Topological susceptibility in the 2d $O(3)$ nonlinear sigma model using the tree-level improved gradient flow

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The model and its topology

- The theory characterized by scalar field ϕ mapping a flat two-dimensional spacetime into the target space S^2 with action

$$\frac{1}{2g_0^2}\int d^2x\, \partial_\mu\phi\cdot\partial_\mu\phi \qquad \text{where } \phi\cdot\phi = 1.$$

- The path integral splits into topological sectors with

$$\mathbb{Z} \ni Q = \int d^2x\, q(x) = \frac{1}{8\pi}\int d^2x\, \epsilon_{\mu\nu}\phi(x)\cdot(\partial_\mu\phi(x)\times\partial_\nu\phi(x)).$$

The model and its topology

- In the lattice regulated theory the susceptibility

$$\chi = \int d^2x \, \langle q(x)q(0) \rangle = \frac{\langle Q^2 \rangle}{V}$$

does not scale as expected from the one loop RG and $\chi\xi^2$ diverges in the continuum limit.[1][2][3][4][5]

[1] B. Berg and M. Luscher. In: *Nucl. Phys. B* 190 (1981).
[2] M. Luscher. In: *Nucl. Phys. B* 200 (1982).
[3] Claudio Bonanno et al. In: *Phys. Rev. D* 107.1 (2023).
[4] Marc Blatter et al. In: *Phys. Rev. D* 53 (1996).
[5] W. Bietenholz et al. In: *JHEP* 12 (2010).

- Possible way out: Use the gradient flow

$$\partial_t e(t,x)^i = (\delta^{ij} - e^i(t,x)e^j(t,x))\Delta e^j(t,x), \qquad e(0,x) = \phi(x)$$

- As Q is a global topological quantity and the flow is a local smearing of the field χ is left invariant.
- Correlation functions of operators built from flowed fields are UV-finite[6].
- But: previous studies suggest that even when flowing the fields $\chi\xi^2$ does not converge.[7][8].

[6] Hiroki Makino and Hiroshi Suzuki. In: *PTEP* 2015.3 (2015).

[7] Wolfgang Bietenholz et al. In: *Physical Review D* 98.11 (Dec. 2018).

[8] Stuart Yi-Thomas and Christopher Monahan. In: *PoS* LATTICE2021 (2022).

A Puzzle

- The only place where divergence is possible is
  - in the IR due to large distance behaviour of the correlator.
  - in the UV at $x = 0$[9].
  - Somewhere inbetween.
- One of the following has to be wrong:
  - Theory has a mass gap (this prevents IR divergences).
  - Flow has no contact terms (this prevents UV divergences).
  - Twopoint function of flowed local operator is finite at non-coinciding points.
  - $\chi\xi^2$ is divergent when flowed.

[9] Daniel Nogradi. In: *JHEP* 05 (2012).

Large cutoff effects

- The answer to this puzzle is complicated by large cutoff effects in the theory
- Naively, one would assume for any operator O a dependence

$$\langle O \rangle_{lat} = \langle O \rangle_{cont} + c_1 \frac{a^2}{\xi^2} + \ldots$$

- But: large anomalous dimensions appear in c_1 (logarithms with large power)[10] and it might be needed to look for a dependence

$$\langle O \rangle_{lat} = \langle O \rangle_{cont} + \tilde{c}_1 \frac{a^2}{\xi^2} \log(\xi/a)^p + \ldots$$

- If p is large, $\chi\xi^2$ might look divergent but is actually finite with large cutoff effects
- Tree level improvement lowers $p \to p - 1$.

[10] Janos Balog et al. In: *Nucl. Phys. B* 824 (2010).

Our plan

We want to tackle this issue, extending previous investigations by:

1. Tree-level improving the the theory and flow.
2. Going to much larger correlation lengths ξ and flowtimes t.
3. Explicitly quantifying systematic errors.

