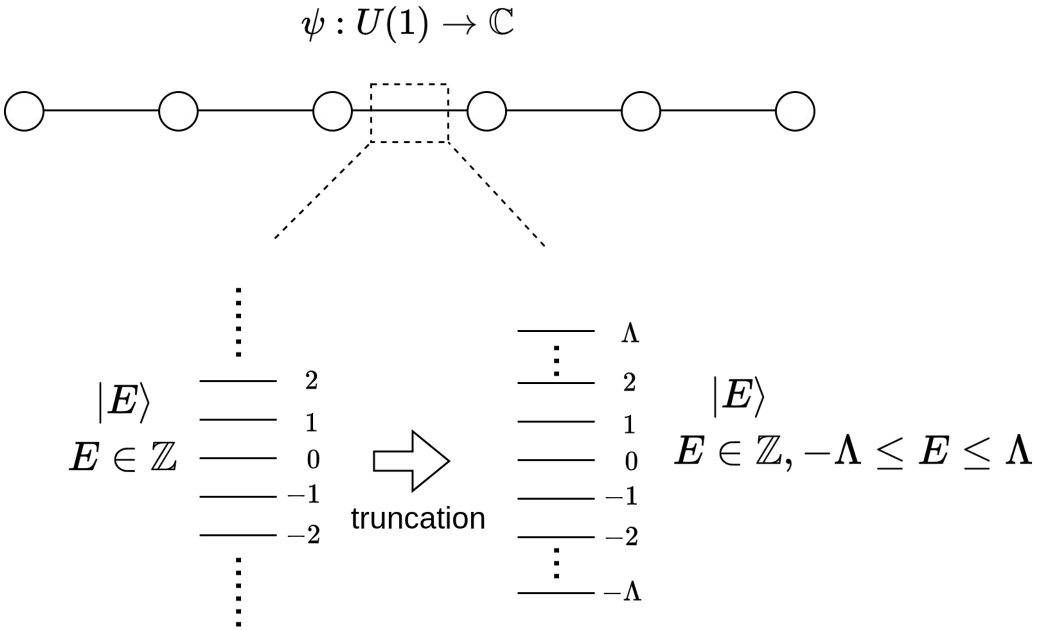
(e.g. group element basis)

We need to digitize U(1) into discrete points.



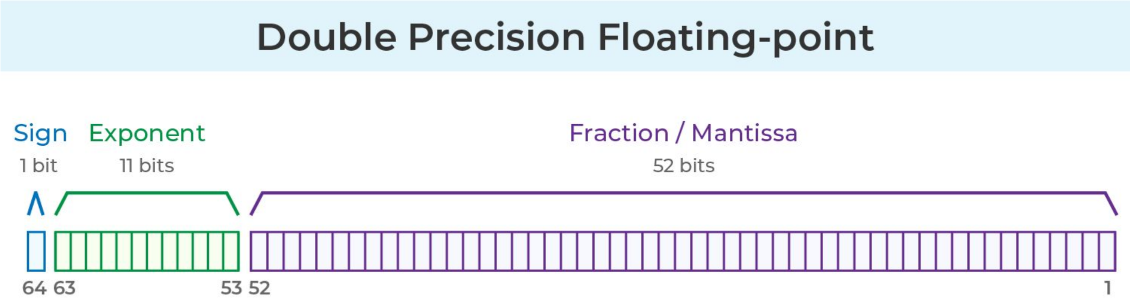
Truncation of Kogut-Susskind Hamiltonian

In electric basis, we similarly require a truncation



Why it's not a problem for traditional Lattice

In Euclidean path integral, the numbers are represented by float64 or float32, which provides great precision and range.



credit: Brown

The situation does not carry directly to quantum simulation, as the cost/error bounds of many quantum algorithms depend polynomially on the truncation cutoffs.

Truncation bound and trotterization

For trotterization (Childs et al.), the trotter error depends on the commutators.

