Large-N Phase Structure of q-Deformed Yang-Mills Theory in 2+1 Dimensions

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Based on work w/ Tomoya Hayata and Yoshimasa Hidaka
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Introduction

A dream:

first-principle computation of QCD in Hamiltonian formalism!

- Different & fresh viewpoint from path-integral formalism
- No sign problem observed in Monte Carlo is expected
- Stimulated by the recent innovations of quantum technology

Many issues to be clarified for implementation on (quantum) computer.

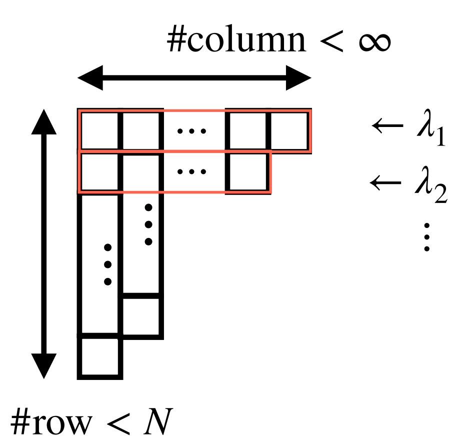
Truncation of Hilbert space

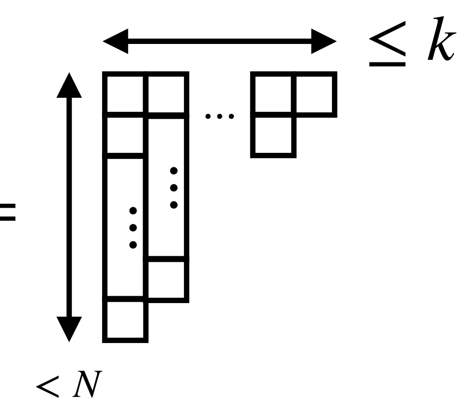
- From theoretical side: how to regularize the Hilbert space of gauge field?
  - Hilbert space of gauge field (:boson) is infinite dimensional, and naive truncation spoils the symmetry. (c.f., angular momentum and rotation)
  - How should the cutoff be removed (i.e., continuum limit)?

This talk : q deformation

c.f. Klco's talk in Wilson Award Session

Roughly, $\mathrm{SU}(N) \longrightarrow \mathrm{SU}(N)_k$ ← cutoff (integer)

$\lambda =$ 

$\lambda =$ 

: Young diagram (irreps.)

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} \geq \lambda_N = 0)$$

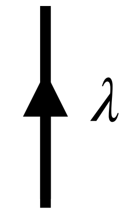
$$P_k = \left\{\lambda \;\middle|\; k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} \geq \lambda_N = 0\right\}, \quad |P_k| = \binom{k+N-1}{N-1}$$

c.f. RCFT such as Wess-Zumino-Novikov-Witten model

q-deformed Yang-Mills theory in 2+1 dimensions

q-deformed gauge theory

- Wilson line

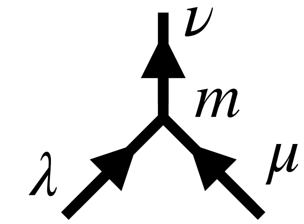


$$= \sqrt{d_\lambda}\, U_\lambda$$

d_λ : quantum dimension

- Fusion rule of irreps. :

$$\lambda \times \mu = \sum_{\nu} N^{\nu}_{\lambda\mu}\, \nu$$

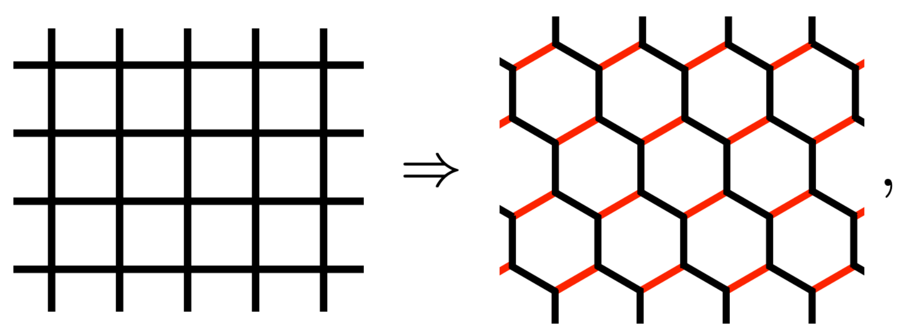


$m = \left\{1, \cdots, N^{\nu}_{\lambda\mu}\right\}$

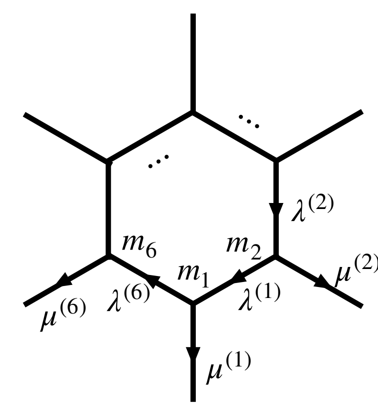
q-deformed: $\nu_1 \leq k$

Ex) SU(3) case: $\mathbf{8} \otimes \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27}$ $\quad N^{\mathbf{1}}_{\mathbf{88}} = N^{\mathbf{10}}_{\mathbf{88}} = N^{\overline{\mathbf{10}}}_{\mathbf{88}} = N^{\mathbf{27}}_{\mathbf{88}} = 1, \quad N^{\mathbf{8}}_{\mathbf{88}} = 2$

- 2+1d theory on the square lattice



quantum states as a network of Wilson lines



$$= \prod_{j=1}^{6} |\lambda^{(j)}\rangle |\mu^{(j)}\rangle |m_j\rangle$$

state around a plaquette

- Lattice Kogut-Susskind Hamiltonian ($K = 1/g^4$)

[Kogut, Susskind, '75, Robson, Webber, '82]

$$H_{\mathrm{KS}} = \frac{1}{2} \sum_{e \in \mathscr{E}'} E^2(e) - \frac{K}{2} \sum_{f \in \mathscr{F}} \left(\mathrm{tr}\, U_{\mathrm{fund}}(f) + \mathrm{tr}\, U_{\overline{\mathrm{fund}}}(f)\right)$$

$\mathscr{E}'$: set of black edges

$\mathscr{F}$: set of honeycomb faces

Mean-field analysis

[Zache, González-Cuadra, Zoller, ’23; Hayata, Hidaka, ’23; ’23]

- Variational ansatz:

$$|\Psi\rangle = \prod_f \sum_{\lambda_f} \psi(\lambda_f)\ \mathrm{tr}\, U_{\lambda_f} |\Omega\rangle$$

