Systematic uncertainties in spectral reconstruction from lattice correlators

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加速器だから見える世界。

Introduction

For related works, see

- S. Hashimoto, July 27, 11:00-
- S. Itatani, July 29, 9:20-

Euclidean correlator and spectral function

Euclidean correlation function

- evaluate with lattice QCD

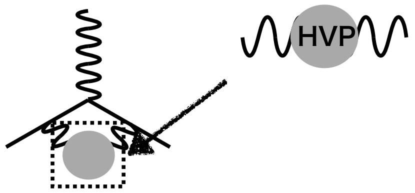
$$\underline{C(t)} = \int_\omega \underline{\rho(\omega)} \mathrm{e}^{-\omega t}$$

Spectral function

Can we reconstruct $\rho(\omega)$ from $C(t)$?

Many applications for inclusive observables

- Hadronic vacuum polarization in $g-2$

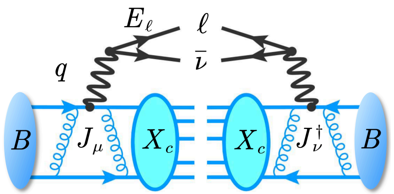


$$= \Pi_{\mu\nu}(q) = (q_\mu q_\nu - \delta_{\mu\nu} q^2)\Pi(q^2)$$

$$= \int \boldsymbol{d^4 x e^{iqx} \langle V_\mu^{\mathrm{em}}(x) V_\nu^{\mathrm{em}}(0)\rangle}$$

$$\longrightarrow \quad \Pi(q^2) - \Pi(0) = q^2 \int_0^\infty ds \frac{\rho(\sqrt{s})}{s(s+q^2)}$$

- Inclusive semileptonic decays



$$\boldsymbol{C_{\mu\nu}(q, t)} = \int_0^\infty d\omega W^{\mu\nu}(\boldsymbol{q}, \omega) e^{-\omega t}$$

$$\longrightarrow \quad \Gamma \sim \int_0^{\boldsymbol{q}^2_{\max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2} \sum_{l=0}^{2} \bar{X}^{(l)}(\boldsymbol{q}^2)$$

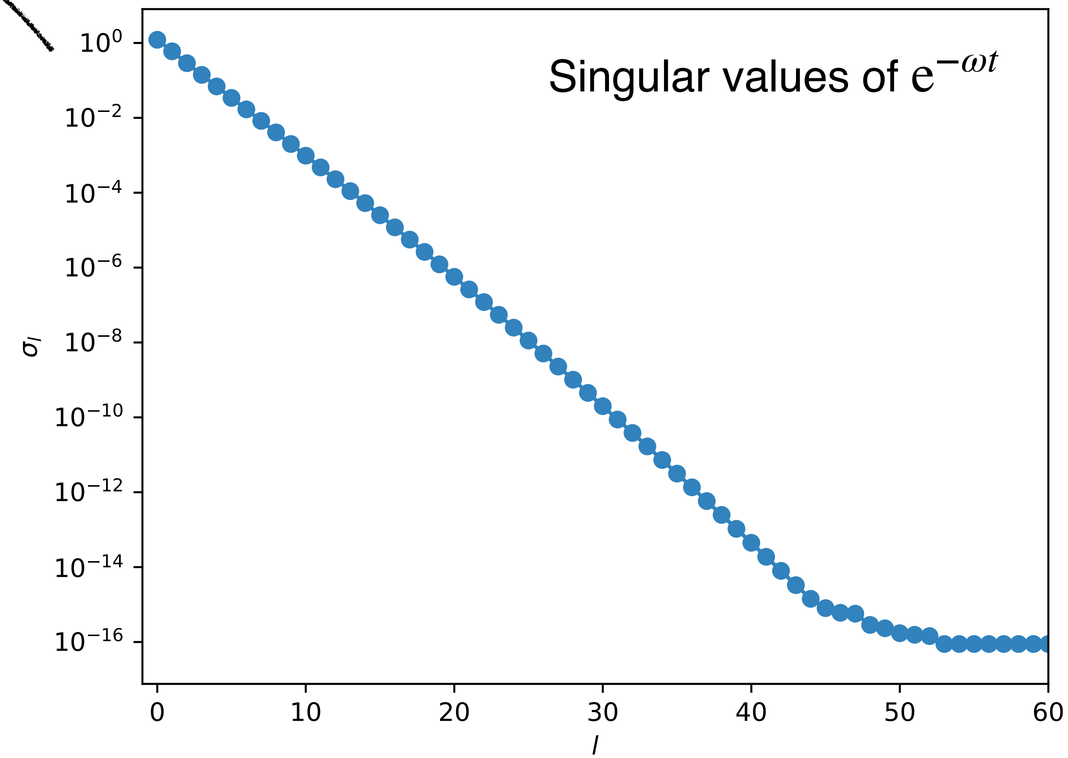
$$\bar{X}^{(l)}(\boldsymbol{q}^2) = \int_0^\infty d\omega W^{\mu\nu}(\boldsymbol{q}, \omega) K^{(l)}_{\mu\nu}(\boldsymbol{q}, \omega)$$

What data can tell us

$$C(t) = \int_\omega \rho(\omega) \mathrm{e}^{-\omega t} \simeq \sum_{l=0}^{N-1} U_l(t) \sigma_l \int_\omega \rho(\omega) V_l(\omega)$$

SVD

$$e^{-\omega t} \simeq \sum_{l=0}^{N-1} U_l(t) \sigma_l V_l(\omega)$$



$C(t)$ can't have all information of $\rho(\omega)$ through the Laplace

$\longrightarrow$ $\rho(\omega)$ can't be fully reconstructed from $C(t)$

Ill-posed inverse problem

Keep only the "relevant" components to approximate $\rho_{\sigma_{\mathrm{smr}}}(\omega)$ from $C(t)$

Rough sketch

Spectral reconstruction through SVD

$$\rho_l = \int_\omega \rho(\omega) V_l(\omega)$$

$$C(t) = \int_\omega \rho(\omega) \mathrm{e}^{-\omega t} \simeq \sum_{l=0}^{N-1} U_l(t)\sigma_l \int_\omega \rho(\omega) V_l(\omega) = \sum_{l=0}^{N-1} U_l(t)\sigma_l \rho_l$$

SVD

$$e^{-\omega t} \simeq \sum_{l=0}^{N-1} U_l(t)\sigma_l V_l(\omega)$$ on space projected by $P_U(i,j) = \sum_{l=0}^{N-1} \tilde{U}_l(t_i)\tilde{U}_l(t_j)$,$P_V(i,j) = \sum_{l=0}^{N-1} \tilde{V}_l(\omega_i)\tilde{V}_l(\omega_j)$

$$\longrightarrow \quad \rho^{(N)}(\omega) \approx \sum_{l=0}^{N_{\mathrm{tr}}} V_l(\omega) \cdot \rho_l \quad \text{with } \rho_l = \frac{1}{\sigma_l}\int_t U_l(t) C(t)$$

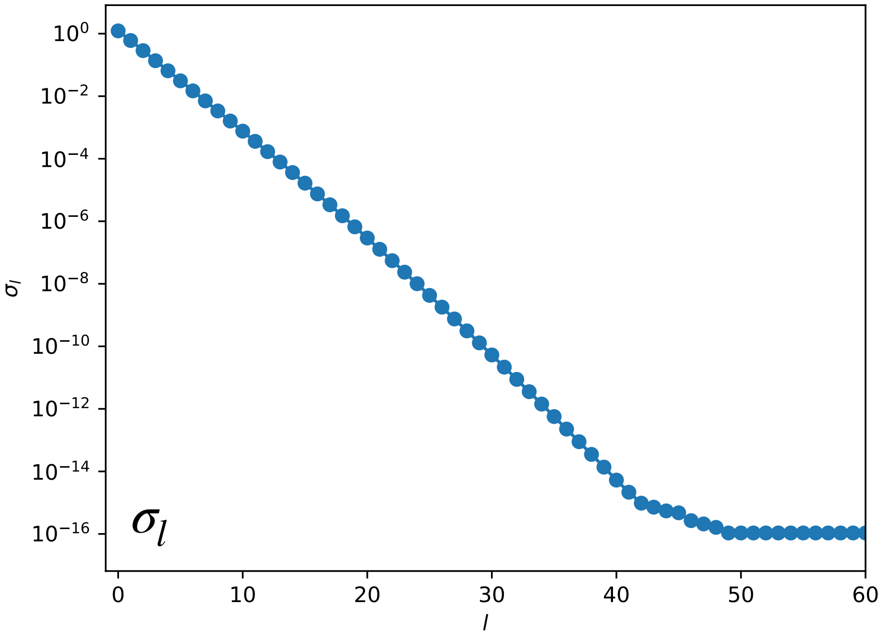
$$|\rho^{(N)}(\omega) - \rho^{(N_{\mathrm{tr}})}(\omega)| = \left| \sum_{l=N_{\mathrm{tr}}}^{N-1} V_l(\omega)\rho_l \right| \le \sum_{l=N_{\mathrm{tr}}}^{N-1} \left| V_l(\omega) \right| \cdot |\rho_l|$$

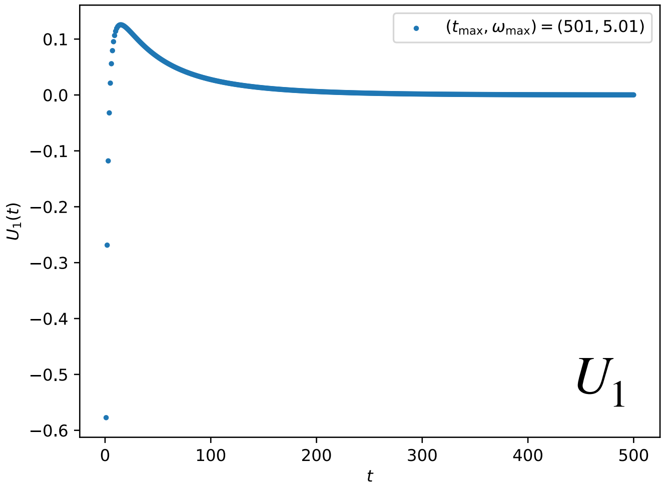
Singular value decomposition approach

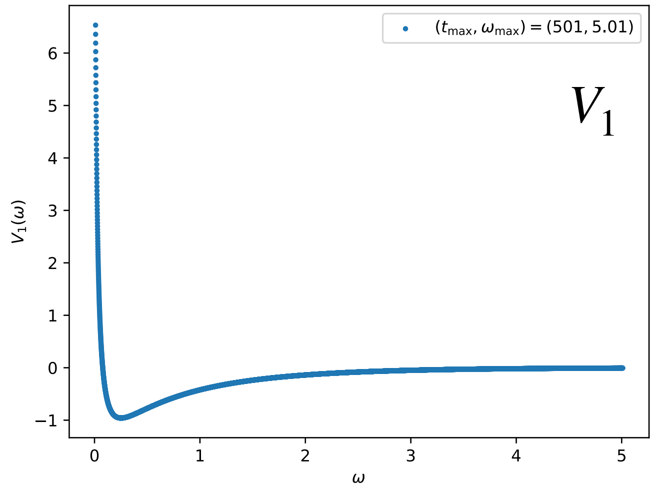
Singular value decomposition

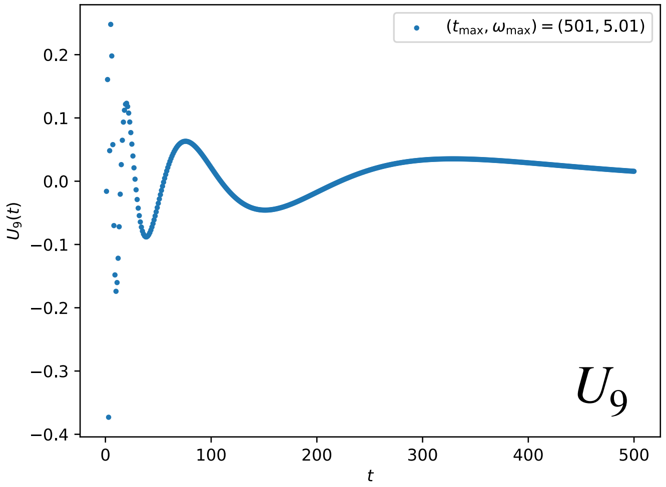
$$e^{-\omega t} \simeq \sum_{l=0}^{N-1} U_l(t)\sigma_l V_l(\omega)$$

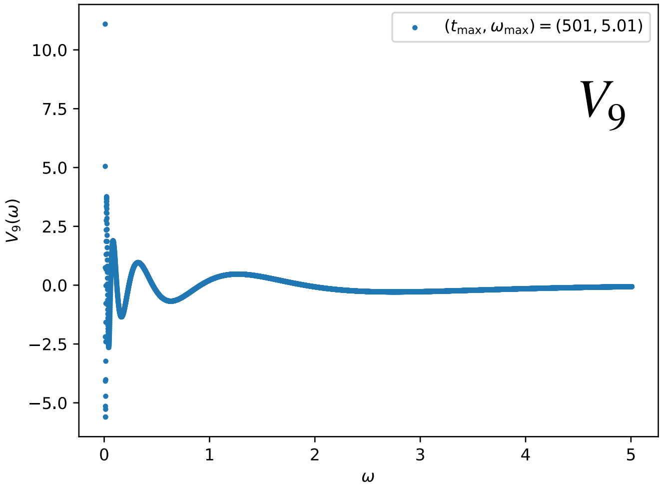
$t \in [1{,}501]$ with $\Delta_t = 1$
$\omega \in [0.01{,}5.0095]$ with $\Delta_\omega = 0.0005$











We can prepare the σ_l, $U_l(t)$, and $V_l(\omega)$ numerically…But what are they?