Tree-level improvement

- Tree-level improvement is rather straight-forward. Only expand lattice derivatives:

$$\text{Flow:}\quad \partial_t e^i = (\delta^{ij} - e^i e^j)\,\boxed{\Delta e^j},\qquad e^i(0,x) = \phi^i(x)\qquad \text{to be expanded}$$

$$\text{Action:}\quad S = \frac{1}{2g_0^2}\sum_x \boxed{\partial^f_\mu\phi}\cdot\boxed{\partial^f_\mu\phi},\qquad \text{Charge:}\quad Q = \frac{1}{8\pi}\sum_x \epsilon_{\mu\nu}\,\phi\cdot\left(\boxed{\partial_\mu\phi}\times\boxed{\partial_\nu\phi}\right)$$

- Job is done by replacing lattice derivatives with improved versions. With

$$\delta^{f(n)}_\mu f = f(x+n\hat\mu) - f(x),\qquad \delta^{(n)}_\mu f = f(x+n\hat\mu) - f(x-n\hat\mu),$$
$$\Delta^{(n)}_\mu f = f(x+n\hat\mu) + f(x-n\hat\mu) - 2f(x)$$

we have

$$\partial^f_\mu\phi\cdot\partial^f_\mu\phi \to \tfrac{4}{3}\,\delta^{f(1)}_\mu\phi\cdot\delta^{f(1)}_\mu\phi - \tfrac{1}{12}\,\delta^{f(2)}_\mu\phi\cdot\delta^{f(2)}_\mu\phi$$
$$\partial_\mu \to \tfrac{2}{3}\delta^{(1)}_\mu - \tfrac{1}{12}\delta^{(2)}_\mu,\qquad \nabla^2 \to \sum_\mu \tfrac{4}{3}\Delta^{(1)}_\mu - \tfrac{1}{12}\Delta^{(2)}_\mu$$

Scale setting

- To determine ξ we systematically fit multiple plateaus at $L = 8\xi, 10\xi, 12\xi$ and then select the one that starts at ξ for $L = 12\xi$ if they all agree.

Improved	β	1.583	1.603	1.630	1.645	1.670	1.690	1.717	1.743
	ξ	47.25(2)	53.25(3)	62.70(3)	68.71(4)	80.06(7)	90.38(8)	106.4(1)	124.7(2)
Standard	β	1.750	1.769	1.795	1.811	1.832	1.850	1.873	1.906
	ξ	47.24(1)	53.26(2)	62.83(2)	69.57(2)	79.59(5)	89.20(6)	103.1(1)	127.1(1)

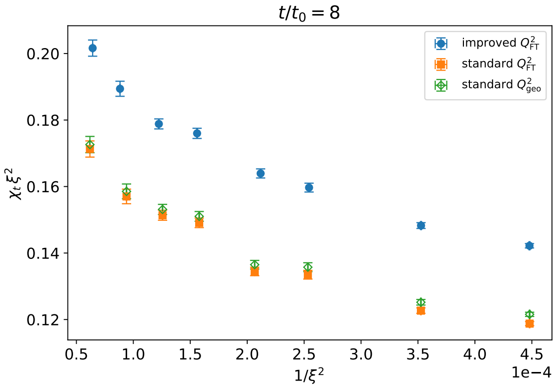
- We choose the reference flowtime via $t_0 = \xi^2/32$.
- We flow up to $8t_0$ or a radius of ξ.
- We choose $L = 10\xi$ for the measurement of χ.

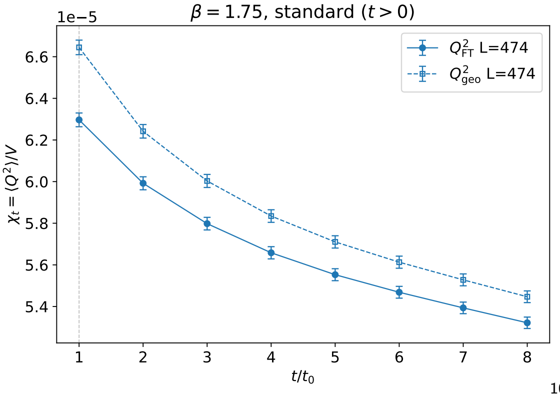
Systematics

- We have quantified all systematics explicitly.
  - Finite volume effects on ξ/a are $\ll 0.1\%$ for $L \geq 8\xi$.
  - Finite volume effects on the susceptibility are $\ll 1\%$ for $L \geq 8\xi$.
  - The systematic error from setting the flowtime via the correlation length is $\ll 0.1\%$.
  - Flowtime integration errors are at machine precision.