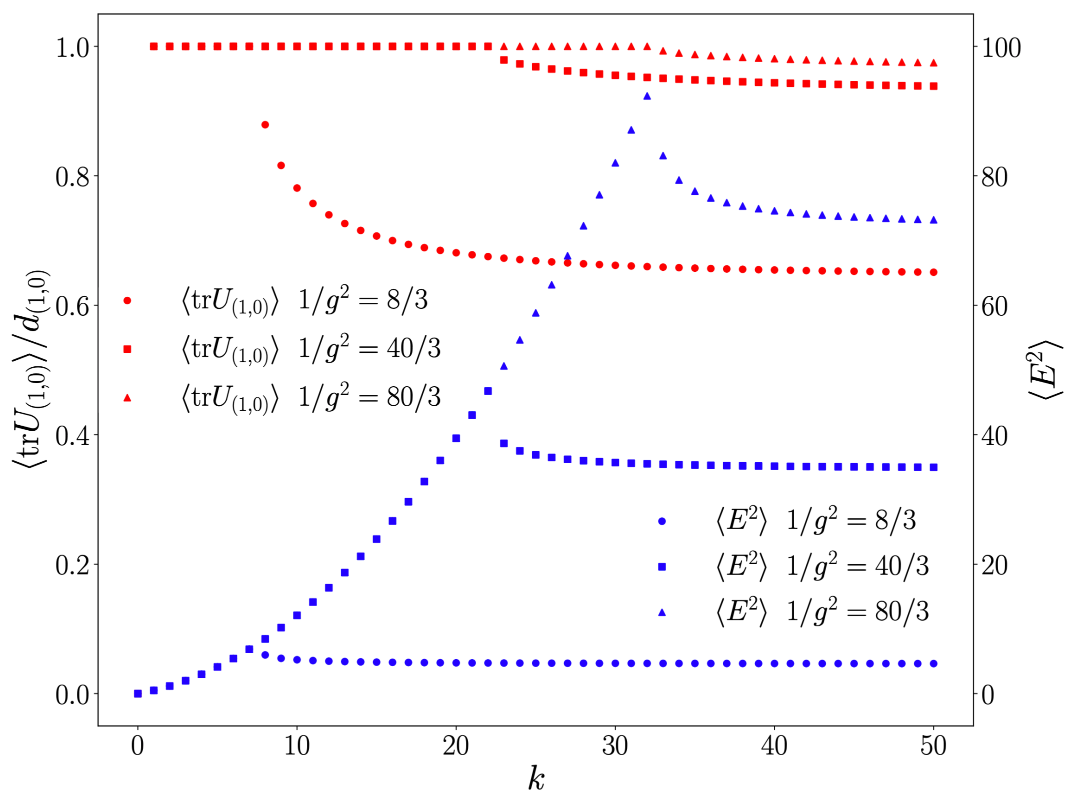
$|\Omega\rangle$: confining vacuum (strong coupling limit)

[Dusuel, Vidal, ’15; Zache, González-Cuadra, Zoller, ’23]

- VEV of Hamiltonian density: (under $V \to \infty$ w/ open b.c. w/ assuming translational invariance)

$$H = \sum_{\lambda,\mu,\nu} C_2(\nu) N^{\mu}_{\lambda\nu} \frac{d_\nu}{d_\lambda d_\mu} |\psi(\lambda)|^2 |\psi(\mu)|^2 - \frac{K}{2} \sum_{\lambda,\mu} \psi^*(\lambda) \left(N^{\lambda}_{\mathrm{fund}\,\mu} + N^{\lambda}_{\overline{\mathrm{fund}}\,\mu} \right) \psi(\mu)$$

- For $\mathrm{SU}(3)_k$ YM, the ground state was evaluated by imag.-time evol. of $\psi(\lambda)$, [Hayata, Hidaka, ’23]



$$\langle E^2 \rangle = \sum_{\lambda,\mu,\nu} C_2(\nu) N^{\mu}_{\lambda\nu} \frac{d_\nu}{d_\lambda d_\mu} |\psi(\lambda)|^2 |\psi(\mu)|^2$$

$$\langle \mathrm{tr}\, U_{\mathrm{fund}} \rangle = \sum_{\lambda,\mu} N^{\lambda}_{\mathrm{fund}\,\mu} \psi^*(\lambda) \psi(\mu)$$

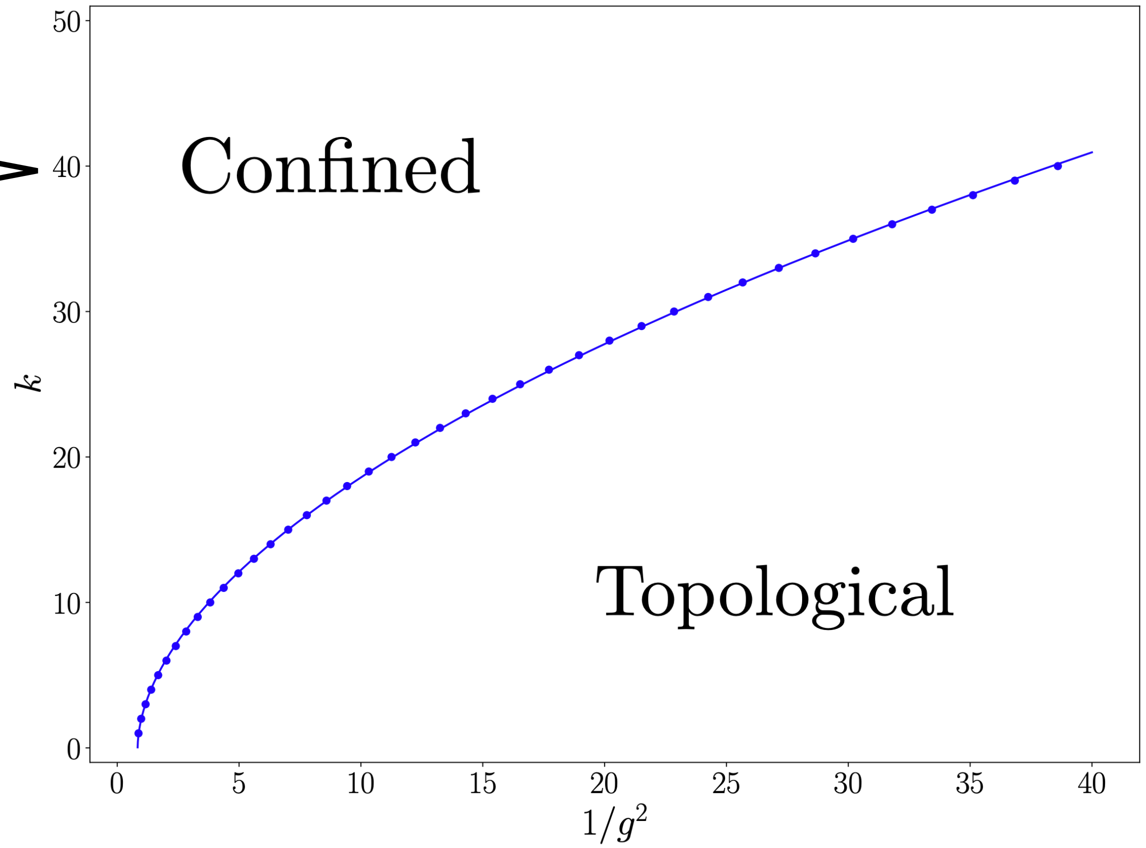
Change of behavior at some k for fixed $1/g^2$

→ Phase transition occurs!