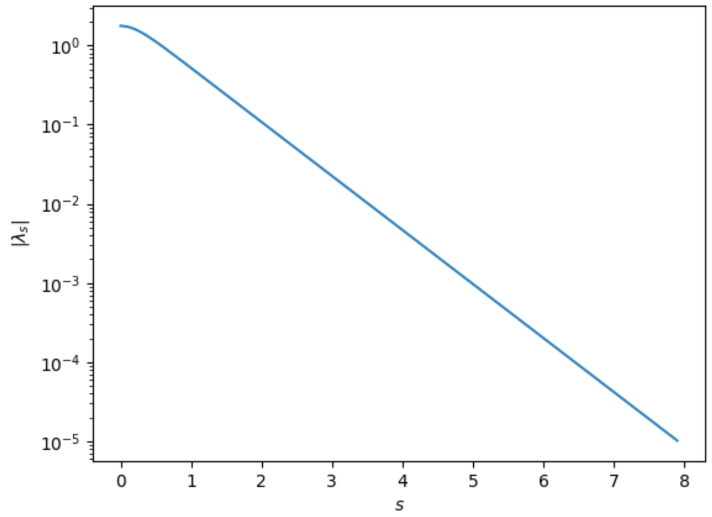
Mellin transform

M. Bruno, L. Giusti and M. Saccardi, Phys. Rev. D 111 (2025) 9, 094515.
L. Giusti, M. Saccardi, and D. Toniolo, arXiv:2606.28167 (2026).

For $t \in [0,\infty)$, $\omega \in [0,\infty)$ and $s \in \mathbb{R}$,

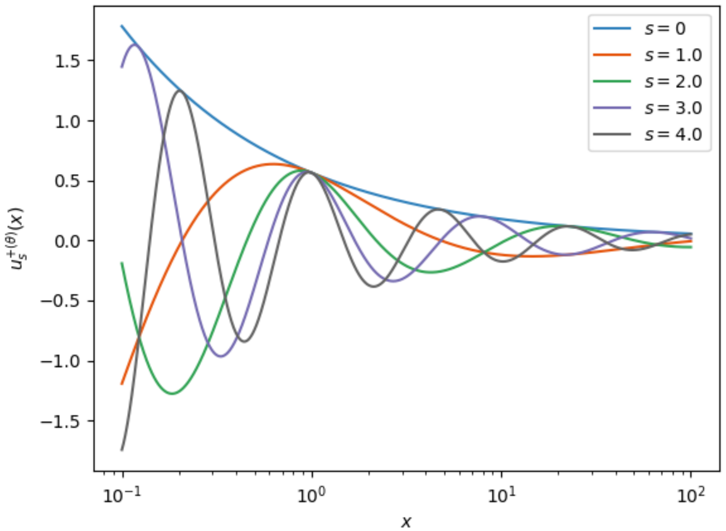
$$e^{-\omega t} = \int_0^\infty ds\,|\lambda_s|\left(u_s^{+(\theta)}(\omega)u_s^{+(\theta)}(t) - u_s^{-(\theta)}(\omega)u_s^{-(\theta)}(t)\right)$$

with $\quad |\lambda_s| = \left|\Gamma(\frac{1}{2}+is)\right| = \sqrt{\frac{\pi}{\cosh(\pi s)}}$



$$u_s^{+(\theta)}(x) = \frac{\cos\left(s\ln(x) - \frac{\theta(s)}{2}\right)}{\sqrt{\pi x}},$$

$$u_s^{-(\theta)}(x) = \frac{\sin\left(s\ln(x) - \frac{\theta(s)}{2}\right)}{\sqrt{\pi x}},$$



where $x = \omega$ or t, and $\theta(s) = \arg(\lambda_s)$

Similar to the Fourier-transform

$\longrightarrow$ The $s \in \mathbb{R}$ acts as the “wave number”

Connection to the SVD basis functions

For $t \in [0,\infty)$, $\omega \in [0,\infty)$ and $s \in \mathbb{R}$,

$$e^{-\omega t} = \int_0^\infty ds\, |\lambda_s| \left(u_s^{+(\theta)}(\omega)u_s^{+(\theta)}(t) - u_s^{-(\theta)}(\omega)u_s^{-(\theta)}(t)\right)$$

$$\boxed{e^{-\omega t} = \sum_l U_l(t)\sigma_l V_l(\omega)}$$

with $$|\lambda_s| = \left|\Gamma(\frac{1}{2} + is)\right| = \sqrt{\frac{\pi}{\cosh(\pi s)}} \longrightarrow \sigma_l\ ?$$

with Finite grid of t, ω

$s \to s_l \sim l\Delta s$

with $\Delta s \sim \dfrac{\pi}{\ln(x_{\max}/x_{\min})}$

$$u_s^{+(\theta)}(x) = \frac{\cos\left(s\ln(x) - \frac{\theta(s)}{2}\right)}{\sqrt{\pi x}},$$

$$u_s^{-(\theta)}(x) = \frac{\sin\left(s\ln(x) - \frac{\theta(s)}{2}\right)}{\sqrt{\pi x}}, \longrightarrow U_l(t), V_l(\omega)\ ?$$

where $x = \omega$ or t, and $\theta(s) = \arg(\lambda_s)$

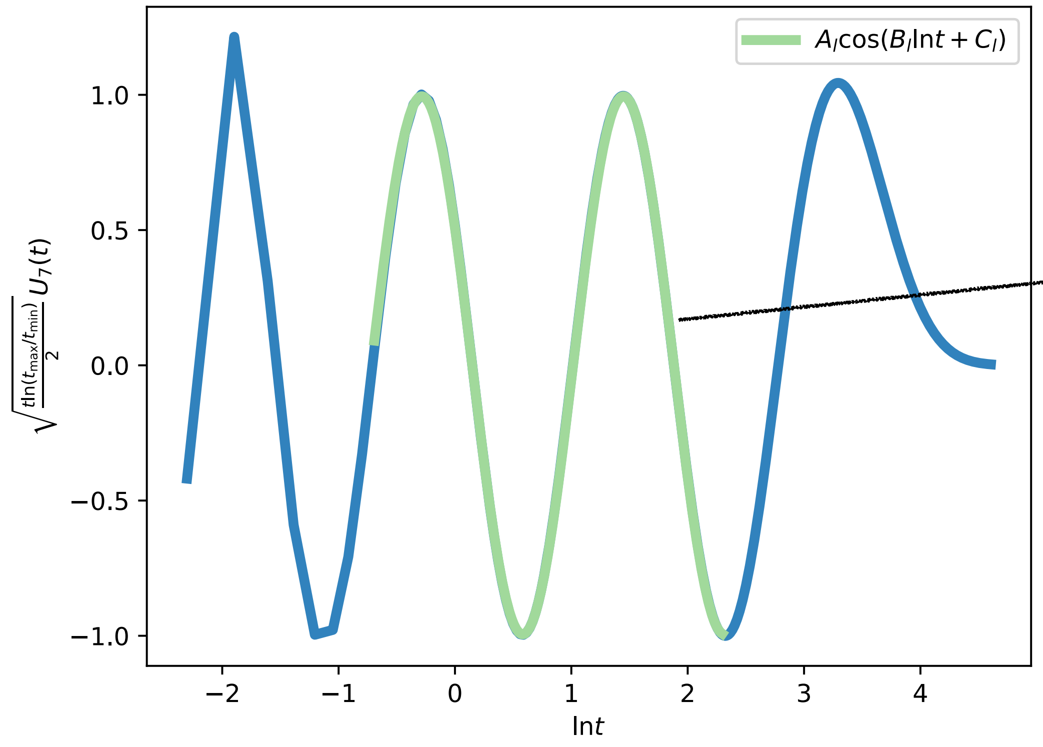
$\longrightarrow$ *Does the σ_l, $U_l(t)$, and $V_l(\omega)$ from SVD give numerical discretization of the Mellin?*

SVD and Mellin

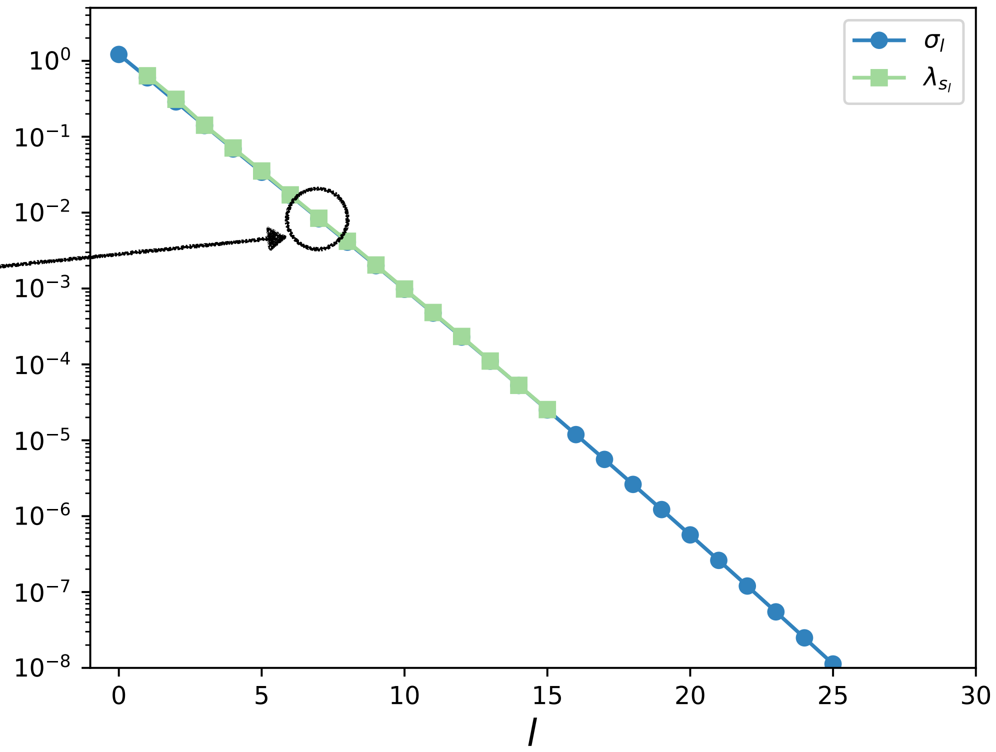
$$e^{-\omega t} \simeq \sum_{l=0}^{N-1} U_l(t)\sigma_l V_l(\omega)$$

For $\omega, t \in [0.1, 100.1]$, $\Delta = 0.05$, **Numerical data of U_l reproduce $u_{s_l}^{\pm(\theta)}$ and λ_{s_l}**

$\sqrt{t}U_7 \propto \cos(s_7 \ln(t) + C_7)$



σ_l or $|\lambda_{s_l}| = \sqrt{\pi/\cosh(\pi s_l)}$



For small l, SVD with finite grid of $t, \omega \approx$ the Mellin transform

l in the SVD corresponds to s in the Mellin $\rightarrow$ Fine structure of $\rho(\omega)$ is described by large l

Chart sheet

M. Bruno, L. Giusti and M. Saccardi, Phys. Rev. D 111 (2025) 9, 094515.
L. Giusti, M. Saccardi, and D. Toniolo, arXiv:2606.28167 (2026).
R. Tsuji, and S. Hashimoto, arXiv:2605.14652.