Results

- All errors are at or slightly below the percentage level.
- Improved action disagrees with standard action even at small lattice spacings.
- On the coarse lattices there are obvious discretization artifacts for the standard action at percentage-level: two discretizations of the topological charge disagree



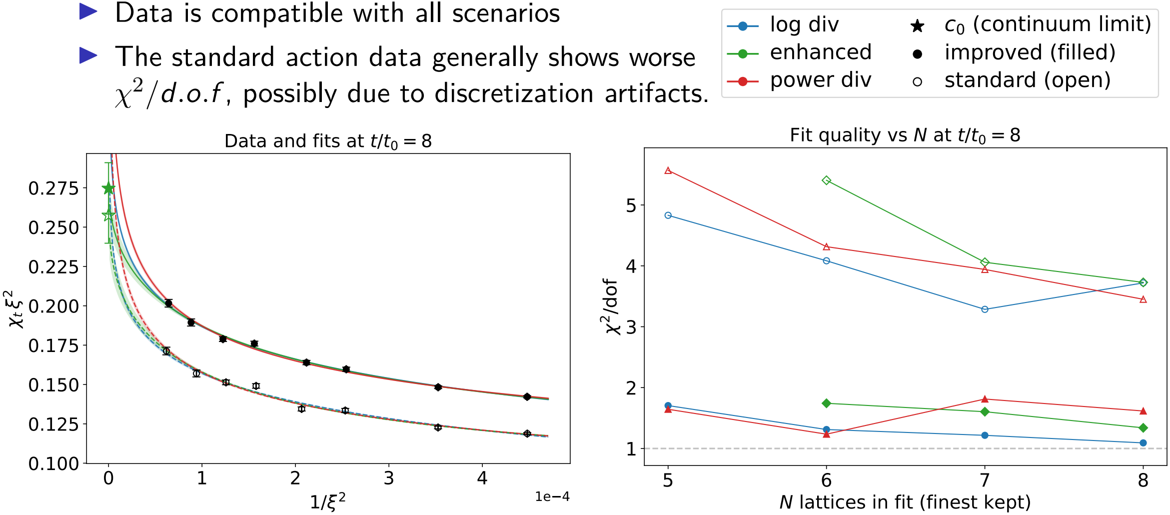


Fitting scenarios

- 3 scenarios suggest themselves:

  1. Enhanced discretization artifacts $\chi\xi^2 = c_0 + c_1 \frac{a^2}{\xi^2} \log\left(\xi/a\right)^p$
  2. Log divergence $\chi\xi^2 = b + c \log\left(\frac{\xi}{a}\right)$
  3. Power-law divergence $\chi\xi^2 = b \left(\frac{\xi}{a}\right)^c$

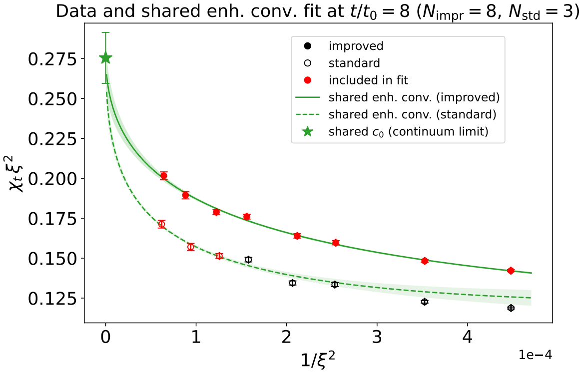
- Data is compatible with all scenarios
- The standard action data generally shows worse $\chi^2/d.o.f$, possibly due to discretization artifacts.

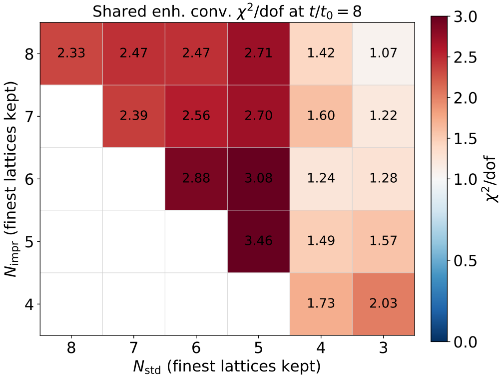


Fitting scenarios

- In the enhanced artifacts scenario we have (s: standard, i: improved)