For scalar field theory, if we write the Hamiltonian as

$$\hat{H} = \hat{H}_\phi + \hat{H}_\pi$$

then the trotter error at first-order trotterization can be expressed as:

$$\epsilon_{\text{sim}} \leq \frac{\tilde{\alpha}_{\text{comm}} t^2}{N_{\text{steps}}}$$

with

$$\widetilde{\alpha}_{\text{comm}} = \mathcal{O}(\|[\hat{H}_\phi, \hat{H}_\pi]\|) = \mathcal{O}\left(\mathcal{V}\left\|[\hat{\pi}^2, \hat{\phi}^4]\right\|\right) = \mathcal{O}(\mathcal{V}\pi_{\max}\phi_{\max}^3)$$

and N_{steps} being the number of trotter steps.

Truncation bound and quantum signal processing

Quantum signal processing:

Embedding Hamiltonian $\hat{H}$ into a larger unitary matrix $U = \begin{pmatrix} H & \cdots \\ \vdots & \ddots \end{pmatrix}$

To do this, we need to rescale it by β with the condition:

$$\left\| \hat{H}/\beta \right\| \leq 1$$

Gate cost of this approach: $N_U = \mathcal{O}\left(\beta t + \log\left(\frac{1}{\epsilon_{\text{sim}}}\right)\right)$

In our case, we have $\beta = \mathcal{O}\left(\mathcal{V}\left[\pi^2_{\max} + \phi^4_{\max}\right]\right)$

JLP energy-based bound

For a given state $|\psi\rangle$, we can expand it as

$$|\psi\rangle = \int_{-\infty}^{\infty} d\phi_1 ... \int_{-\infty}^{\infty} d\phi_{\mathcal{V}}\, \psi(\phi_1, ..., \phi_{\mathcal{V}})\, |\phi_1, ..., \phi_{\mathcal{V}}\rangle$$

After truncation, we get a truncated state

$$|\psi_{\phi_{\max}}\rangle = \int_{-\phi_{\max}}^{\phi_{\max}} d\phi_1 ... \int_{-\phi_{\max}}^{\phi_{\max}} d\phi_{\mathcal{V}}\, \psi(\phi_1, ..., \phi_{\mathcal{V}})\, |\phi_1, ..., \phi_{\mathcal{V}}\rangle$$

$$\epsilon = 1 - |\langle\psi|\psi_{\phi_{\max}}\rangle|$$

$$\epsilon \lesssim ?$$

JLP energy-based bound

Recall $\epsilon = 1 - |\langle\psi|\psi_{\phi_{\max}}\rangle|$

$$|\psi\rangle = \int_{-\infty}^{\infty} d\phi_1 ... \int_{-\infty}^{\infty} d\phi_{\mathcal{V}} \, \psi(\phi_1, ..., \phi_{\mathcal{V}}) \, |\phi_1, ..., \phi_{\mathcal{V}}\rangle$$

We then have

$$\epsilon = 1 - \int_{-\phi_{\max}}^{\phi_{\max}} d\phi_1 ... \int_{-\phi_{\max}}^{\phi_{\max}} d\phi_{\mathcal{V}} \, |\psi(\phi_1, ..., \phi_{\mathcal{V}})|^2$$

Thus, ϵ is the probability that at least one of $|\phi_1|$, $|\phi_2|$, ..., $|\phi_{\mathcal{V}}|$ exceeds $\phi_{\max}$

JLP energy-based bound

Let's consider distribution for single site field value $\phi(\mathbf{x})$, we have

$$\mu_{\phi(\mathbf{x})} = \langle\psi|\,\hat{\phi}(\mathbf{x})\,|\psi\rangle \leq \sqrt{\langle\psi|\,\hat{\phi}^2(\mathbf{x})\,|\psi\rangle}$$

$$\sigma_{\phi(\mathbf{x})} = \sqrt{\langle\psi|\,\hat{\phi}^2(\mathbf{x})\,|\psi\rangle - \langle\psi|\,\hat{\phi}(\mathbf{x})\,|\psi\rangle^2} \leq \sqrt{\langle\psi|\,\hat{\phi}^2(\mathbf{x})\,|\psi\rangle}$$

By Chebyshev inequality, the probability for $|\phi(\mathbf{x})|$ to exceed $\phi_{\max} = |\mu_{\phi(\mathbf{x})}| + c\,\sigma_{\phi(\mathbf{x})}$ is upper bounded by $1/c^2$, thus

$$p_{\text{out}} \leq \frac{\sigma^2_{\phi(\mathbf{x})}}{\left(\phi_{\max} - |\mu_{\phi(\mathbf{x})}|\right)^2} \leq \frac{\langle\psi|\hat{\phi}^2(\mathbf{x})|\psi\rangle}{\left(\phi_{\max} - \sqrt{\langle\psi|\hat{\phi}^2(\mathbf{x})|\psi\rangle}\right)^2}$$