SVD provides a discretization of the Mellin based reconstruction

SVD		Mellin
$\sum_{l=0}^{N-1} U_l(t)\sigma_l V_l(\omega)$	$e^{-\omega t}$	$\int_0^\infty ds\,\lvert\lambda_s\rvert\left(u_s^{+(\theta)}(\omega)u_s^{+(\theta)}(t) - u_s^{-(\theta)}(\omega)u_s^{-(\theta)}(t)\right)$
$\sum_{i=1}^{N_t} \tilde{U}_l(t_i)\tilde{U}_{l'}(t_i) = \delta_{l,l'}\,,\ \sum_{j=1}^{N_\omega} \tilde{V}_l(\omega_j)\tilde{V}_{l'}(\omega_j) = \delta_{l,l'}$	**Orthogonality and Completeness**	$\int_x u_s^{\pm(\theta)}(x)u_{s'}^{\pm(\theta)}(x) = \delta(s-s')\,,\ \int_x u_s^{\pm(\theta)}(x)u_{s'}^{\mp(\theta)}(x) = 0$
$\sum_{l=0}^{N-1} \tilde{U}_l(t_i)\tilde{U}_l(t_j) = P_U(i,j)\,,\ \sum_{l=0}^{N-1} \tilde{V}_l(\omega_i)\tilde{V}_l(\omega_j) = P_V(i,j)$		$\int_0^\infty ds\left[u_s^{+(\theta)}(x)u_s^{+(\theta)}(x') \pm u_s^{-(\theta)}(x)u_s^{-(\theta)}(x')\right] = \delta(x-x')$
$\rho_l = \sum_{j=1}^{N_\omega} \tilde{V}_l(\omega_j)\tilde{\rho}(\omega_j) = \frac{1}{\sigma_l}\sum_{i=1}^{N_t} \tilde{U}_l(t_i)\tilde{C}(t_i)$	**Modal decomposition**	$\rho_s^\pm = \int_0^\infty d\omega\, u_s^{\pm(\theta)}(\omega)\rho(\omega) = \pm\frac{1}{\lvert\lambda_s\rvert}\int_0^\infty dt\, u_s^{\pm(\theta)}(t)C(t)$
$\rho(\omega) \approx \sum_{l=0}^{N_{\mathrm{tr}}} \tilde{V}_l(\omega)\rho_l$	**Approximation**	$\rho(\omega) \approx \int_0^{\Lambda_s} ds\, u_s^{+(\theta)}(\omega)\rho_s^+ + \int_0^{\Lambda_s} ds\, u_s^{-(\theta)}(\omega)\rho_s^-$

Systematic error of inversion

$$\rho^{(N_{\rm tr})}(\omega) \approx \sum_{l=0}^{N_{\rm tr}} V_l(\omega) \cdot \rho_l$$

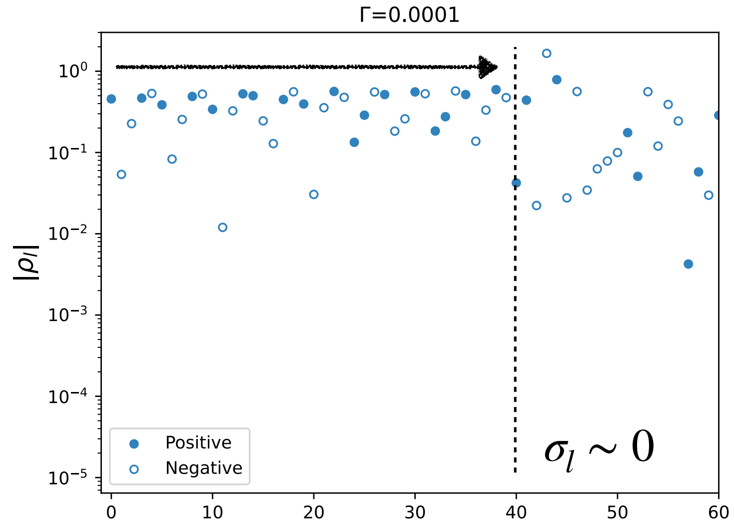
$$|\underbrace{\rho^{(N)}(\omega)}_{\approx \rho(\omega) \text{ for large } N} - \rho^{(N_{\rm tr})}(\omega)| = \left| \sum_{l=N_{\rm tr}}^{N-1} V_l(\omega) \cdot \rho_l \right|$$

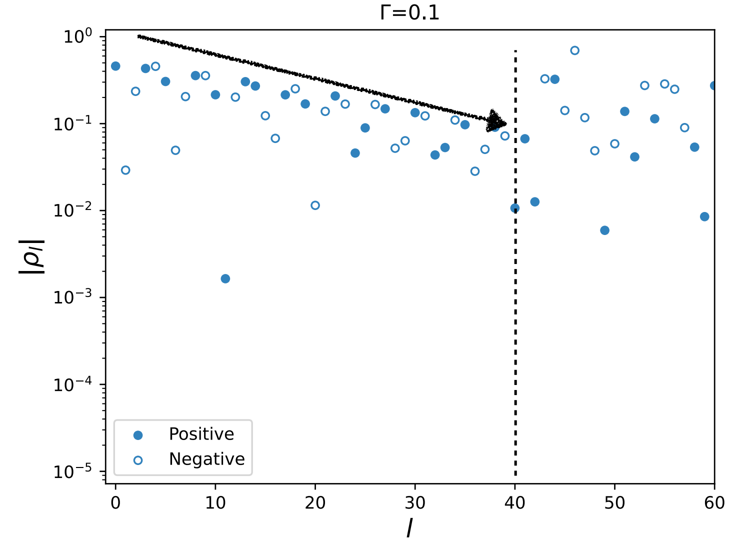
$$\leq \sum_{l=N_{\rm tr}}^{N-1} \left| V_l(\omega) \right| \cdot |\rho_l|$$

$$\lesssim \sum_{l=N_{\rm tr}}^{N-1} \left| V_l(\omega) \right| \cdot |A_\rho e^{-B_\rho l}|$$

Data fit (assumption)

e.g. $\rho(\omega) = \dfrac{1}{\pi}\dfrac{2\Gamma}{(\omega - 0.1)^2 + \Gamma^2}$





Systematic error = truncation of higher l modes

(※) Mathematical rigorous bounds is given by R. Abott 7/27 15:00- and W. Jay 7/27 15:20-.

Smeared spectral function

$$\rho^{(N_{\rm tr})}_{\sigma_{\rm smr}}(\omega) \approx \sum_{l=0}^{N_{\rm tr}} S_l(\omega) \cdot \rho_l \qquad \text{with } S_l(\omega) = \int d\omega' \, S_{\sigma_{\rm smr}}(\omega,\omega') V_l(\omega')$$

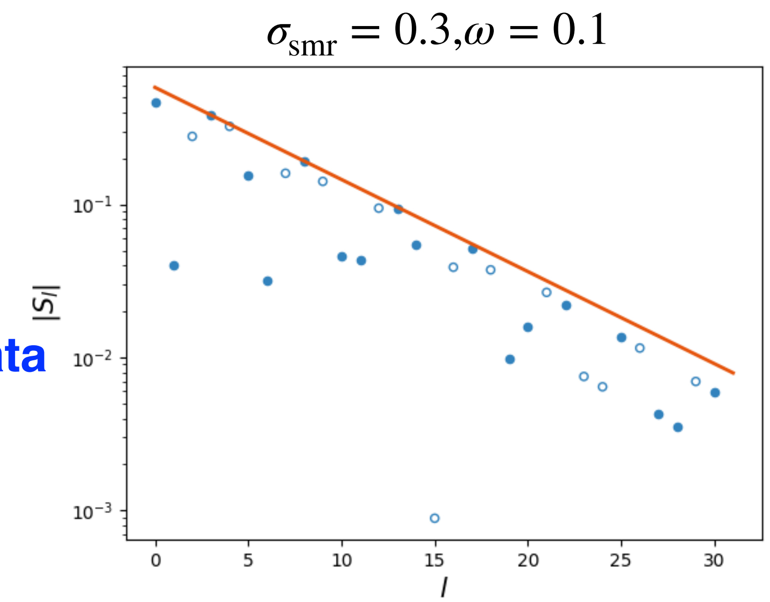
$$S_{\sigma_{\rm smr}}(\omega,\omega') = \frac{1}{\pi} \frac{2\sigma_{\rm smr}}{(\omega-\omega')^2 + \sigma_{\rm smr}^2}$$

$$|\rho^{(N)}_{\sigma_{\rm smr}}(\omega) - \rho^{(N_{\rm tr})}_{\sigma_{\rm smr}}(\omega)| \le \sum_{l=N_{\rm tr}}^{N-1} \left| S_l(\omega) \right| \cdot |\rho_l|$$

$$\lesssim \sum_{l=N_{\rm tr}}^{N-1} \left| A_{\sigma_{\rm smr}}(\omega) e^{-B_{\sigma_{\rm smr}}(\omega) l} \right| \cdot |A_\rho e^{-B_\rho l}|$$

Fixed for given data

Controllable for any $\sigma_{\rm smr}$



Smearing = Reduction of higher l modes contribution

For a real data from lattice

Can we reconstruct δ-function?

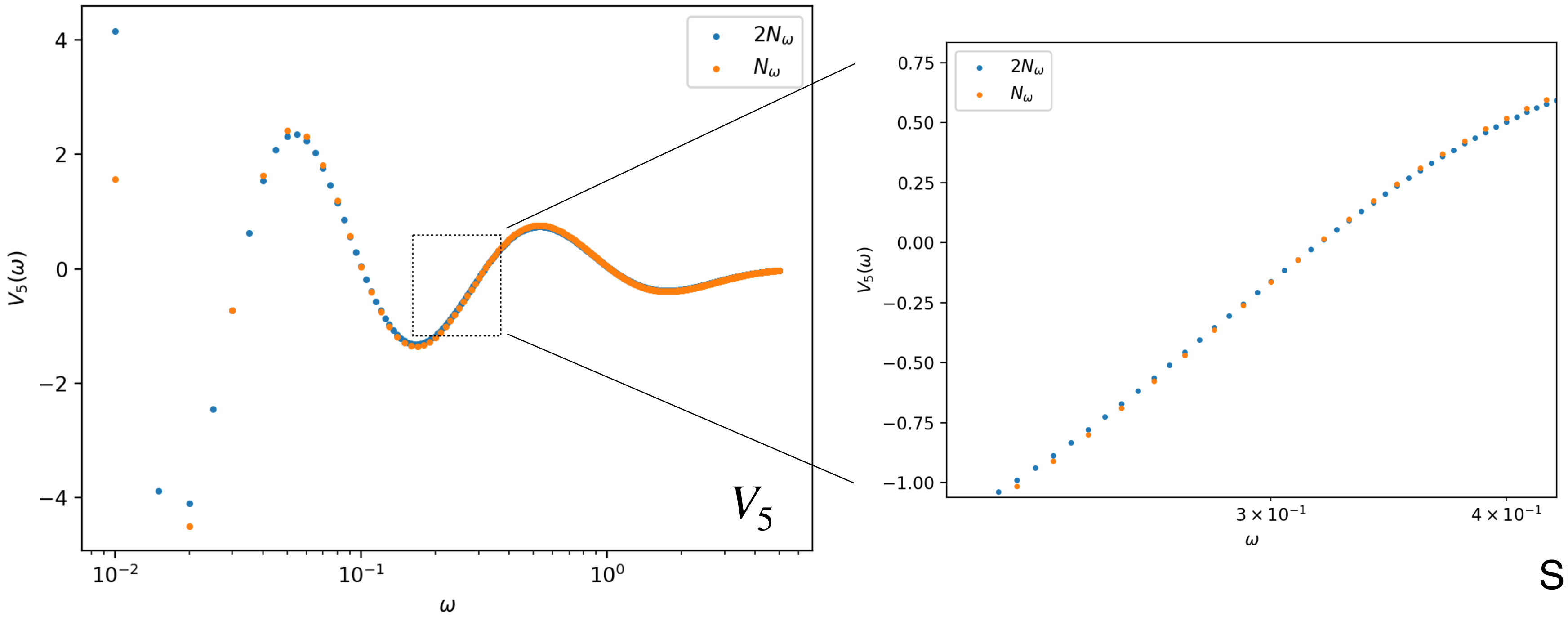
A. NO. it requires the Mellin with $s \in [0,\infty)$, i.e. $\mathrm{rank} = \infty$