Phase diagram of $\mathrm{SU}(3)_k$

Strong coupling limit

$$\psi_0(\lambda) = \delta_{\lambda,\mathrm{trivial}}$$



$k = A\left(1/g^2 - 1/g_0^2\right)^B$

Weak coupling limit

$$\psi_0(\lambda) = \frac{d_\lambda}{\sqrt{\sum_\lambda d_\lambda^2}}$$

c.f. string-net condensed state [Levin, Wen '05]

from [Hayata, Hidaka, '23]

Taking $k \to \infty$ and weak-coupling limit $(1/g^2 \to \infty)$ in order corresponds to continuum limit.

Q., Possibility of non-trivial continuum limit for a more generic case?

We investigated $\mathrm{SU}(N)_k$ theories for $N \leq 10$ and discussed the fate at large N.

Phase structure at large N

Stability of mean-field analysis

[Hayata, Hidaka, HW, '26]

Strategy: stability analysis of mean field in the topological phase

- Mean-field Hamiltonian density: $\psi(\lambda) = \psi_{\rm top}(\lambda) + \delta\psi(\lambda)$

$$E[\psi,\psi^*] := \frac{\langle \Psi | H_{\rm KS} | \Psi \rangle}{V} = E[\psi_{\rm top},\psi_{\rm top}^*] + \cancel{\left.\frac{\partial E}{\partial \psi_A}\right|_{\psi=\psi_{\rm top}}} \delta\psi_A + \frac{1}{2}\left.\frac{\partial^2 E}{\partial \psi_A \partial \psi_B}\right|_{\psi=\psi_{\rm top}} \delta\psi_A \delta\psi_B + \cdots$$

- Second-order perturbation: for complex $\delta\psi$, by using $\mathbf{v}(\lambda)^\top = (\delta\psi(\lambda), \delta\psi^*(\lambda))$

$$E^{(2)} = \sum_{\lambda,\mu} \mathbf{v}(\lambda)^\dagger \mathscr{M}_{\lambda\mu} \mathbf{v}(\mu), \qquad \mathscr{M}_{\lambda\mu} = \begin{pmatrix} H_{\psi^*(\lambda)\psi(\mu)} & H_{\psi^*(\lambda)\psi^*(\mu)} \\ H_{\psi(\lambda)\psi(\mu)} & H_{\psi(\lambda)\psi^*(\mu)} \end{pmatrix}, \qquad H_{\psi_1\psi_2} = \left.\frac{\partial^2 E}{\partial \psi_1 \partial \psi_2}\right|_{\psi=\psi_{\rm top}}$$

$$H_{\psi(\lambda)\psi(\mu)} = 2\sum_\nu C_2(\nu)\frac{d_\nu}{D^2}\left(N^\mu_{\lambda\nu} - \frac{d_\lambda d_\mu d_\nu}{D^2}\right) = H_{\psi^*(\lambda)\psi^*(\mu)}, \quad H_{\psi(\lambda)\psi^*(\mu)} = H_{\psi(\lambda)\psi(\mu)} - \frac{K}{2}\left(M_{\lambda\mu} - 2d_{\rm fund}\delta_{\lambda\mu}\right) = H_{\psi^*(\lambda)\psi(\mu)},$$

$\leftarrow$ $C_2(\lambda)$, d_λ, $N^\nu_{\lambda\mu}$, as input data ($D^2 = \sum_\lambda d_\lambda^2$, $M_{\lambda\mu} := N^\lambda_{{\rm fund}\,\mu} + N^\lambda_{\overline{\rm fund}\,\mu}$)

Phase boundary in $(1/g^2, k)$-plane is where the negative eigenvalue of $\mathscr{M}_{\lambda\mu}$ (dis)appears.

Algebraic data for stability analysis

- We used quadratic Casimir of $\mathrm{SU}(N)$ $(k \to \infty)$

$$C_2(\lambda) = \frac{1}{2}\left(nN + \tilde{C}_2(\lambda) - \frac{n^2}{N}\right), \qquad \tilde{C}_2(\lambda) = \sum_{j=1}^{N-1} \lambda_j(\lambda_j - 2j + 1), \qquad n = \sum_j \lambda_j \quad \text{:\#box}$$

- Fusion coefficients are computed through Verlinde formula: [Verlinde, '88]

$$N^{\nu}_{\lambda\mu} = \sum_{\sigma\in P_k} \frac{S_{\lambda\sigma} S_{\mu\sigma} S^{-1}_{\nu\sigma}}{S_{\varnothing\sigma}}, \qquad S_{\lambda\mu} = \frac{\mathrm{e}^{\mathrm{i}\pi N(N-1)/4}}{\sqrt{N(N+k)^{N-1}}} \sum_{w\in W} (-1)^{\ell(w)} \mathrm{e}^{-\frac{2\pi\mathrm{i}}{N+k}\langle \mu+\rho, w(\lambda+\rho)\rangle}$$

:modular S-matrix [Kac, Peterson, '84]

+ many techniques to reduce computational cost
Code available on GitHub (https://github.com/Hiromasa-Watanabe/fusioncoeff_qdefSUN)