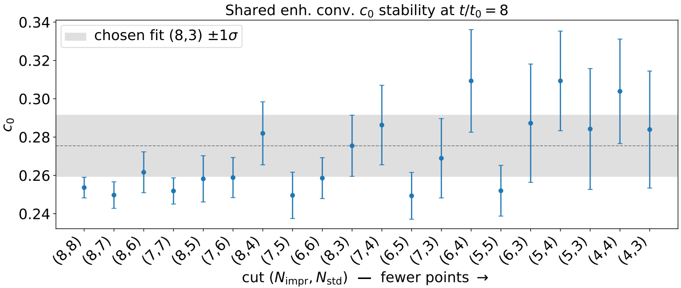
$$\chi_s\xi_s^2 = c_0 + c_{1,s}\frac{a^2}{\xi_s^2}\log\left(\xi_s/a\right)^p \qquad \chi_i\xi_i^2 = c_0 + c_{1,i}\frac{a^2}{\xi_i^2}\log\left(\xi_i/a\right)^{p-1}$$

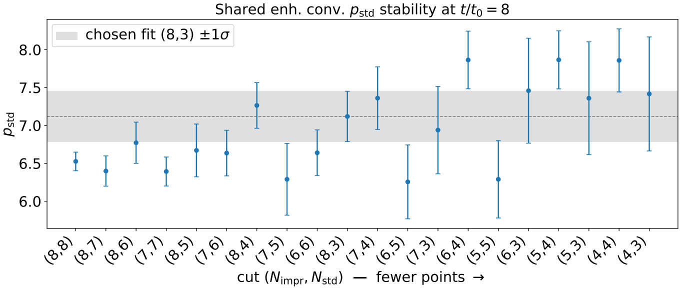




N_{impr} \ N_{std}	8	7	6	5	4	3
8	2.33	2.47	2.47	2.71	1.42	1.07
7		2.39	2.56	2.70	1.60	1.22
6			2.88	3.08	1.24	1.28
5				3.46	1.49	1.57
4					1.73	2.03

Fitting scenarios





Summary and the future

Summary:

- We have tree-level improved the gradient flow and probed $\chi\xi^2$ at large flowtimes and correlation lengths.
- We have quantified all systematic errors to make sure they are under control.
- While the divergence of $\chi\xi^2$ might be settled, the flowed case is not. We would like to "reopen" this question.

Future steps:

- Consider the correlator $\langle q(x)q(y)\rangle$ at positive flowtime to find out whether there are contact terms.

EFT analysis beyond tree-level

- Beyond tree-level, a proper EFT analysis is needed.
- Strategy (analogous to the YM-theory case):
  1. Consider theory in $(2+1)D$ and enforce flow equation through lagrange multiplier λ with partition function
  $$\mathcal{Z} = \int D[\phi]\ D[e]\ D[\lambda]\ \exp\left(-S[\phi] - S_{\text{fl}}[\phi, e, \lambda]\right),$$
  where flowtime is regulated via forward discretization and $[\lambda] = 2$.
  2. Classify operators up to mass dimension 4 by their $\mathcal{O}(\lambda)$.
  3. Use integration by parts and e.o.m insertions to reduce to desired operator basis.

EFT analysis beyond tree-level

- Only the boundary action matters as in the bulk at $t > 0$ there are no loop diagrams and tree-level improvement does the job[11].
- After reduction the most general boundary Symanzik action at $O(a^2)$ is

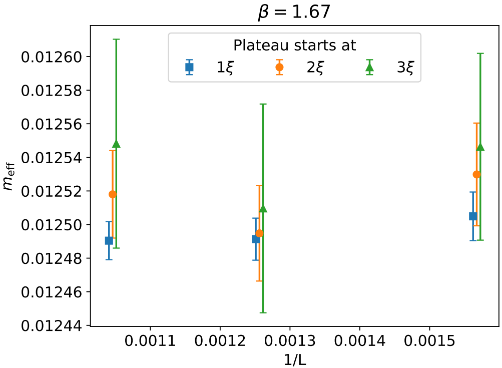
$$S_2^{(\mathrm{bdy})} = \int d^2x \left[c_b\, \lambda^i(0,x)\, \Delta\phi^i(x) + \sum_\alpha d_\alpha\, \mathcal{O}_\alpha^{(\mathrm{A})}\right],$$