$\longrightarrow$ SVD only reconstructs $\rho(\omega)$ projected onto SVD space

Resolution of $\rho(\omega)$ is controlled by rank

i.e. $\Delta_\omega \to 0$ with fixed Δ_t (=fixed rank) cannot give δ-function

e.g. Compare N_ω v.s. $2N_\omega$ for $V_l(\omega)$



Small Δ_ω fills the gap

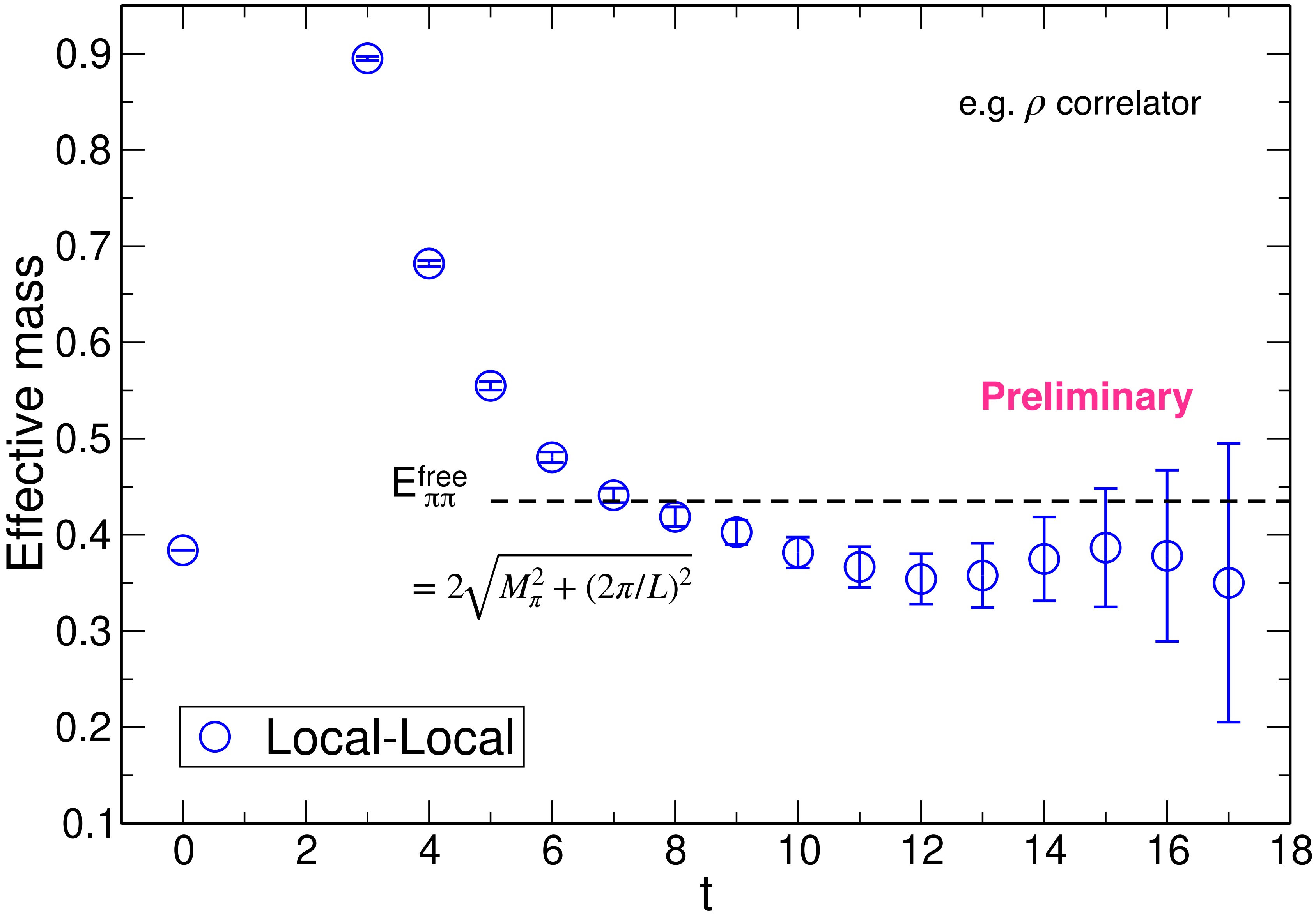
Numerical application

- $N_f = 2 + 1$ JLQCD ensemble for numerical simulation
- The tree-level improved Symanzik gauge action + Möbius domain-wall for all relevant quarks

$L^3 \times T \times L_s$	β	a^{-1} GeV	m_{ud}	m_s	M_π MeV	M_K MeV	N	B_s
$32^3 \times 64 \times 12$	4.17	2.453(44)	0.0035	0.0040	226(1)	525(1)	100	500

Simulation conducted on Fugaku using Grid [P. Boyle et al., *https://github.com/pabole/Grid*] and Hadrons [A. Portelli et al, *https://github.com/aportelli/Hadrons*] software packages

For a real data from lattice



Normalization

$$\overline{C}(t) = \frac{C(t+2)}{C(2)}$$

$$\longrightarrow \overline{C}(t) = \int_0^\infty d\omega e^{-\omega t}\bar{\rho}(\omega)$$

How the $\rho_{\sigma_{\rm smr}}(\omega)$ looks like?

Modal decomposition of $\rho(\omega)$

$$e^{-\omega t} \simeq \sum_{l=0}^{N-1} U_l(t)\sigma_l V_l(\omega)$$

$t \in [0{,}21]\ \Delta_t = 1$
$\omega \in [0.01{,}3.097]\ \Delta_\omega = 0.003$

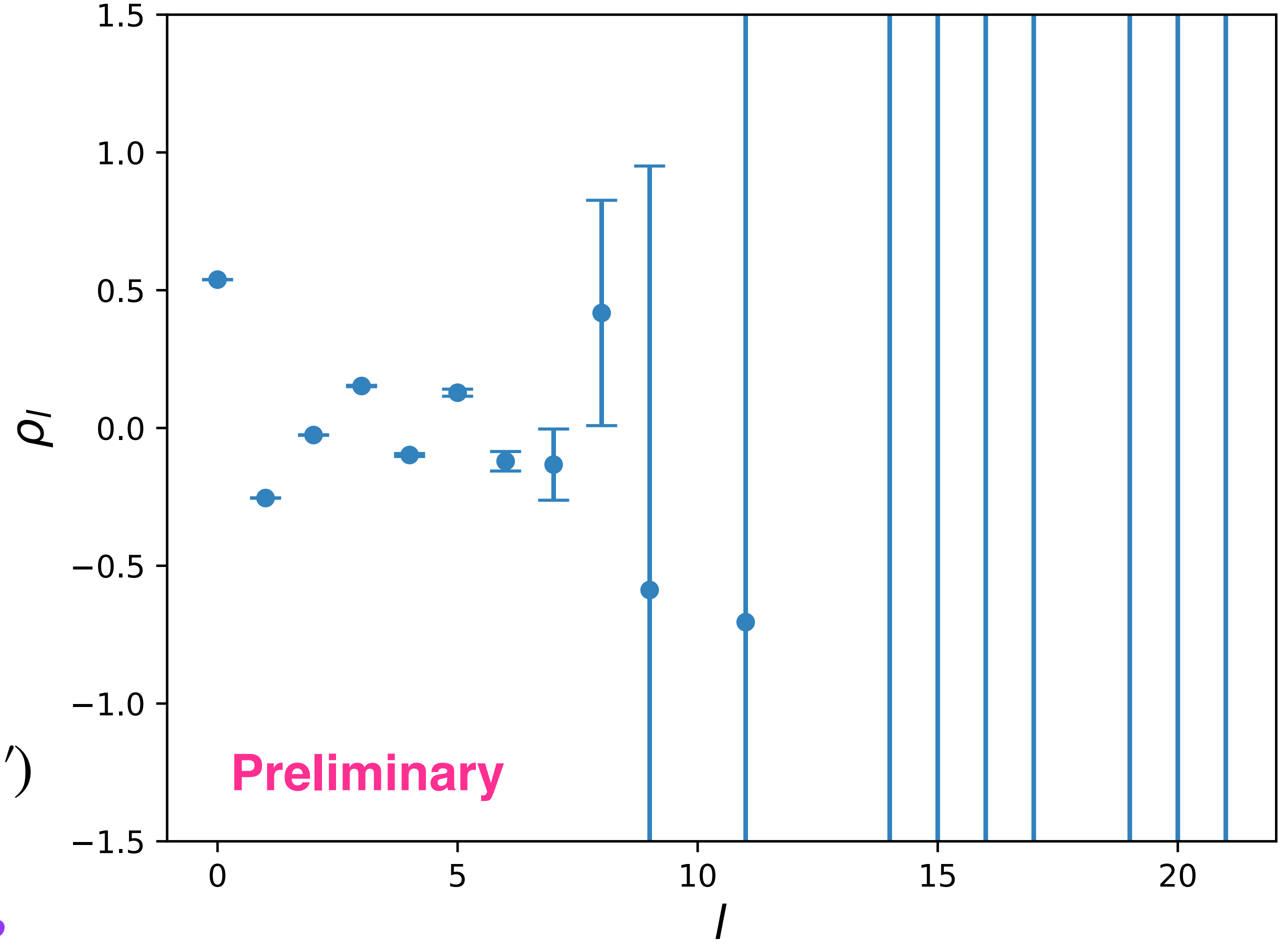
$$\longrightarrow \quad \rho_l = \int_\omega \rho(\omega) V_l(\omega) = \frac{1}{\sigma_l}\int_t U_l(t) C(t)$$

$$\longrightarrow \quad \rho^{(N_{\rm tr})}_{\sigma_{\rm smr}}(\omega) \approx \sum_{l=0}^{N_{\rm tr}} S_l(\omega)\cdot\rho_l$$