JLP energy-based bound

Recall for a single site

$$p_{\text{out}} \leq \frac{\langle\psi|\hat{\phi}^2(\mathbf{x})|\psi\rangle}{\left(\phi_{\max} - \sqrt{\langle\psi|\hat{\phi}^2(\mathbf{x})|\psi\rangle}\right)^2}$$

There are $\mathcal{V}$ sites in total, by Union bound ($\Pr(A \cup B) \leq \Pr(A) + \Pr(B)$), we have

$$\epsilon \leq \frac{\mathcal{V}\langle\psi|\hat{\phi}^2(\mathbf{x})|\psi\rangle}{\left(\phi_{\max} - \sqrt{\langle\psi|\hat{\phi}^2(\mathbf{x})|\psi\rangle}\right)^2} \sim \frac{\mathcal{V}\langle\psi|\hat{\phi}^2(\mathbf{x})|\psi\rangle}{(\phi_{\max})^2}$$

Improved bound for conjugate field

For the conjugate field π, it's not straightforward to combine both Monte-Carlo trick and *p*-norm trick.

We can use either, but not both.

For *p*-norm trick, we use $p = 2$

Notice,

$$\frac{1}{2} \langle\psi| (\pi^{(2)})^2 |\psi\rangle = \frac{1}{2} \langle\psi| \sum_i (\pi_i)^2 |\psi\rangle \leq E$$

Thus,

$$\pi_{\max} \sim \sqrt{\frac{\langle\psi|(\hat{\pi}^{(2)})^2|\psi\rangle}{\epsilon}} \leq \sqrt{\frac{2aE}{\epsilon}}$$

Derivation of p-norm trick

Error probability

$$\epsilon = \Pr\left[\max_i |\phi_i| > \phi_{\max}\right]$$

We have $|\phi_i| \leq \phi^{(p)}$ for all i.

Thus,

$$\epsilon = \Pr\left[\max_i |\phi_i| > \phi_{\max}\right] \leq \Pr\left[\phi^{(p)} > \phi_{\max}\right]$$

By Chebyshev inequality,

$$\epsilon \leq \Pr\left[\phi^{(p)} > \phi_{\max}\right] \leq \frac{\langle\psi|\,(\hat{\phi}^{(p)})^2\,|\psi\rangle}{\left(\phi_{\max} - \sqrt{\langle\psi|(\hat{\phi}^{(p)})^2|\psi\rangle}\right)^2} \sim \frac{\langle\psi|\,(\hat{\phi}^{(p)})^2\,|\psi\rangle}{(\phi_{\max})^2}$$

Bounding inf-norm expectation value

Let $\hat{H}$ denote the original Hamiltonian and $\hat{H}'_{\phi^{(\infty)},\eta} = \hat{H} - \eta\frac{1}{a}\frac{1}{2}m_0^2(\hat{\phi}^{(\infty)})^2$ denote a modified Hamiltonian.

Denote their ground state energies as E_0 and $E'_{0,\phi^{(\infty)},\eta}$, respectively.

Suppose $\langle\psi|\,\hat{H}\,|\psi\rangle \leq E = E_0 + \Delta E$, then

$$\begin{aligned}\langle\psi|\,(\hat{\phi}^{(\infty)})^2\,|\psi\rangle =& \frac{2}{\eta m_0^2}\left(a\,\langle\psi|\,\hat{H}\,|\psi\rangle - a\,\langle\psi|\,\hat{H}'_{\phi^{(\infty)},\eta}\,|\psi\rangle\right)\\ \leq& \frac{2}{\eta m_0^2}a\left(E_0 - E'_{0,\phi^{(\infty)},\eta} + \Delta E\right),\end{aligned}$$

Having chosen ΔE, one can bound $\langle\psi|\,(\hat{\phi}^{(\infty)})^2\,|\psi\rangle$ by evaluating E_0 and $E'_{0,\phi^{(\infty)},\eta}$ through path-integral Monte Carlo simulation.