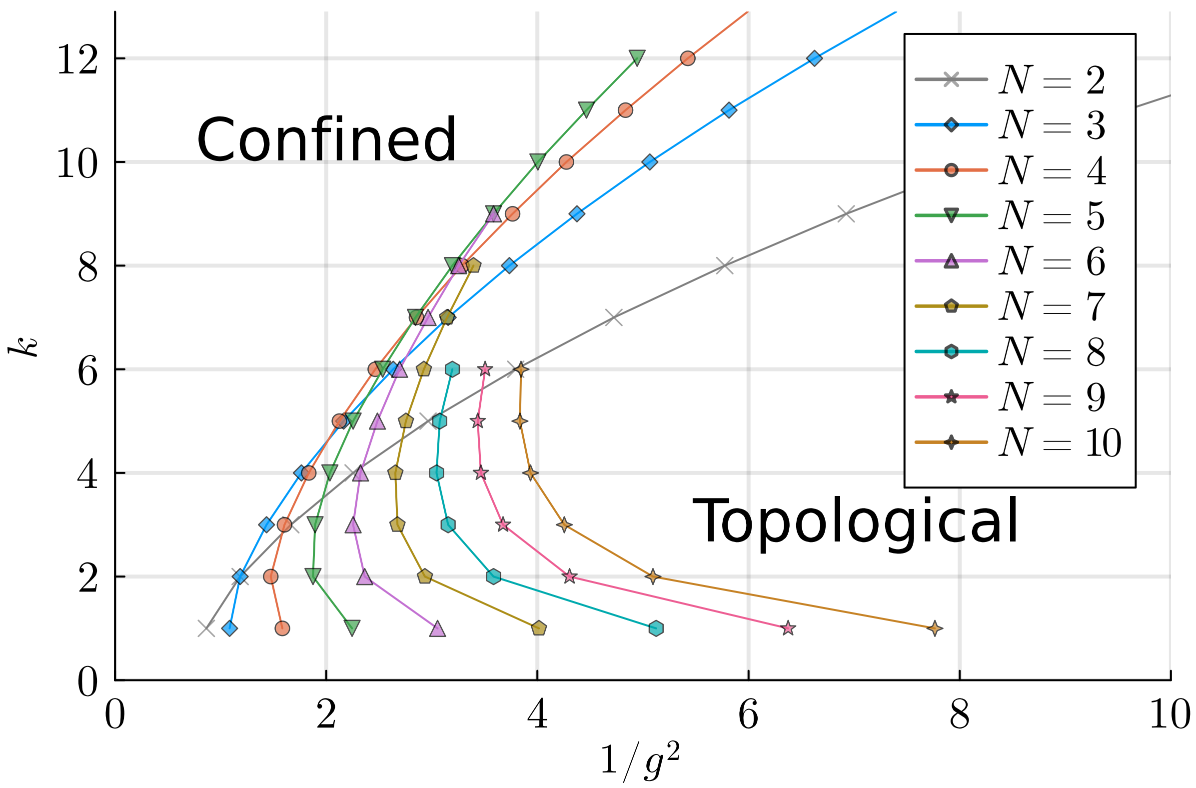
→ In this work, we numerically compute $N^{\nu}_{\lambda\mu}$ of various N, k for the region $|P_k| = \binom{k+N-1}{N-1} \lesssim 3000.$

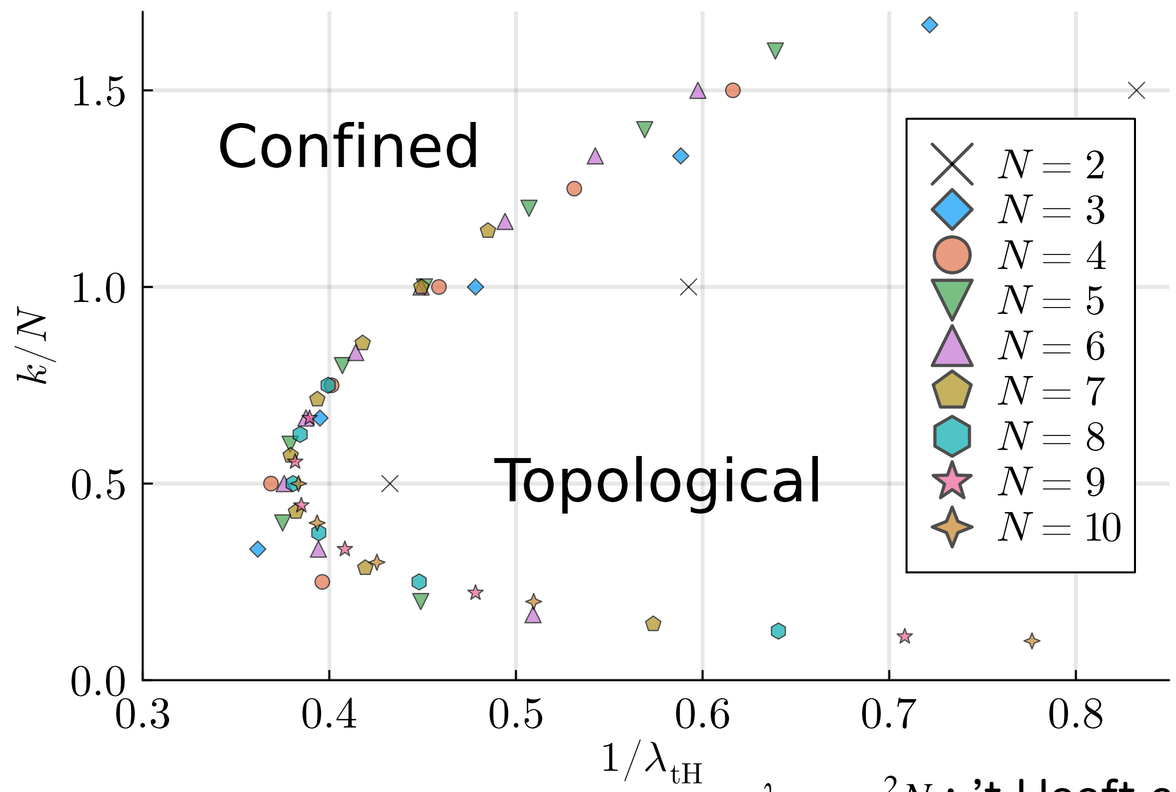
- quantum dimension of irreps.

$$d_\lambda = \frac{S_{\lambda\varnothing}}{S_{\varnothing\varnothing}} = \prod_{1\le j<k\le N} \frac{[\lambda_j - \lambda_k + k - j]}{[k-j]}, \qquad q = \exp\left(\frac{2\pi\mathrm{i}}{k+N}\right), \qquad [x] = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \xrightarrow{k\to\infty} x$$

c.f. [Gross, Taylor, '93; Naculich, Schnitzer, '07]