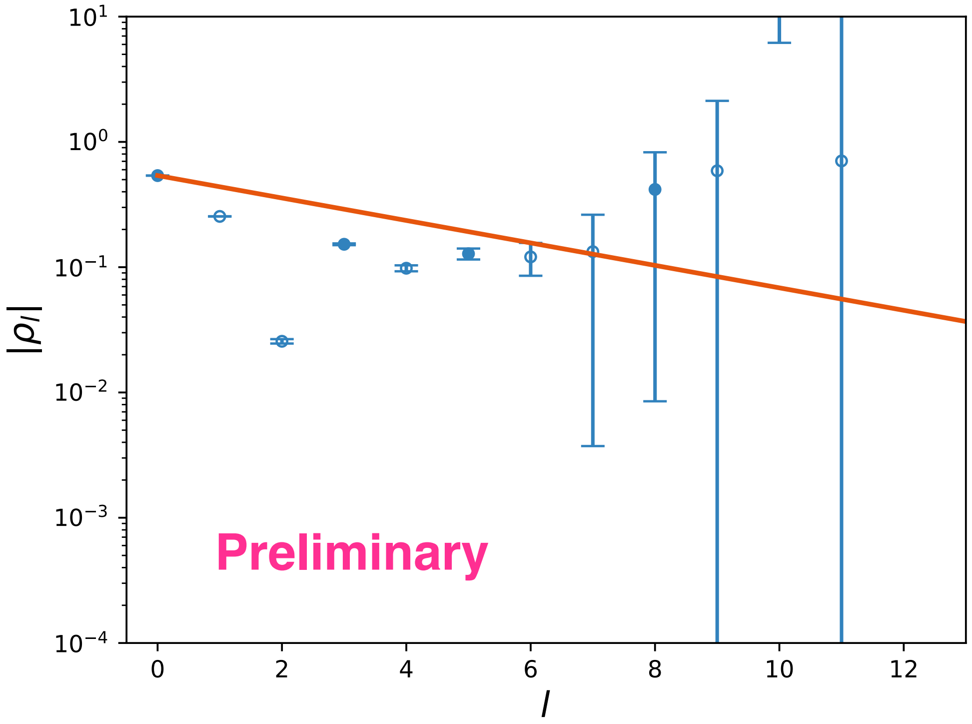
with $S_l(\omega) = \int d\omega'\, S_{\sigma_{\rm smr}}(\omega,\omega')V_l(\omega')$

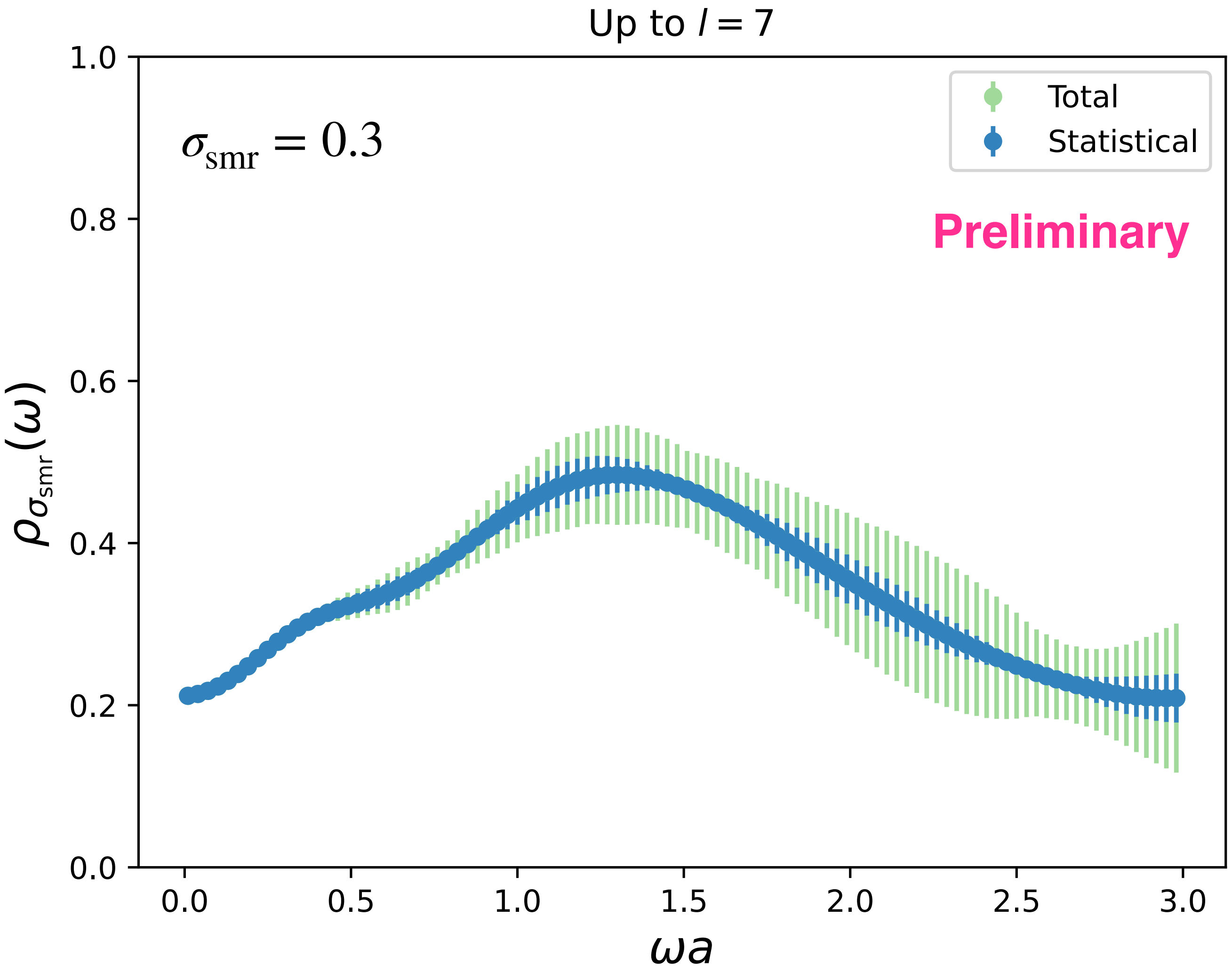
Truncation error?

Data only carries "8" SVD components



Smeared Spectrum



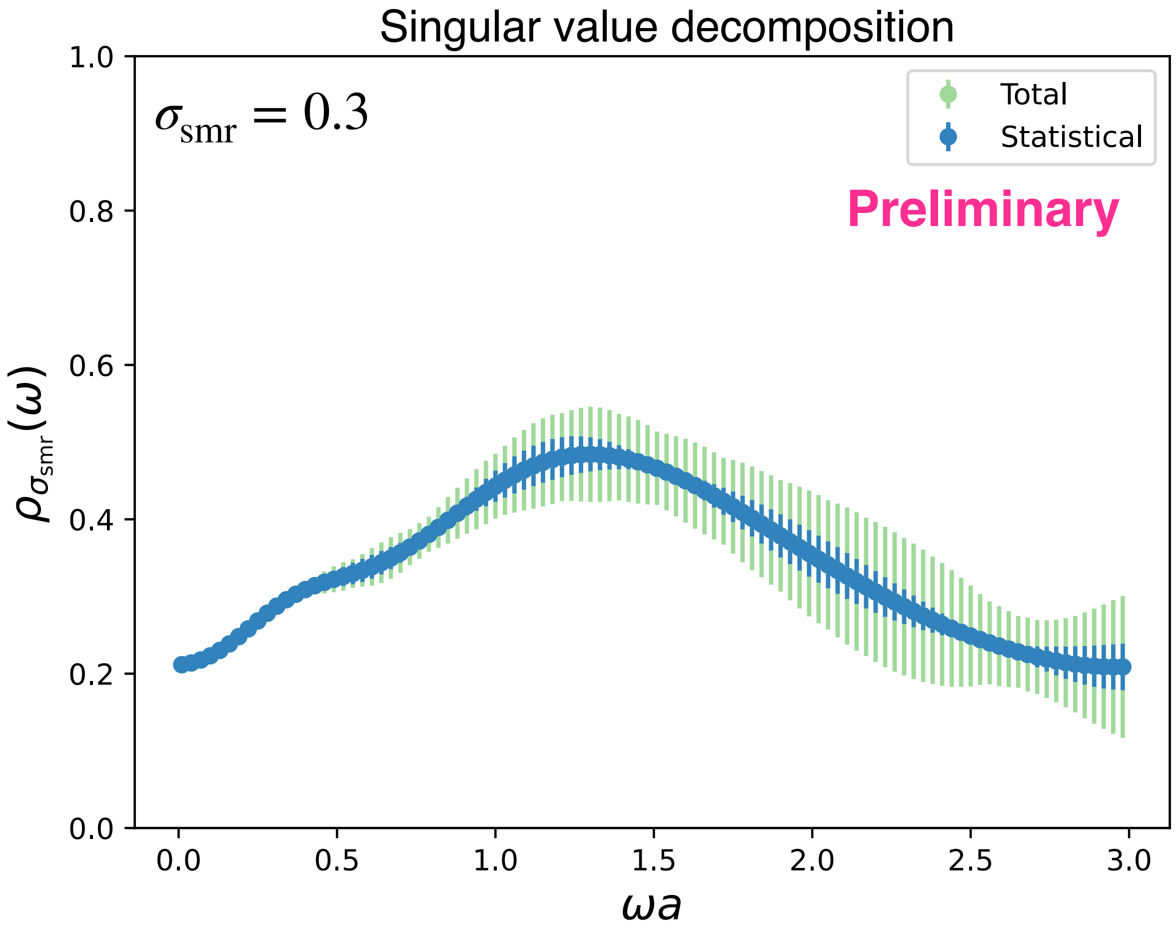


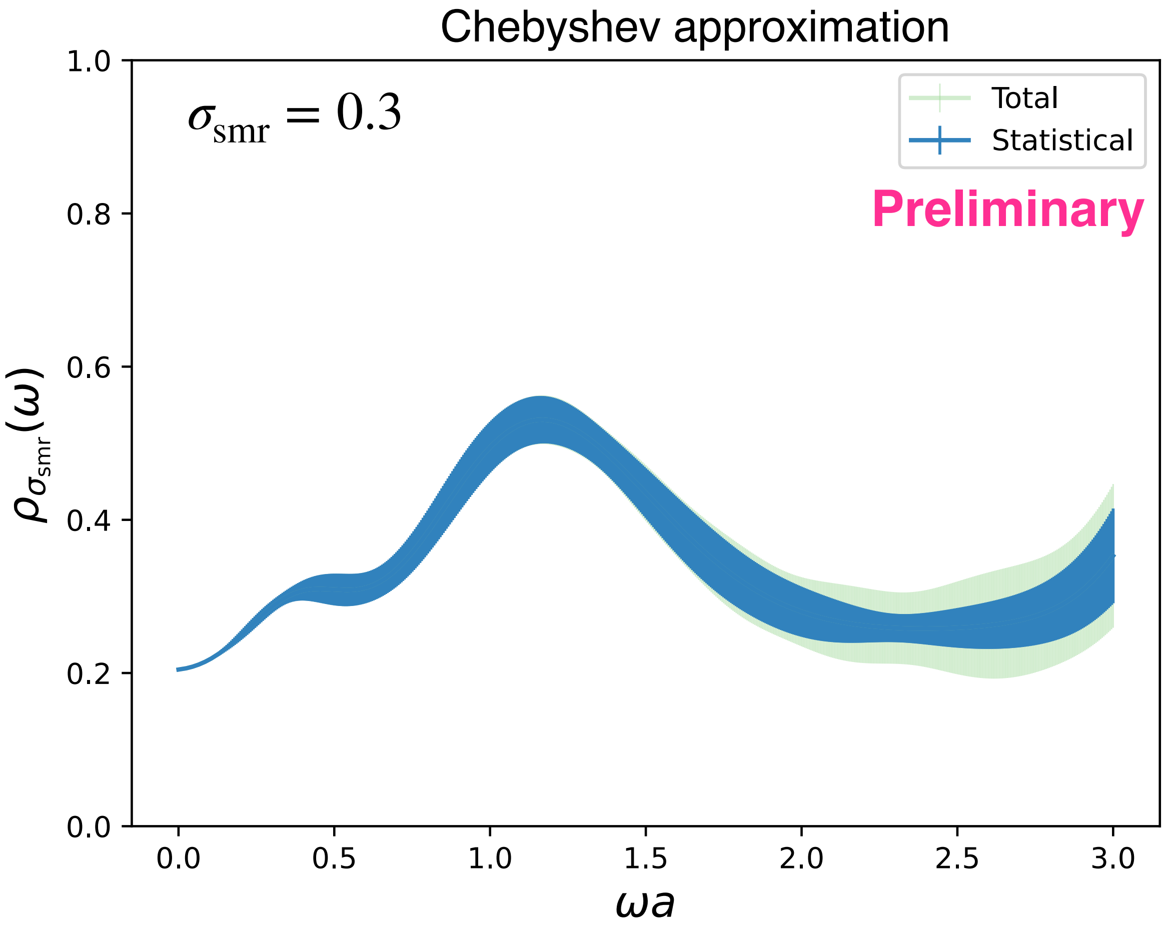
$$\rho_{\sigma_{\rm smr}}^{(N_{\rm tr})}(\omega) \approx \sum_{l=0}^{N_{\rm tr}} S_l(\omega) \cdot \rho_l$$

$$|\rho_{\sigma_{\rm smr}}^{(N)}(\omega) - \rho_{\sigma_{\rm smr}}^{(N_{\rm tr})}(\omega)| \leq \sum_{l=N_{\rm tr}}^{N-1} \left| S_l(\omega) \right| \cdot |\rho_l|$$