Bounding inf-norm expectation value (cont.)

Alternative, denote the ground state of $\hat{H}'_{\phi^{(\infty)},\eta}$ as $\left|\Omega_{\phi^{(\infty)},\eta}\right\rangle$. Notice,

$$
\begin{aligned}
E_0 &= \min_{|\alpha\rangle} \langle\alpha| \hat{H} |\alpha\rangle \\
&\leq \left\langle\Omega_{\phi^{(\infty)},\eta}\right| \hat{H} \left|\Omega_{\phi^{(\infty)},\eta}\right\rangle \\
&= \left\langle\Omega_{\phi^{(\infty)},\eta}\right| \hat{H}'_{\phi^{(\infty)},\eta} \left|\Omega_{\phi^{(\infty)},\eta}\right\rangle + \left\langle\Omega_{\phi^{(\infty)},\eta}\right| \frac{1}{a}\frac{\eta}{2} m_0^2 \hat{\phi}(\mathbf{x}_i)^2 \left|\Omega_{\phi^{(\infty)},\eta}\right\rangle \\
&= E'_{\phi^{(\infty)},\eta} + \left\langle\Omega_{\phi^{(\infty)},\eta}\right| \frac{1}{a}\frac{\eta}{2} m_0^2 (\hat{\phi}^{(\infty)})^2 \left|\Omega_{\phi^{(\infty)},\eta}\right\rangle
\end{aligned}
$$

Hence,

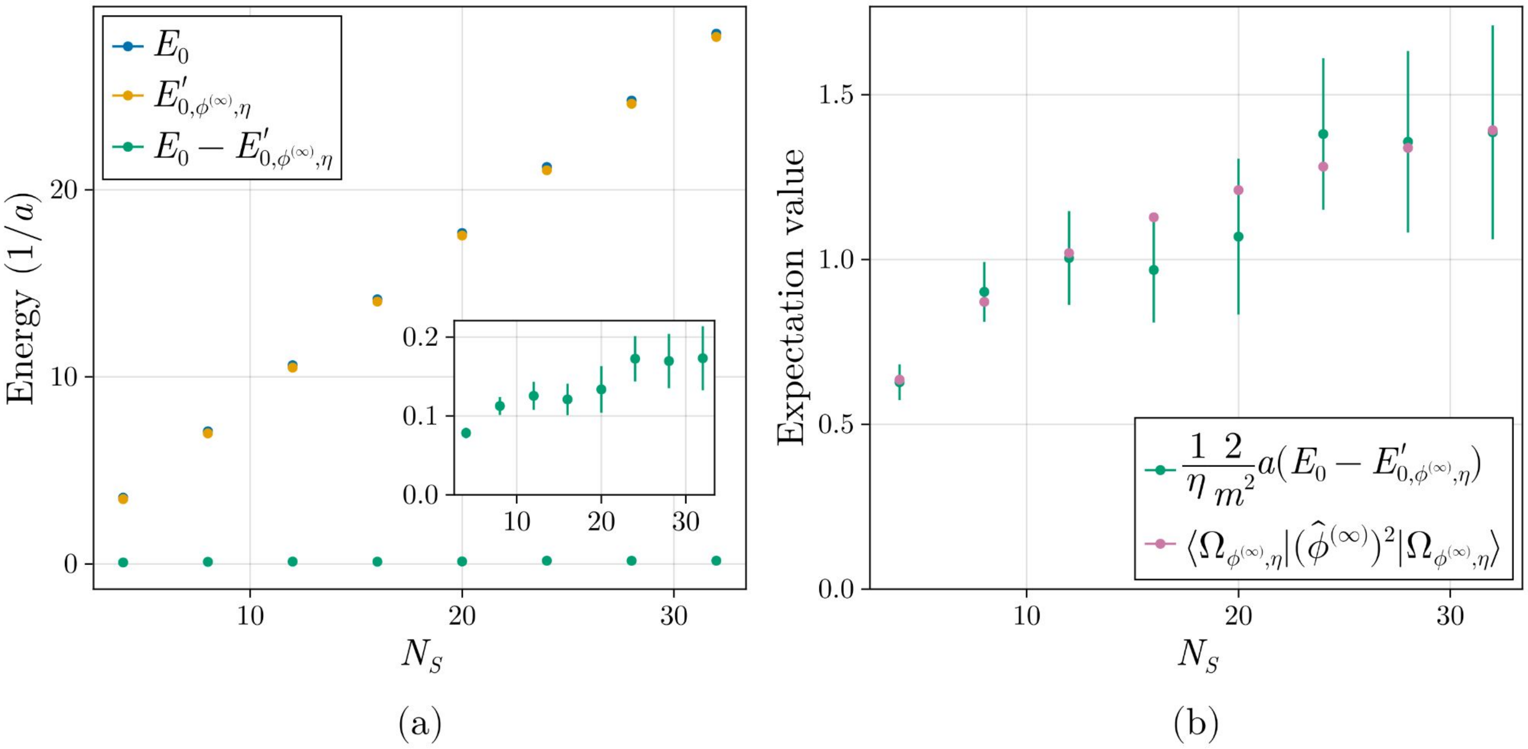
$$
\frac{2}{\eta m_0^2} a \left(E_0 - E'_{0,\phi^{(\infty)},\eta}\right) \leq \left\langle\Omega_{\phi^{(\infty)},\eta}\right| (\hat{\phi}^{(\infty)})^2 \left|\Omega_{\phi^{(\infty)},\eta}\right\rangle
$$