Phase diagram of $\mathrm{SU}(N)_k$





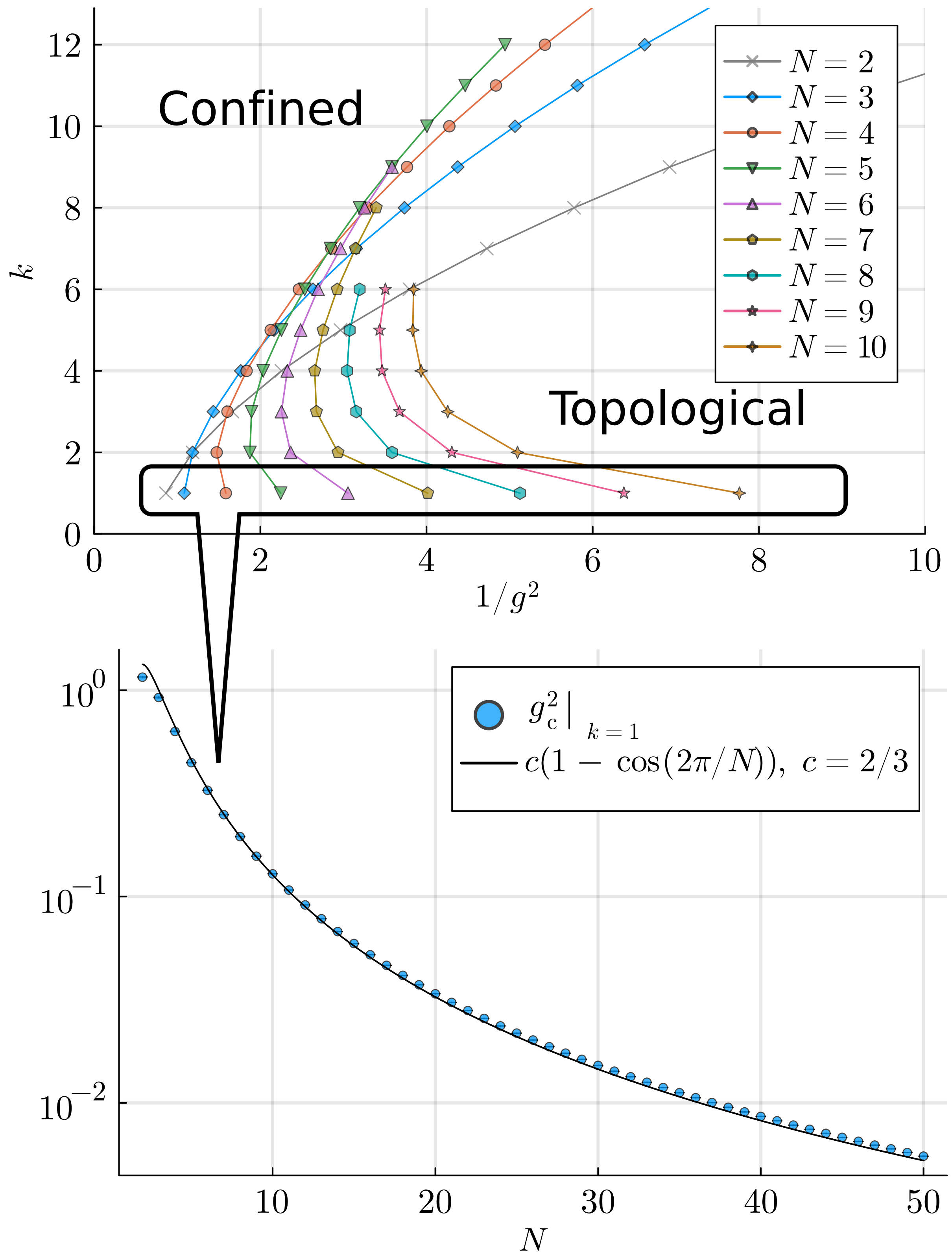
$\lambda_{\mathrm{tH}} = g^2 N$: 't Hooft coupling

- [Left] Boundaries for $N \geq 4$ exhibit more nontrivial behavior than $N = 3$.
  - The confined phase is broaden in the region $k/N \lesssim 0.5$.

    $\rightarrow$ necessity of a delicate treatment for continuum limit
- [Right] Critical λ_{tH}'s align with a specific band region, particularly for $k/N \gtrsim 0.5$.
  - Not the case for $N = 2$; $\rightarrow$ "$N = 2$ is not large" for the present case.

Let's see one by one.

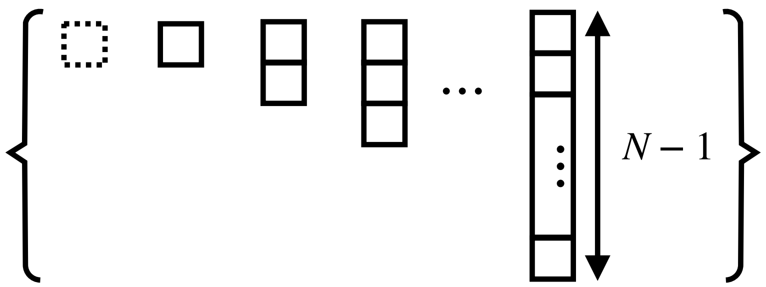
Phase structure of $\mathrm{SU}(N)_{k=1}$