- Approximate $\rho_{\sigma_{\rm smr}}(\omega)$ with SVD
- Data based error for $\rho_{\sigma_{\rm smr}}(\omega)$

Systematic uncertainty





- Both SVD and Chebyshev approximation reconstruct roughly same structure of $\rho(\omega)$
- **Systematic error = Truncation of higher l modes, Smearing = Reduction of higher l modes**

Summery and Perspectives

Summary

•**NOTE: all numerical results are PRELIMINARY!**

✓ The singular value decomposition for the Laplace kernel

$$\boxed{e^{-\omega t} \simeq \sum_{l=0}^{N-1} U_l(t)\sigma_l V_l(\omega)} \longrightarrow C(t) \simeq \sum_{l=0}^{N-1} U_l(t)\sigma_l \int_\omega \rho(\omega)V_l(\omega)$$

-$C(t)$ only carries 5-15 significant modes ρ_l on a restricted space due to the noise

✓SVD based reconstruction

-SVD projects the inverse problem onto a restricted space

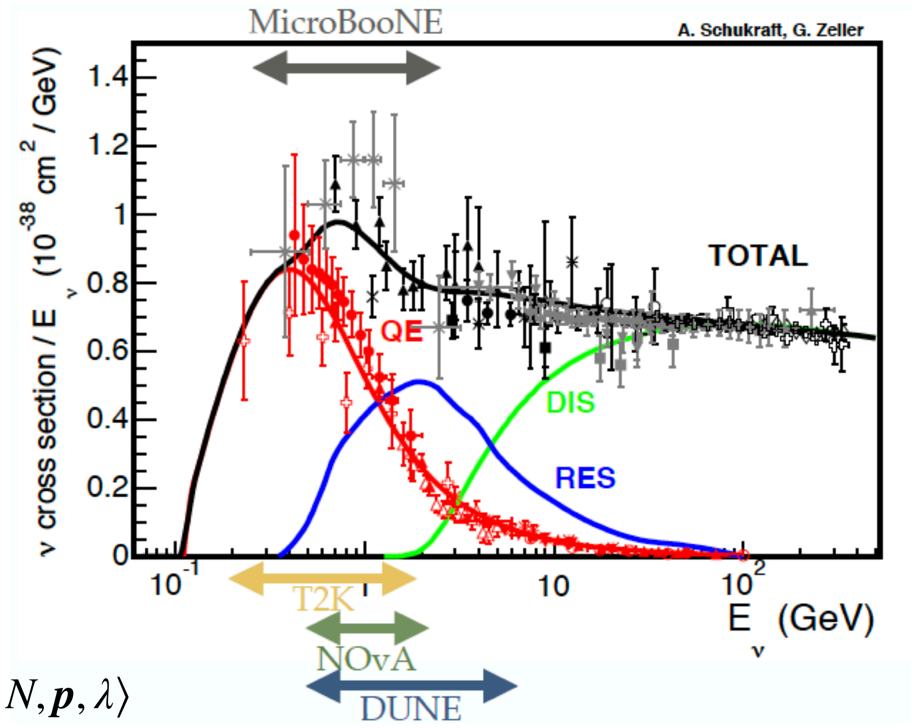
→ Coarse structure $\rho(\omega)$ described by lower l basis functions can be reconstructed

✓ **Systematic error = Truncation of higher l modes, Smearing = Reduction of higher l modes**

✓ **Application to inclusive observables will be considered!**

Nucleon-lepton inclusive scattering

H. Fukaya. S. Hashimoto, T. Kaneko and H. Ohki Phys. Rev. D 102, 114516 (2020)



e.g. Deep inelastic scattering

$$d\sigma = \frac{e^4}{Q^4}\int \frac{d^3\boldsymbol{k}'}{(2\pi)^3 2\omega_{\boldsymbol{k}'}} \frac{4\pi L^{\mu\nu}W_{\mu\nu}(p, q=k-k')}{2k^0 2M}$$

with $L^{\mu\nu} = 2(k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu} k\cdot k')$

$$W^{\mu\nu} = \frac{1}{4\pi n_\lambda}\sum_\lambda \int d^4x e^{iqx}\langle N,\boldsymbol{p},\lambda\,|\,V_\mu^{\rm em}(x)V_\nu^{\rm em}(0)\,|\,N,\boldsymbol{p},\lambda\rangle$$

$$\longleftrightarrow \; C_{\mu\nu}(\tau,\boldsymbol{p_x};\boldsymbol{p}) = \frac{1}{4\pi n_\lambda}e^{-\omega_p\tau}\int d^3\boldsymbol{x} e^{i(\boldsymbol{p}-\boldsymbol{p}_x)\cdot\boldsymbol{x}}\sum_\lambda \langle N,\boldsymbol{p},\lambda\,|\,V_\mu^{\rm em}(\tau,\boldsymbol{x})V_\nu^{\rm em}(0)\,|\,N,\boldsymbol{p},\lambda\rangle$$

Inverse problem **Hadronic tensor**

$$C_{\mu\nu}(\tau,\boldsymbol{p_x};\boldsymbol{p}) = \int_0^\infty dp_x^0 e^{-p_x^0\tau}W_{\mu\nu}(p,p-p_x)$$

Simulation on lattice **Inverse Laplace transform**

1. Signal-noise ratio problem
   → *Transfer matrix GEVP approach?*
   R. Tsuji et al., Phys. Rev. D 113 (2026) 054514.
2. Inverse problem
   → *Data based error with SVD?*
   R. Tsuji et al., arXiv:2605.15674.

Backup

Finite grid of t and ω

$$u_s^{+(\theta)}(x) = \frac{\cos\left(s\ln(x) - \frac{\theta(s)}{2}\right)}{\sqrt{\pi x}} \text{ and } u_s^{-(\theta)}(x) = \frac{\sin\left(s\ln(x) - \frac{\theta(s)}{2}\right)}{\sqrt{\pi x}},$$ where $x = \omega$ or t, and $\theta(s) = \arg(\lambda_s)$

$\downarrow$ **Finite grid ($x \in [x_{\min}, x_{\max}]$)**

Ansatz

$$U_l(x) \propto \frac{1}{\sqrt{x}}\cos(B_l\ln(x) + C_l) \;\; ; B_l = s_l, \;\; C_l = \begin{cases} -\frac{\theta(s_l)}{2}, & \text{even } l \\ -\frac{\theta(s_l)}{2} + \frac{\pi}{2}, & \text{odd } l \end{cases}$$

for $e^{-\omega t} = \sum_l U_l(t)\sigma_l U_l(\omega)$ (symmetric matrix)

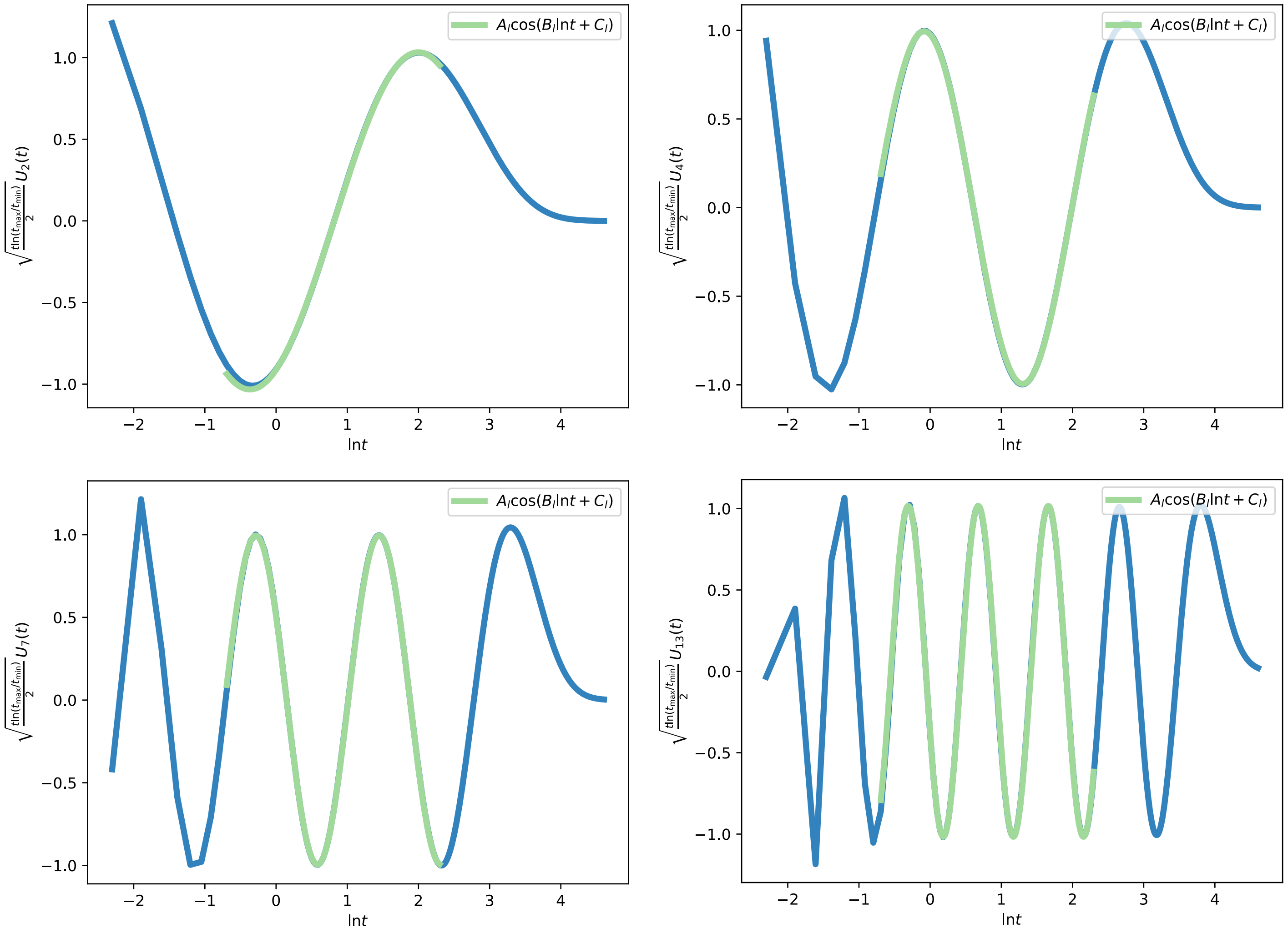
$\longrightarrow$ With numerical data of U_l, one can read s_l and $\theta(s_l)$ through a fitting

Then, examine $|\lambda_{s_l}| = \sqrt{\frac{\pi}{\cosh(\pi s_l)}} \approx \sigma_l$ and $\arg\left(\Gamma(\frac{1}{2} + is_l)\right) \approx \theta(s_l)$