Thus,

$$
\langle\psi| (\hat{\phi}^{(\infty)})^2 |\psi\rangle \leq \left\langle\Omega_{\phi^{(\infty)},\eta}\right| (\hat{\phi}^{(\infty)})^2 \left|\Omega_{\phi^{(\infty)},\eta}\right\rangle + \frac{2}{\eta m_0^2} a\Delta E
$$

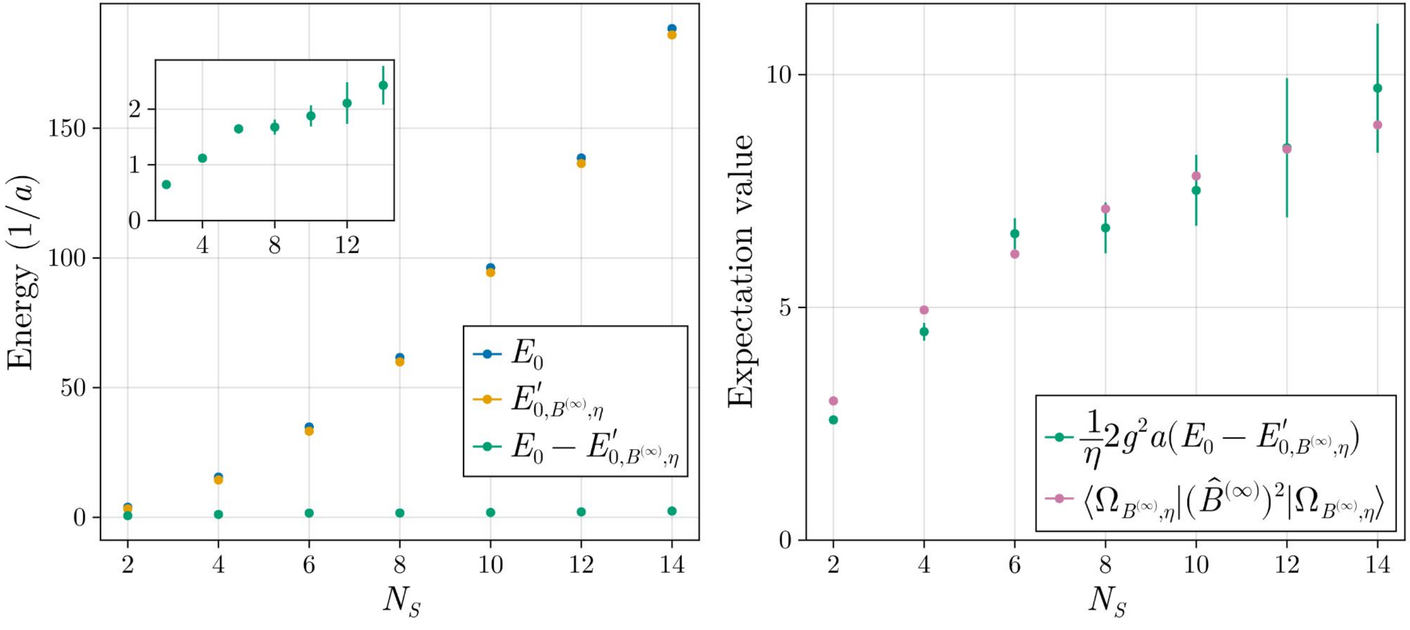
Energy scales

1+1D ϕ^4 theory



Energy scales

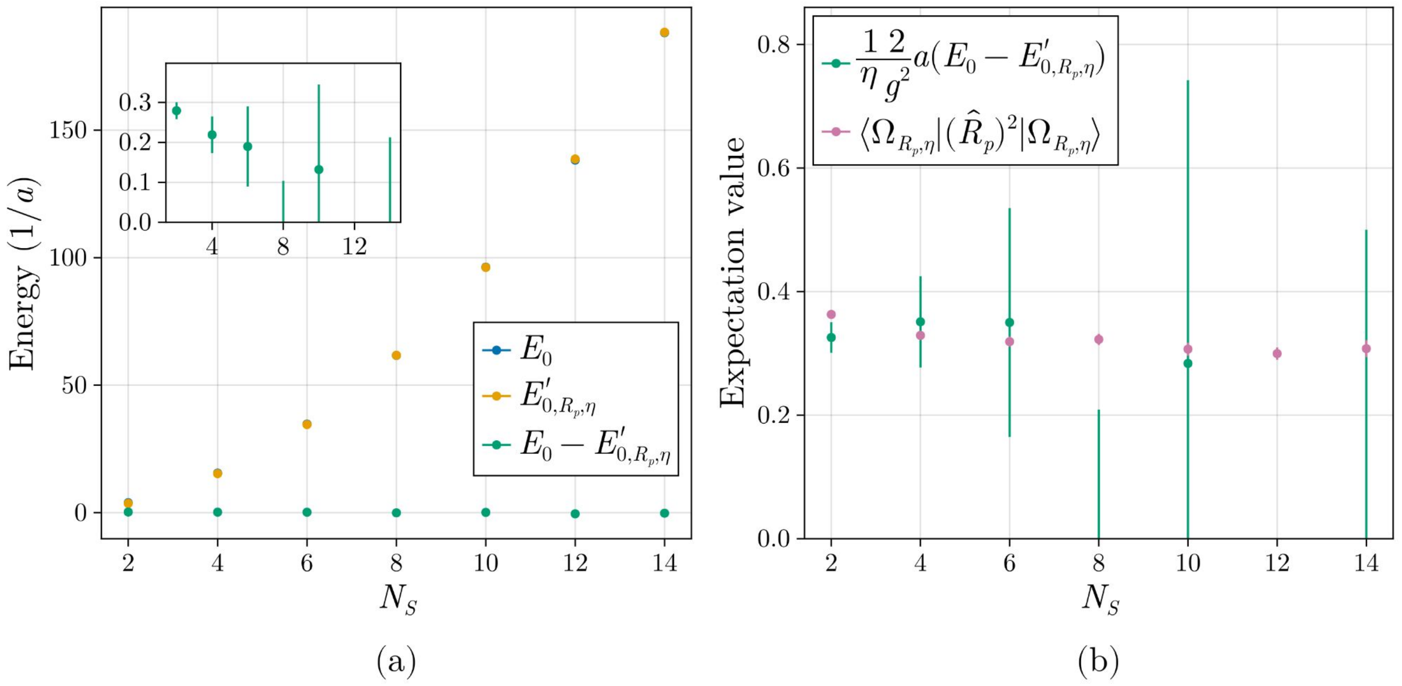
2+1D non-compact U(1) theory in dual formalism



(a) (b)