- Effective structure of $\mathbb{Z}_N$ gauge theory

  The irreps. of $\mathrm{SU}(N)_{k=1}$ are

  $\lambda \in$

  $d_\lambda = 1$

  fusion: $$\lambda_n \times \lambda_m = \lambda_{n+m,\ \mathrm{mod}\ N}$$

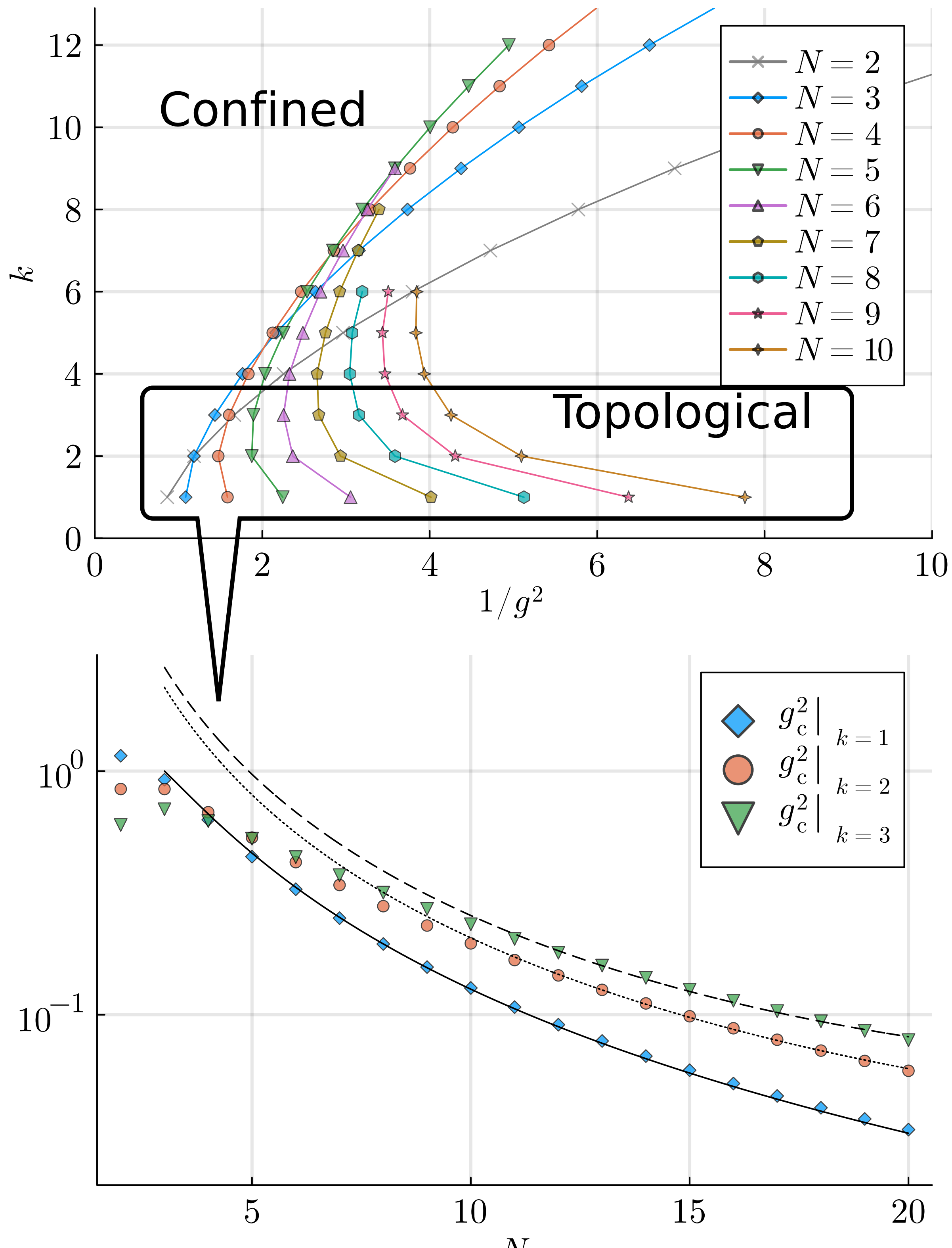
  → abelian topological order

- Remarkable agreement w/ lattice simulation of $\mathbb{Z}_N$ gauge theory [Bhanot, Creutz, '80]

- A naive large-N limit leads to U(1) gauge theory, which admits only a confining vacuum in 2+1d. [Polyakov, '75; '77]

  → expanded confined phases for k=1

Phase structure of $\mathrm{SU}(N)_k$ $(k \leq 3)$



- Unexpectedly, confined phase extends for $k > 1$.
  - $k = 1$: only abelian anyons ($d_\lambda = 1$)
  - $k > 1$: nonabelian anyons ($d_\lambda > 1$) enters
  - For $N \geq 3$, nontrivial multiplicity ($N^{\nu}_{\lambda\mu} \geq 2$) starts to appear at $k = 3$.

→ Abelian nature will not persist for $k > 1$.

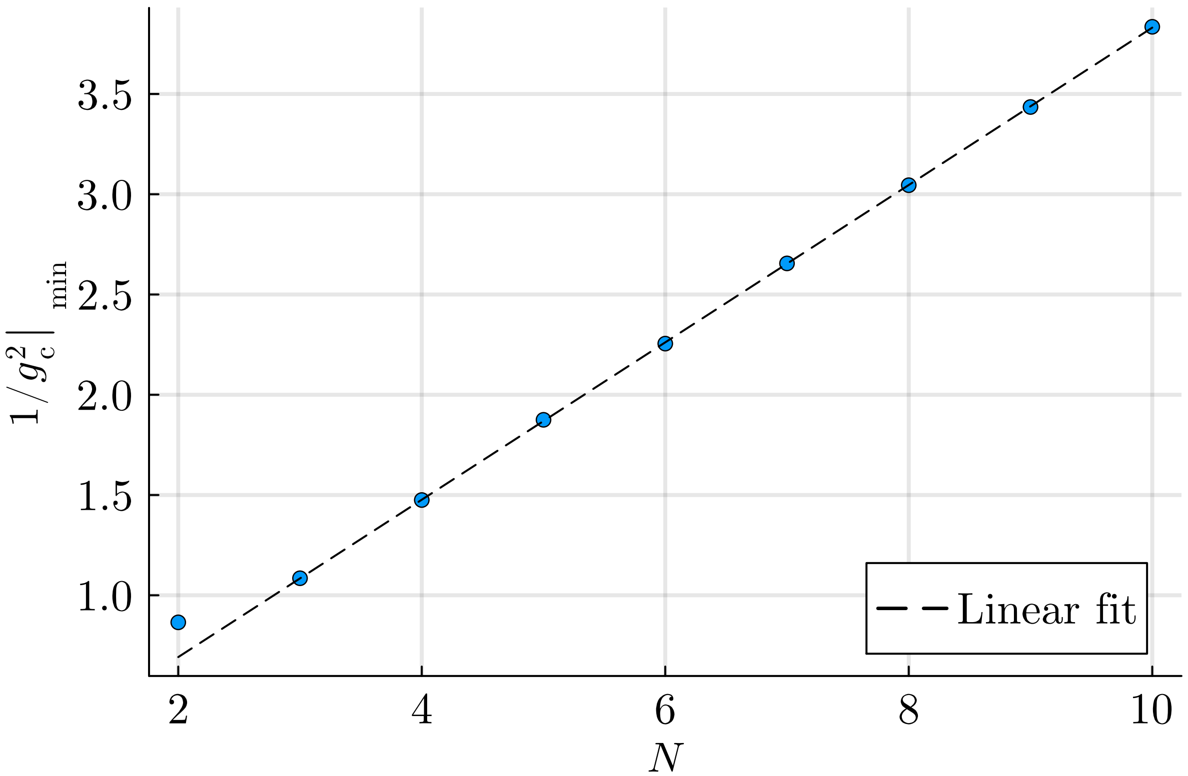
- Electric-field contribution must be taken into account, which breaks topological order.