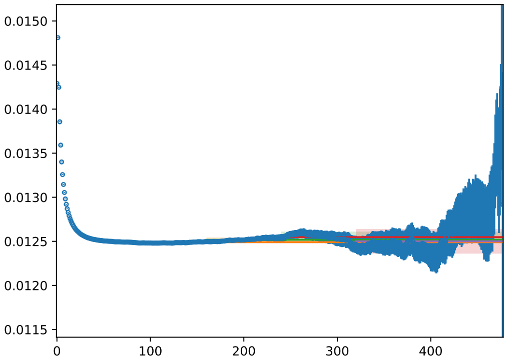
  where $\mathcal{O}_\alpha^{(\mathrm{A})}$ show up in the standard improvement of the action
- As in the YM case, c_b can be incorporated by a change of initial condition

$$e^i(0,x) = \phi^i(x) + a^2\, c_b\, (\delta^{ij} - \phi^i(t,x)\phi^j(t,x))\Delta\phi^j(t,x) + O(a^4)\,.$$

[11] Hiroki Makino and Hiroshi Suzuki. In: *PTEP* 2015.3 (2015).

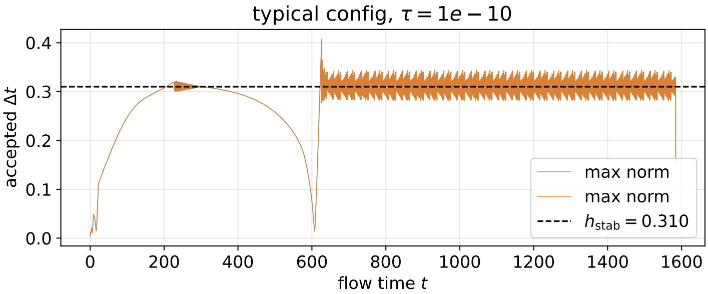
- To determine ξ we systematically fit multiple plateaus at $L = 8\xi, 10\xi, 12\xi$ and then select the one that starts at ξ for $L = 12\xi$ if they all agree.





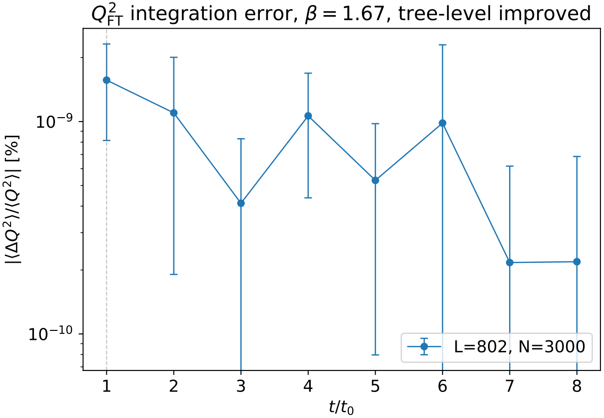
Choice of integrator

- Considered an *RK*4 and a Dormand Prince adaptive *RK*5 scheme.
- The latter turns out to be much more efficient as the integrator is stability bound along the much of the trajectory, even for small tolerances.



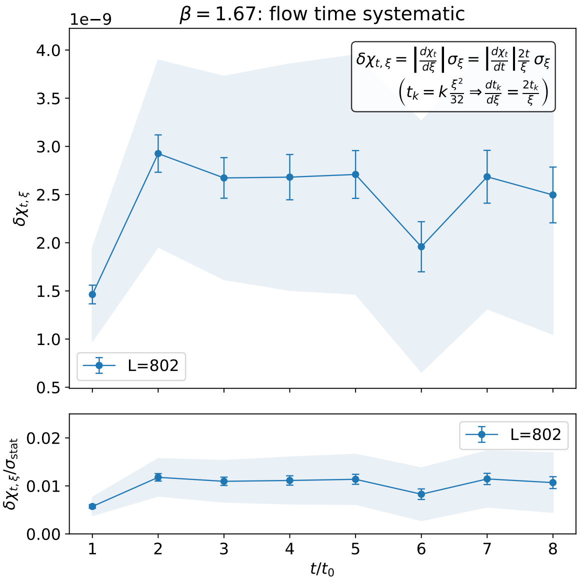
Flowtime integration

- To estimate the flowtime integration error, for all runs done we rerun 10% of the ensemble with stepsizes $(\Delta t)' = \Delta t/2$.
- In all cases, integration errors are at machine precision.