SVD and Mellin

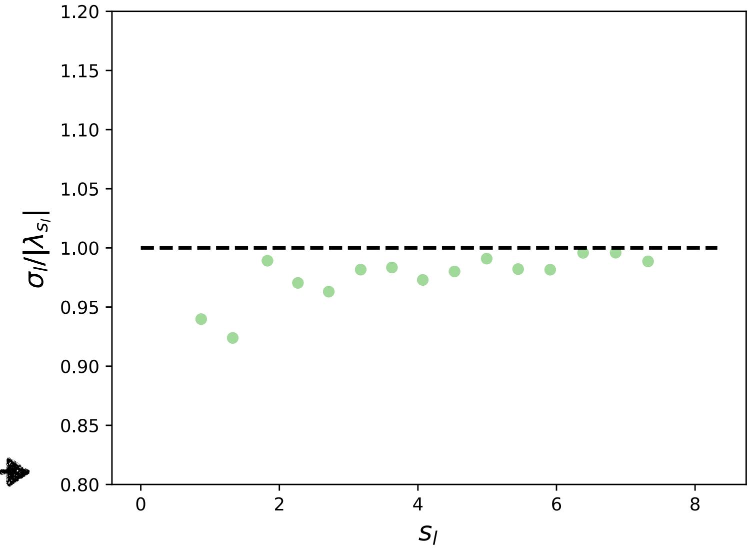
$$e^{-\omega t} \simeq \sum_{l=0}^{N-1} U_l(t)\sigma_l V_l(\omega)$$

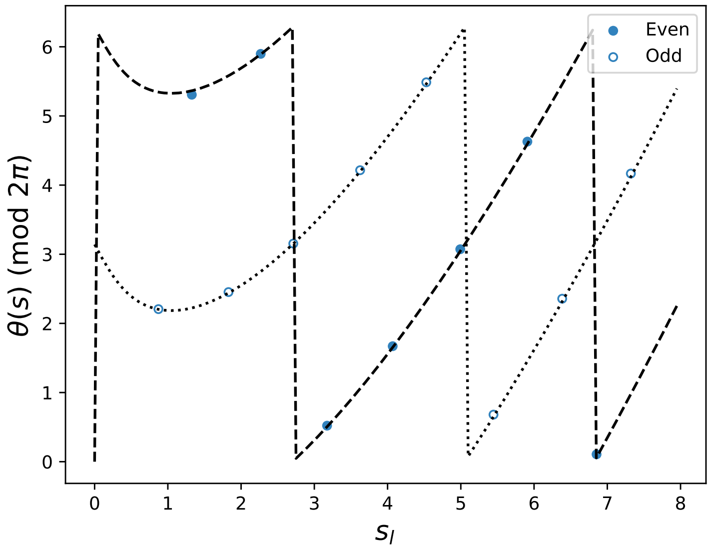
For $\omega, t \in [0.1, 100.1]$, $\Delta = 0.05$, **Numerical data of U_l reproduce $u_{s_l}^{\pm(\theta)}$**



s_l , $\theta(s_l)$

Data reproduce λ_{s_l} as well





For small l, SVD with finite grid of $t, \omega \approx$ the Mellin transform

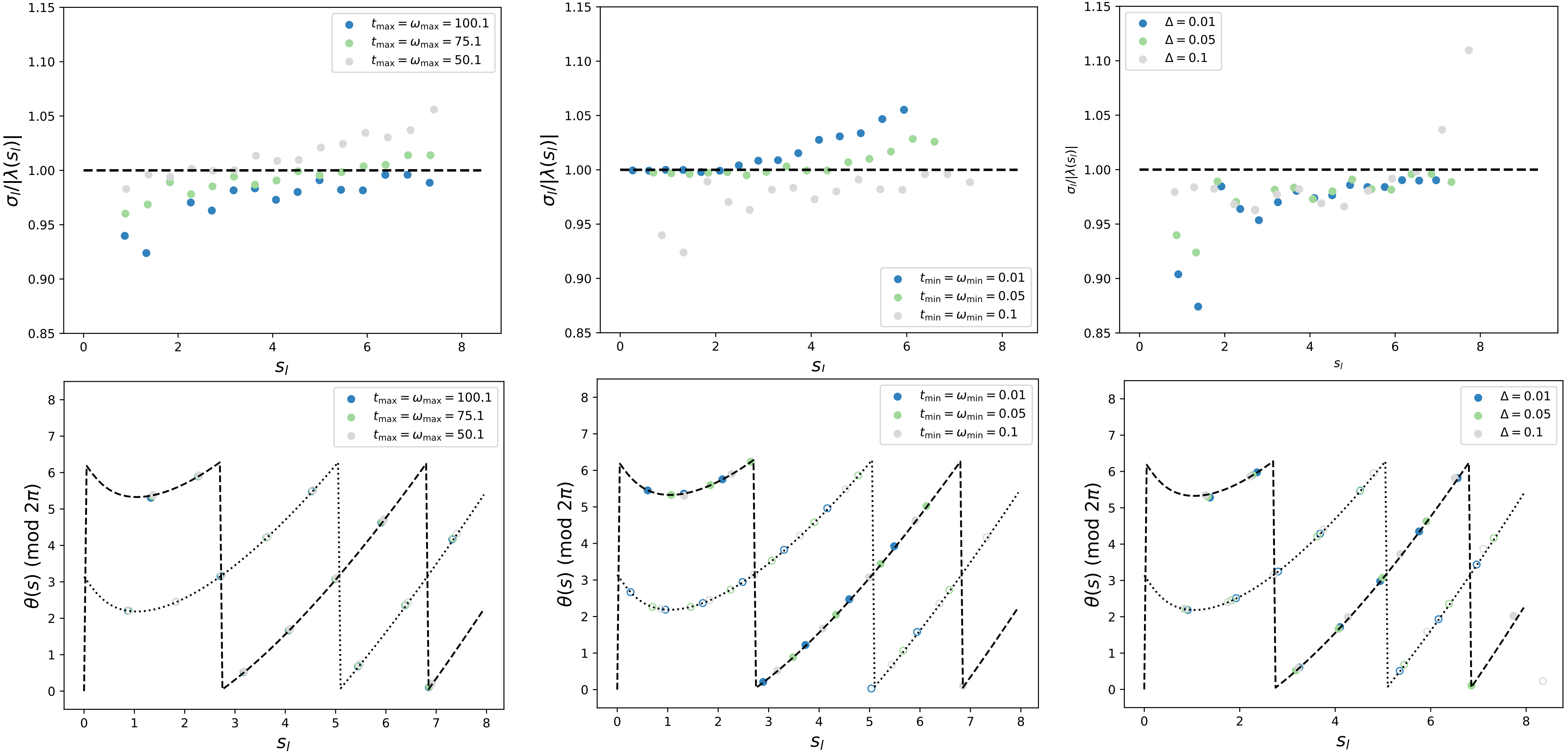
Scaling to Mellin

R. Tsuji, and S. Hashimoto, arXiv:2605.14652.

x_{max} dependence

x_{min} dependence

Δ dependence



Discrete time

R. Tsuji, and S. Hashimoto, arXiv:2605.14652.

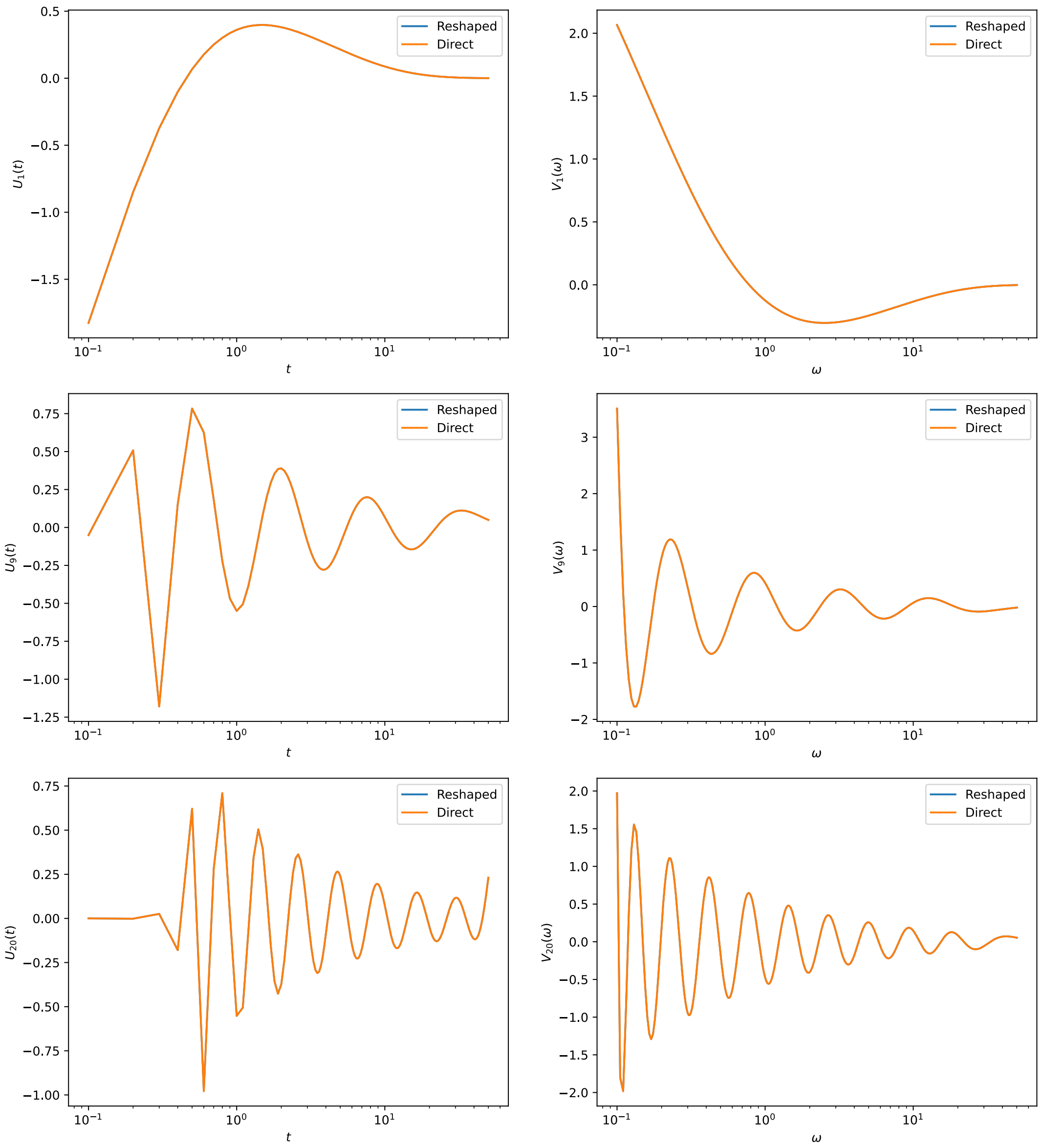
Practically $\Delta_t = 1$ in the unit of lattice, while $\Delta_\omega \ll 1$

→ First, prepare U_l with symmetric and fine $\Delta_t = \Delta_\omega = \Delta$

Then, apply a scale transformation

$$t = \lambda \bar{t}, \omega = \lambda^{-1} \bar{\omega} \text{ so that } \Delta_t = 1 \text{ and } \Delta_\omega = \Delta^2$$

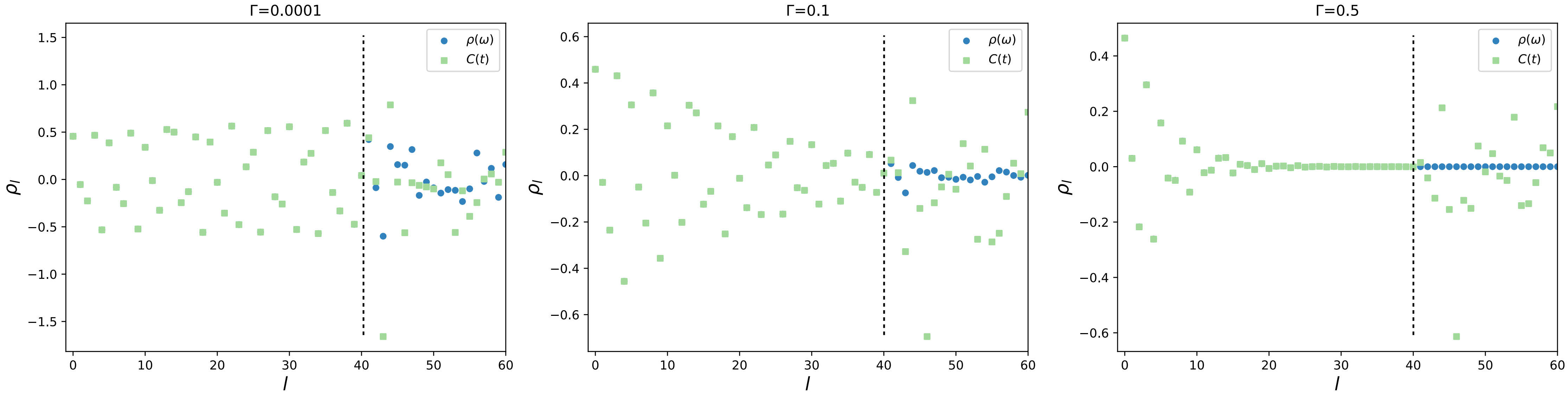
Scale transformation reproduces a direct one with $\Delta_t = 1$