Energy scales

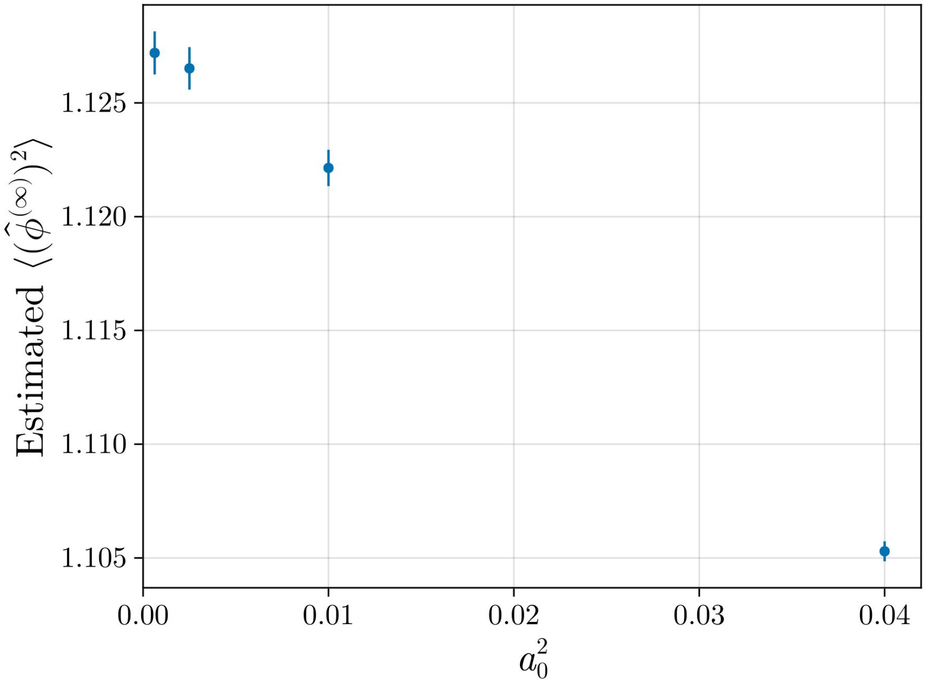
2+1D non-compact U(1) theory in dual formalism



Hamiltonian-limit extrapolation

continuous time extrapolation

$$a_0 \to 0$$



Derivation of path-integral

One can write the identity operator as

$$I = \int \mathcal{D}\phi \, |\phi\rangle \langle\phi| \, ,$$

where $|\phi\rangle := |\phi(\mathbf{x}_1)\rangle \otimes |\phi(\mathbf{x}_2)\rangle \otimes \ldots \otimes |\phi(\mathbf{x}_{\mathcal{V}})\rangle$

Hence, one can write

$$\begin{aligned} \mathrm{Tr}\left[e^{-\hat{H}T}\right] &= \mathrm{Tr}\left[\left(\int \mathcal{D}\phi \, |\phi\rangle \langle\phi| \, e^{-\hat{H}a_0}\right)^N\right] \\ &= \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 \ldots \mathcal{D}\phi_N \, \langle\phi_N| \, e^{-\hat{H}a_0} \, |\phi_{N-1}\rangle \langle\phi_{N-1}| \, e^{-\hat{H}a_0} \, |\phi_{N-2}\rangle \ldots \langle\phi_2| \, e^{-\hat{H}a_0} \, |\phi_1\rangle \langle\phi_1| \, e^{-\hat{H}a_0} \, |\phi_N\rangle \end{aligned}$$

Derivation of path-integral (cont.)

After some derivation, one can show that

$$\mathrm{Tr}\left[e^{-\hat{H}T}\right] \approx \mathcal{N}\int \mathcal{D}\phi\; e^{-\sum_{t,\mathbf{x}}\left[\frac{a}{a_0}\frac{1}{2}(\phi(t+a_0,\mathbf{x})-\phi(t,\mathbf{x}))^2+\frac{a_0}{a}\left(\frac{1}{2}(\nabla\phi)^2(t,\mathbf{x})+\frac{1}{2}m_0^2\phi(t,\mathbf{x})^2+\frac{\lambda_0}{4!}\phi(t,\mathbf{x})^4\right)\right]}$$

and

$$\mathrm{Tr}\left[\hat{O}[\hat{\phi}]e^{-\hat{H}T}\right] \approx \mathcal{N}\int \mathcal{D}\phi\; O[\boldsymbol{\phi}(t_0)]e^{-\sum_{t,\mathbf{x}}\left[\frac{a}{a_0}\frac{1}{2}(\phi(t+a_0,\mathbf{x})-\phi(t,\mathbf{x}))^2+\frac{a_0}{a}\left(\frac{1}{2}(\nabla\phi)^2(t,\mathbf{x})+\frac{1}{2}m_0^2\phi(t,\mathbf{x})^2+\frac{\lambda_0}{4!}\phi(t,\mathbf{x})^4\right)\right]}$$