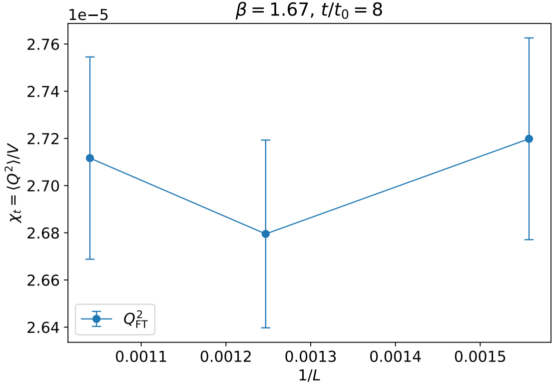
Choice of reference flowtime

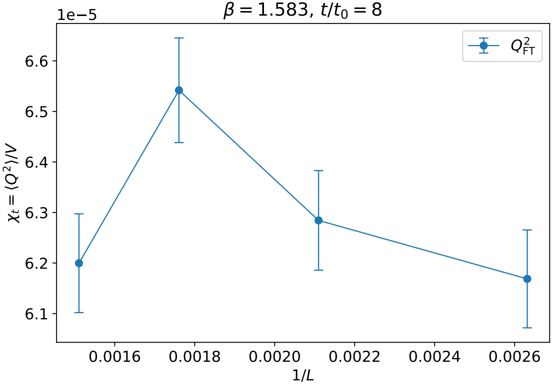
- We estimate the systematic error on χ coming from this by looking at the slope.
- They are negligible in all cases.



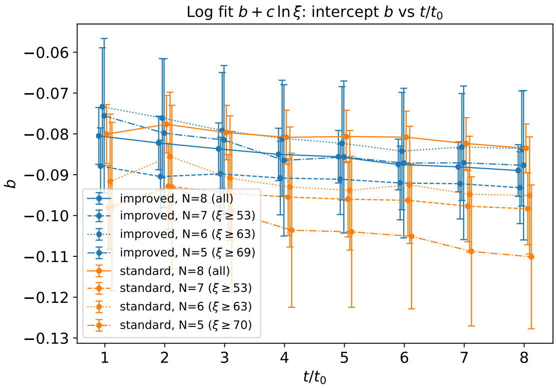
Finite Volume effects

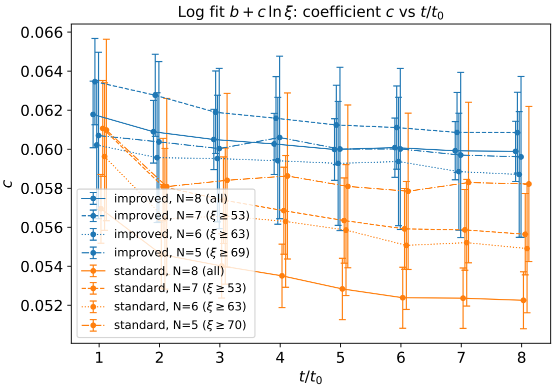
- For the coarsest and one of the finer lattice spacings, we estimate finite volume effects by comparing aspect ratios $8\xi, 10\xi, 12\xi, (14\xi)$



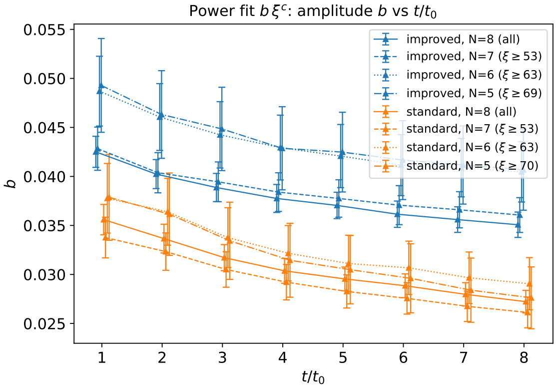


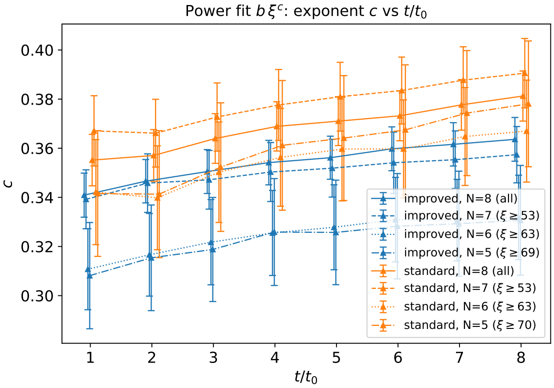
Stability for log fits (impr)