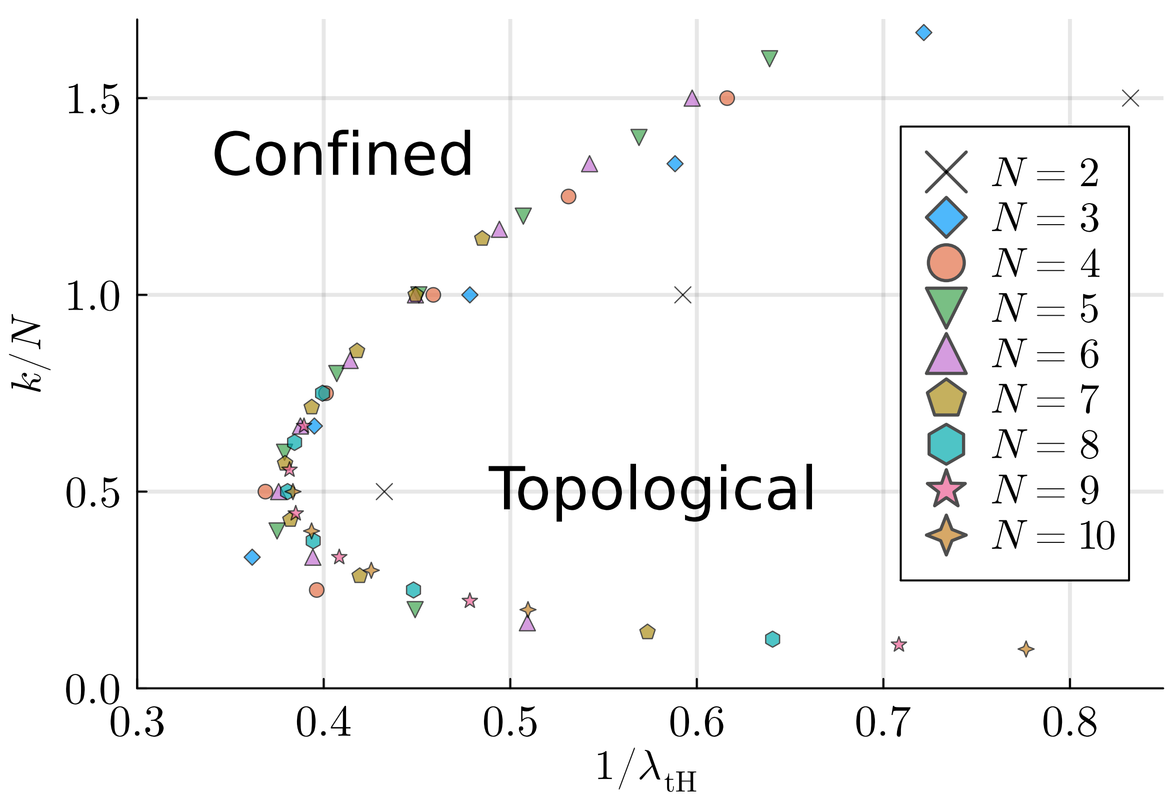
→ may work nontrivially for the realization of phase structure.

← Fit w/ ansatz

$$g_c^2(N)\,|_k = (c_2^{(k)} N^2 + c_4^{(k)} N^4)^{-1}$$

In 't Hooft large-N limit





- $1/g_c^2(N)|_{\min} = \min_k \left[1/g_c^2(N,k)\right]$ grows linearly w/ N.

  $\rightarrow (1/\lambda_{\rm tH})|_{\min} = 1/(g_c^2 N)|_{\min}$ is independent of N.

- $k_{\rm ext}(N) = k\left(1/g_c^2(N)|_{\min}\right)$ also grows w/ N.

TABLE I. List of $k_{\rm ext}$ for each N.

N	2	3	4	5	6	7	8	9	10
$k_{\rm ext}$	1	1	2	2	3	4	4	5	5
$k_{\rm ext}/N$	0.5	0.33	0.5	0.4	0.5	0.57	0.5	0.56	0.5

$\rightarrow$ Point $(1/\lambda_{\rm tH}, k/N) \approx (0.39,\ 0.5)$ as a reference when taking the continuum limit.

- In the region $k/N \gtrsim 0.5$, fitting w/

$$\frac{k(\lambda_{\rm tH})}{N} = A\left(\frac{1}{\lambda_{\rm tH}} - \frac{1}{\lambda_0}\right)^B + \frac{k_{\rm ext}}{N} + C$$

(essentially same ansatz w/ [Hayata, Hidaka, '23])

shows agreement for $A, B, 1/\lambda_0$.

Summary & Discussion

2+1d $\mathrm{SU}(N)$ q-deformed YM based on variational mean-field approach

- Variational mean-field ansatz + stability analysis
  $\leftarrow$ algebraic data ($C_2(\lambda)$ and $N_{\lambda\mu}^{\nu}$ of $\mathrm{SU}(N)_k$ irreps.) as input.

- Continuum limit for generic N is more intricate than naively expected!

- Confined phase expands at small k, partly due to $\mathbb{Z}_N$ structure at $k = 1$

  $\rightarrow$ offering benchmark of (large-N) gauge theory quantum simulator.

  c.f.) A simulation on a trapped-ion quantum computer [Hayata, Hidaka, Kikuchi, '26]

Future prospects

- Analysis beyond mean-field treatment, determination of order of transition
- Adding matter fields, alternative truncation scheme (e.g., choice of $C_2(\lambda)$), $\cdots$
- Refinement from path-integral viewpoint c.f. [Gross, Witten, '80; Wadia, '80; Douglas, Kazakov, '93]

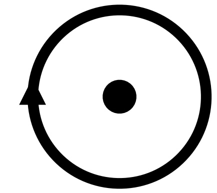
Backup

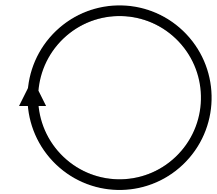
Topological phase

Ground state in the topological phase

$$\psi_0(a) = \frac{d_a}{D} \qquad d_a = d_{\bar{a}}, \qquad D^2 := \sum_a d_a^2 \qquad \sum_a |\psi_0(a)|^2 = 1$$

$$d_a d_b = \sum_c N_{ab}^c d_c, \qquad \sum_{a,c} N_{ac}^b d_a d_c = D^2 d_b$$

$\mathrm{tr}\, U_a =$ a 

$d_a =$ a 

VEV of Wilson loop operator can be represented as quantum dimension:

$$\langle \mathrm{tr}\, U_a \rangle = \sum_{b,c} N_{ac}^b \psi^*(b)\psi(c) = d_a$$

→ yielding a vanishing string tension

Identical to the string-net condensed state in Levin-Wen model [Levin, Wen '05; Zache, González-Cuadra, Zoller, '23]

$$|\Psi_{\text{string-net}}\rangle = \prod_{f\in\mathcal{F}} \frac{1}{D^2} U_{\text{reg}}(f)|0\rangle, \qquad U_{\text{reg}}(f) = \sum_a d_a U_a(f)$$

Stability of mean-field analysis

[Hayata, Hidaka, HW, '26]

Strategy: stability analysis of mean field in the topological phase

- Mean-field Hamiltonian density: $\psi(\lambda) = \psi_{\rm top}(\lambda) + \delta\psi(\lambda)$

$$E[\psi,\psi^*] := \frac{\langle\Psi|H_{\rm KS}|\Psi\rangle}{V} = \frac{1}{\mathcal{N}^2}\sum_{\lambda,\mu,\nu} C_2(\nu) N^{\mu}_{\lambda\nu}\frac{d_\nu}{d_\lambda d_\mu}\left|\psi(\lambda)\right|^2\left|\psi(\mu)\right|^2 - \frac{K}{2\mathcal{N}}\sum_{\lambda,\mu}\psi^*(\lambda)M_{\lambda\mu}\psi(\mu),$$

- $M_{\lambda\mu} := N^{\lambda}_{\mathrm{fund}\,\mu} + N^{\lambda}_{\overline{\mathrm{fund}}\,\mu}$
- $\mathcal{N}(\psi) = \sum_\lambda |\psi(\lambda)|^2 = 1$