Thus, one can interpret this as a path integral and define the corresponding action as

$$S[\phi] = \sum_{t,\mathbf{x}}\left[\frac{a}{a_0}\frac{(\phi(t+a_0,\mathbf{x})-\phi(t,\mathbf{x}))^2}{2} + \frac{a_0}{a}\frac{1}{2}(\nabla\phi)^2(t,\mathbf{x}) + \frac{a_0}{a}\frac{m_0^2\phi(t,\mathbf{x})^2}{2} + \frac{a_0}{a}\frac{\lambda_0}{4!}\phi(t,\mathbf{x})^4\right]$$

Derivation of path-integral (cont.)

Notice,

$$
\begin{aligned}
&\left\langle \boldsymbol{\phi}_{j+1} \right| \hat{\pi}(\mathbf{x}_k)^2 e^{-\hat{H} a_0} \left| \boldsymbol{\phi}_j \right\rangle \\
&\approx \int \mathcal{D}\boldsymbol{\pi} \left\langle \boldsymbol{\phi}_j \middle| \boldsymbol{\pi} \right\rangle \left\langle \boldsymbol{\pi} \right| \hat{\pi}(\mathbf{x}_k)^2 e^{-\hat{H}_{\mathrm{kin}} a_0} e^{-\hat{H}_{\mathrm{pol}} a_0} \left| \boldsymbol{\phi}_{j-1} \right\rangle \\
&= \int \mathcal{D}\boldsymbol{\pi} \left(\frac{1}{2\pi}\right)^{\mathcal{V}} \pi(\mathbf{x}_k)^2 e^{-\frac{a_0}{a} \sum_{\mathbf{x}} \frac{1}{2}\pi(\mathbf{x})^2 + i \sum_{\mathbf{x}} \pi(\mathbf{x})(\phi_{j+1}(\mathbf{x}) - \phi_j(\mathbf{x}))} e^{-\frac{a_0}{a} \sum_{\mathbf{x}} \left(\frac{1}{2}(\nabla\phi_j)^2(\mathbf{x}) + \frac{1}{2} m_0^2 \phi_j(\mathbf{x})^2 + \frac{\lambda_0}{4!}\phi_j(\mathbf{x})^4\right)} \\
&= \left(\sqrt{\frac{a}{2\pi a_0}}\right)^{\mathcal{V}} \left(\frac{a}{a_0} - \frac{a^2}{a_0^2}\left(\phi_{j+1}(\mathbf{x}_k) - \phi_j(\mathbf{x}_k)\right)^2\right) e^{-\frac{a}{a_0} \sum_{\mathbf{x}} \frac{1}{2}(\phi_{j+1}(\mathbf{x}) - \phi_j(\mathbf{x}))^2 - \frac{a_0}{a} \sum_{\mathbf{x}} \left(\frac{1}{2}(\nabla\phi_j)^2(\mathbf{x}) + \frac{1}{2} m_0^2 \phi_j(\mathbf{x})^2 + \frac{\lambda_0}{4!}\phi_j(\mathbf{x})^4\right)},
\end{aligned}
$$

Thus, to evaluate $\hat{\pi}(\mathbf{x}_k)^2$ in a path-integral, one can evaluate

$$
\frac{a}{a_0} - \frac{a^2}{a_0^2}\left(\phi(t + a_0, \mathbf{x}_k) - \phi(t, \mathbf{x}_k)\right)^2
$$

Derivation of path-integral (cont.)

To evaluate ground-state energy, one can evaluate the following expression in path integral

$$\frac{1}{a}\sum_{\mathbf{x}}\left(\frac{1}{2}\left(\frac{a}{a_0}-\frac{a^2}{a_0^2}(\phi(t+a_0,\mathbf{x})-\phi(t,\mathbf{x}))^2\right)+\frac{1}{2}(\nabla\phi)^2(t,\mathbf{x})+\frac{1}{2}m_0^2\phi(t,\mathbf{x})^2+\frac{\lambda_0}{4!}\phi(t,\mathbf{x})^4\right)$$