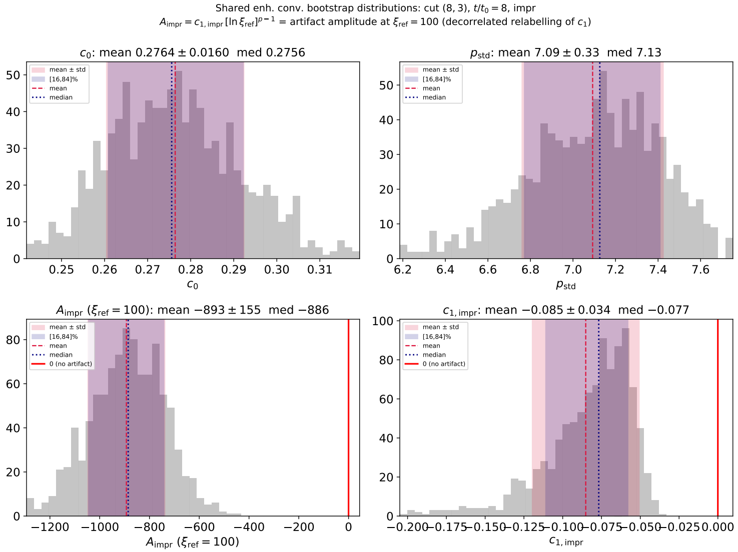


Stability for power law fits (impr)

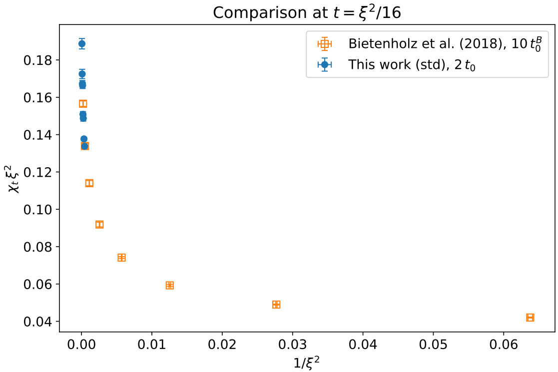


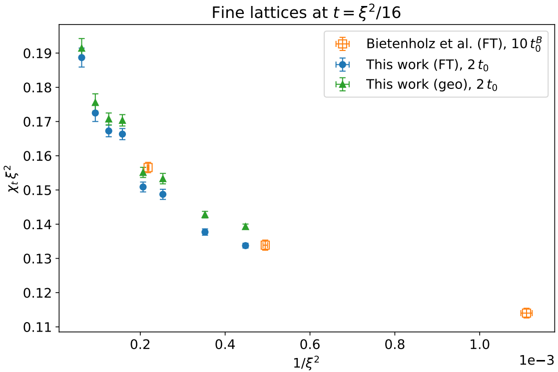


Bootstrap fit distributions for shared enh. conv.



Comparing to[12]



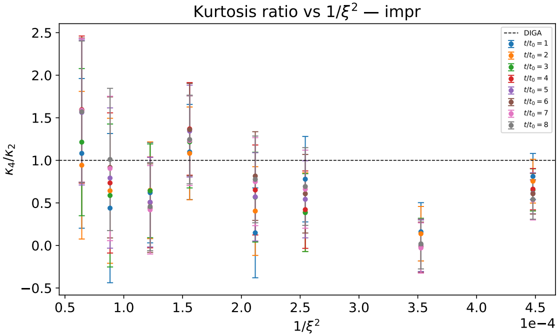


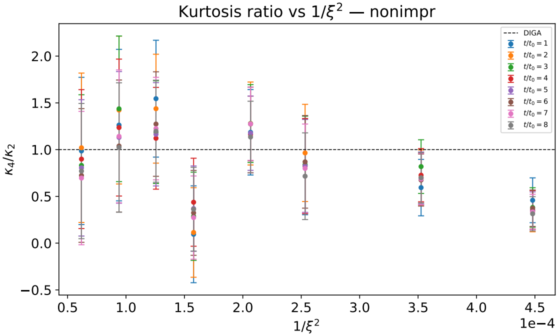
[12] Wolfgang Bietenholz et al. In: *Physical Review D* 98.11 (Dec. 2018).

Kurtosis

- Kurtosis

$$\frac{\kappa_4}{\kappa_2} = \frac{\langle Q^4 \rangle - 3\langle Q^2 \rangle^2}{\langle Q^2 \rangle}$$





RG scaling defect