- Second-order perturbation: for complex $\delta\psi$, by using $\mathbf{v}(\lambda)^\top = (\delta\psi(\lambda), \delta\psi^*(\lambda))$

$$E^{(2)} = \sum_{\lambda,\mu}\mathbf{v}(\lambda)^\dagger \mathcal{M}_{\lambda\mu}\mathbf{v}(\mu), \qquad \mathcal{M}_{\lambda\mu} = \begin{pmatrix} H_{\psi^*(\lambda)\psi(\mu)} & H_{\psi^*(\lambda)\psi^*(\mu)} \\ H_{\psi(\lambda)\psi(\mu)} & H_{\psi(\lambda)\psi^*(\mu)}\end{pmatrix}, \qquad H_{\psi_1\psi_2} = \left.\frac{\partial^2 E}{\partial\psi_1\partial\psi_2}\right|_{\psi=\psi_{\rm top}}$$

$$H_{\psi(\lambda)\psi(\mu)} = 2\sum_\nu C_2(\nu)\frac{d_\nu}{D^2}\left(N^{\mu}_{\lambda\nu} - \frac{d_\lambda d_\mu d_\nu}{D^2}\right) = H_{\psi^*(\lambda)\psi^*(\mu)}, \quad H_{\psi(\lambda)\psi^*(\mu)} = H_{\psi(\lambda)\psi(\mu)} - \frac{K}{2}\left(M_{\lambda\mu} - 2d_{\rm fund}\delta_{\lambda\mu}\right) = H_{\psi^*(\lambda)\psi(\mu)},$$

$\leftarrow C_2(\lambda),\ d_\lambda,\ N^{\nu}_{\lambda\mu},\ D^2 = \sum_\lambda d_\lambda^2$ as inputs

Phase bdry. in $(1/g^2, k)$-plane is where the negative eigenvalue of $\mathcal{M}_{\lambda\mu}$ (dis)appears.

Computing algebraic data (I)

- Modular S-matrix c.f. [Kac, Peterson, '84]

$$\rho = \sum_{\alpha \in \Delta^+} \alpha \quad \text{: Weyl vector}$$

$$S_{\lambda\mu} = \frac{e^{i\pi N(N-1)/4}}{\sqrt{N(N+k)^{N-1}}} \sum_{w \in W} (-1)^{\ell(w)} e^{-\frac{2\pi i}{N+k}\langle \mu+\rho, w(\lambda+\rho)\rangle}$$

$$\langle u, v\rangle = \sum_{j=1}^{N} u_j v_j - \frac{1}{N}\left(\sum_{j=1}^{N} u_j\right)\left(\sum_{j=1}^{N} v_j\right)$$

W: Weyl group is now S_N permutation group → $\sim O(N!)$ computational cost⋯

- Introducing $q := e^{\frac{2\pi i}{k+N}}$, $\quad x_j := q^{-\left(\mu_j + N - j - \frac{1}{N}\sum_{j=1}^{N}(\mu_j + N - j)\right)} = q^{-(\tilde{\mu}_j + N - j)}$, $\quad \prod_{j=1}^{N} x_j = 1$

$$S_{\lambda\mu} \propto \det_{1\le i,j\le N}\left(x_j^{\tilde{\lambda}_i+N-i}\right) = \det_{1\le i,j\le N}\left(x_j^{\lambda_i+N-i}\right) = s_\lambda(X)\cdot\Delta(X)$$

$s_\lambda(X)$: Schur polynomial, $\Delta(X)$: Vandermonde determinant → $\sim O(N^3)$ cost

- Using formulas of symmetric polynomials,

$$s_\lambda(X) = \det_{1\le i,j\le \ell(\lambda)}\left(h_{\lambda_i - i + j}(X)\right), \qquad r h_r(X) = \sum_{m=1}^{r} p_m(X) h_{r-m}(X), \qquad p_m(X) = \sum_{j=1}^{N} x_j^m, \quad h_0 = 1, \quad h_{r<0} = 0$$

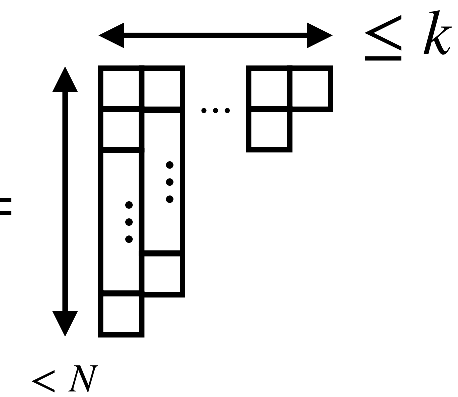
→ $\sim O(\ell(\lambda)^3)$ cost w/ $\ell(\lambda) \le N-1$, & $\{h_r(X)\}$ for given μ is reusable to compute $s_\lambda(X)$.

Computing algebraic data (II)

- Fusion coefficients by Verlinde formula [Verlinde, '88]

$$N_{\lambda\mu}^{\nu} = \sum_{\sigma \in P_k} \frac{S_{\lambda\sigma} S_{\mu\sigma} S_{\nu\sigma}^{-1}}{S_{\varnothing\sigma}}, \qquad P_k = \left\{ \lambda \;\middle|\; k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{N-1} \geq \lambda_N = 0 \right\}, \quad |P_k| = \binom{k+N-1}{N-1}$$

$\lambda =$



- Properties
  - Unitarity: $S^\dagger S = 1, \quad S^\dagger = S^{-1}$
  - Crossing symmetry: $\overline{S_{\lambda\mu}} = S_{\bar{\lambda}\mu}$
  - Charge conjugation matrix $C = S^2$, $C_{\lambda\mu} = \delta_{\lambda\bar{\mu}}$
  - Modular relation: $(ST)^3 = C$, $\qquad T_{\lambda\lambda} = \exp\left[2\pi \mathrm{i} \left(h_\lambda - \frac{c}{24}\right)\right], \quad h_\lambda = \frac{\langle \lambda, \lambda + 2\rho \rangle}{2(k+N)}, \quad c = \frac{k \dim \mathrm{SU}(N)}{k+N}$

- Cost reduction strategies:
  - Using N-ality, $|\lambda| + |\mu| - |\nu| = 0 \mod N$
  - Using symmetry of fusion coefficients, such as $N_{\lambda\mu}^{\nu} = N_{\mu\lambda}^{\nu}$
  - $N_{\lambda\mu}^{\nu} \in \mathbb{Z}_{\geq 0}$ is reusable for irreps. the cut-off dependence is negligible, i.e. $\lambda_1 + \mu_1 < k$.

Phase diagram in $(1/\lambda_{\mathrm{tH}}, k/N)$ plane