Modal decomposition of $\rho(\omega)$

R. Tsuji, and S. Hashimoto, arXiv:2605.14652.

$$\rho_l = \int_\omega \rho(\omega) V_l(\omega) = \frac{1}{\sigma_l}\int_t U_l(t) C(t) \quad \text{with } \rho(\omega) = \frac{1}{\pi}\frac{2\Gamma}{(\omega - 0.1)^2 + \Gamma^2}$$



- ρ_l from $C(t)$ agree with ρ_l from $\rho(\omega)$ for $l \lesssim 40$ ($\sigma_l \simeq 10^{-16}$ for $l \sim 40$)
- Large l describes the fine structure of $\rho(\omega)$

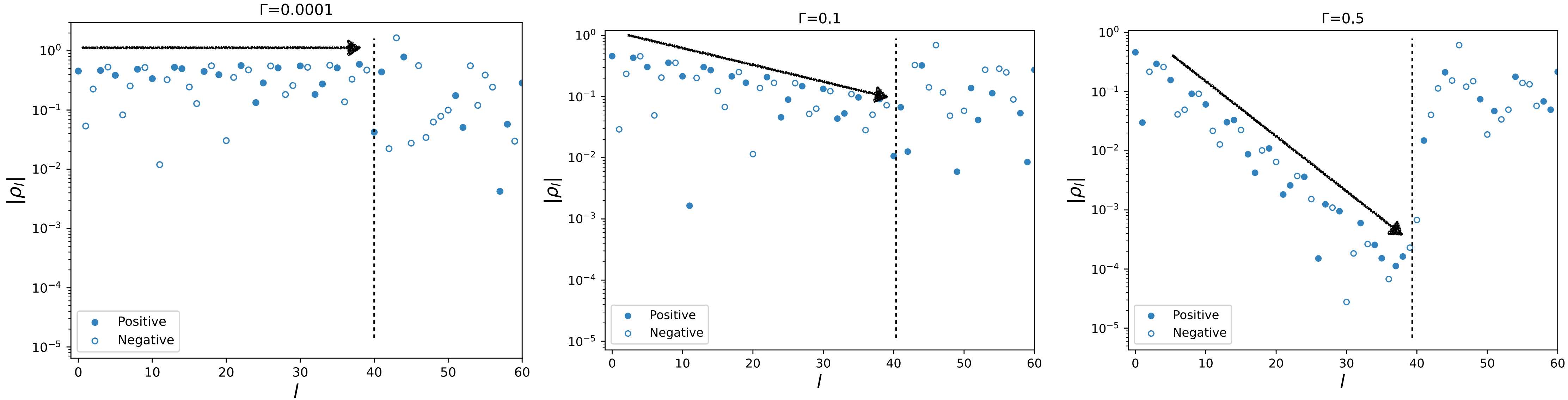
$$\longrightarrow \qquad \rho^{(N_{\rm tr})}(\omega) \approx \sum_{l=0}^{N_{\rm tr}} V_l(\omega)\cdot \frac{1}{\sigma_l}\int_t U_l(t)C(t)$$

Truncation error?

Modal decomposition of $\rho(\omega)$

R. Tsuji, and S. Hashimoto, arXiv:2605.14652.

$$\rho_l = \int_\omega \rho(\omega) V_l(\omega) = \frac{1}{\sigma_l}\int_t U_l(t) C(t) \quad \text{with } \rho(\omega) = \frac{1}{\pi}\frac{2\Gamma}{(\omega - 0.1)^2 + \Gamma^2}$$

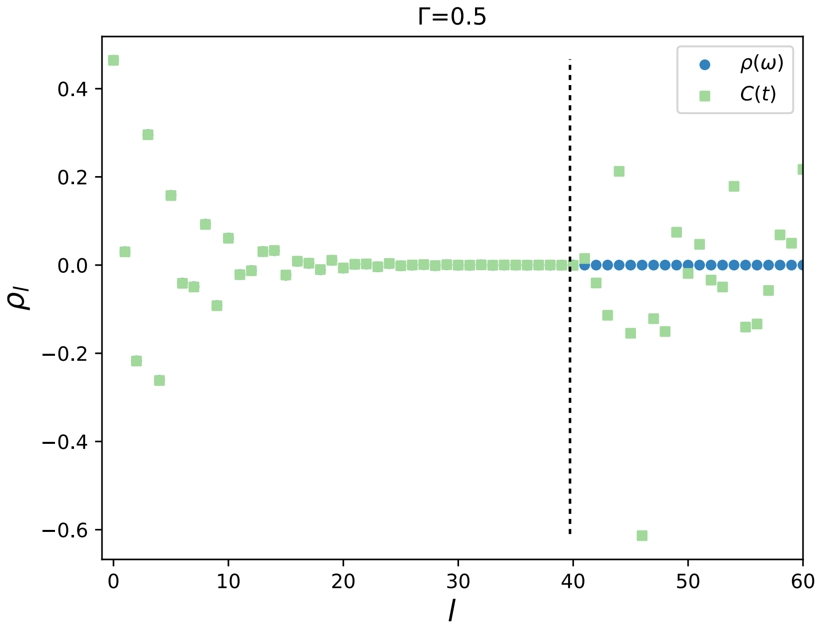


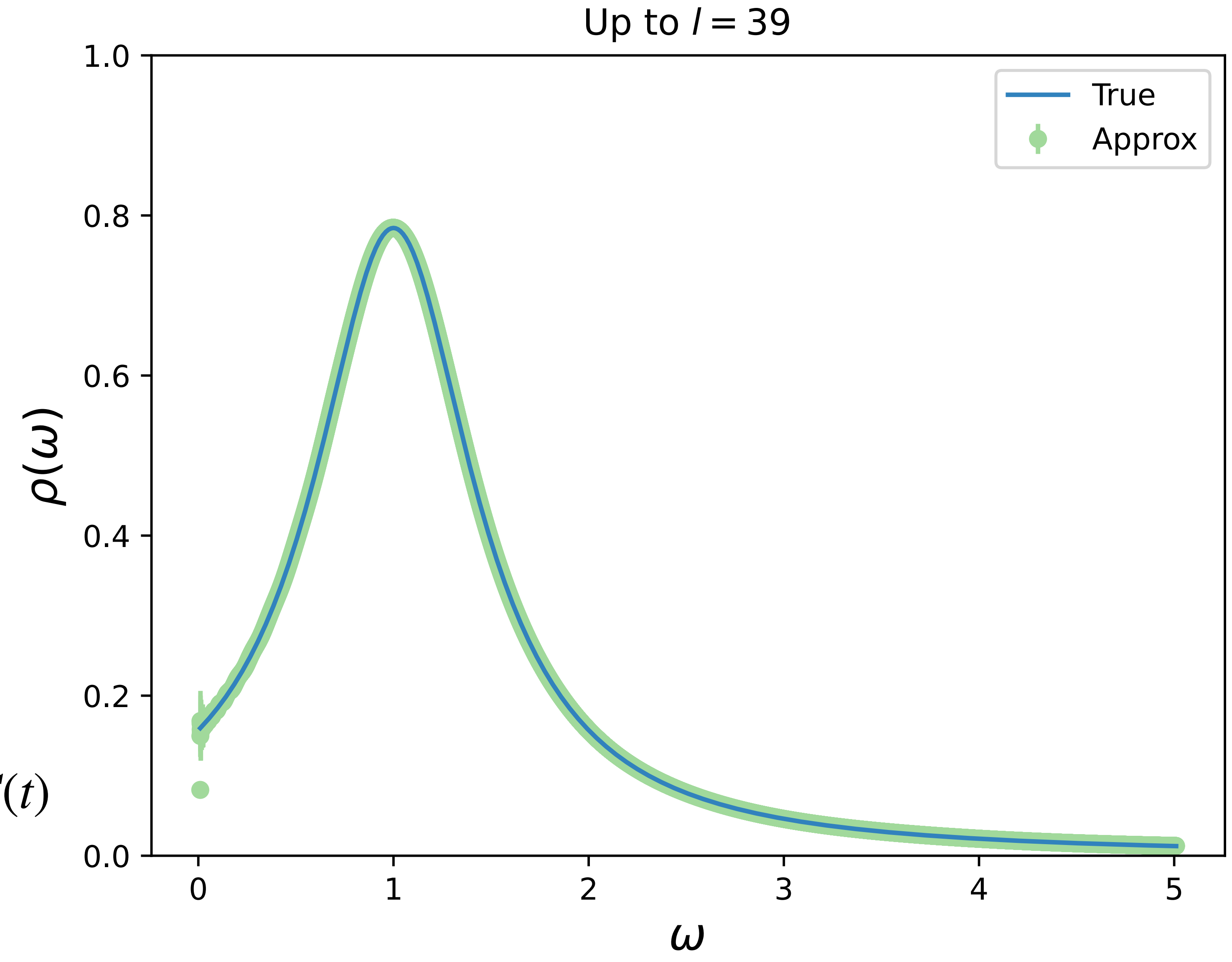
$$|\rho^{(N)}(\omega) - \rho^{(N_{\rm tr})}(\omega)| = \left| \sum_{l=N_{\rm tr}}^{N-1} V_l(\omega)\rho_l \right| \le \sum_{l=N_{\rm tr}}^{N-1} \left| V_l(\omega) \right| \cdot |\rho_l| \lesssim \sum_{l=N_{\rm tr}}^{N-1} \left| V_l(\omega) \right| \cdot |A_\rho e^{-B_\rho l}|$$

Data fit (assumption)

Reconstruction of $\rho(\omega)$

R. Tsuji, and S. Hashimoto, arXiv:2605.14652.



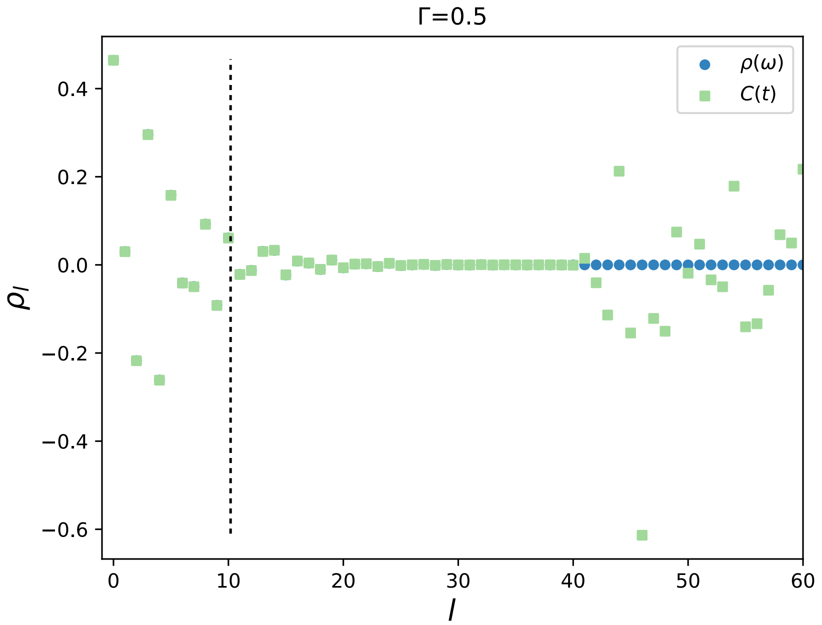


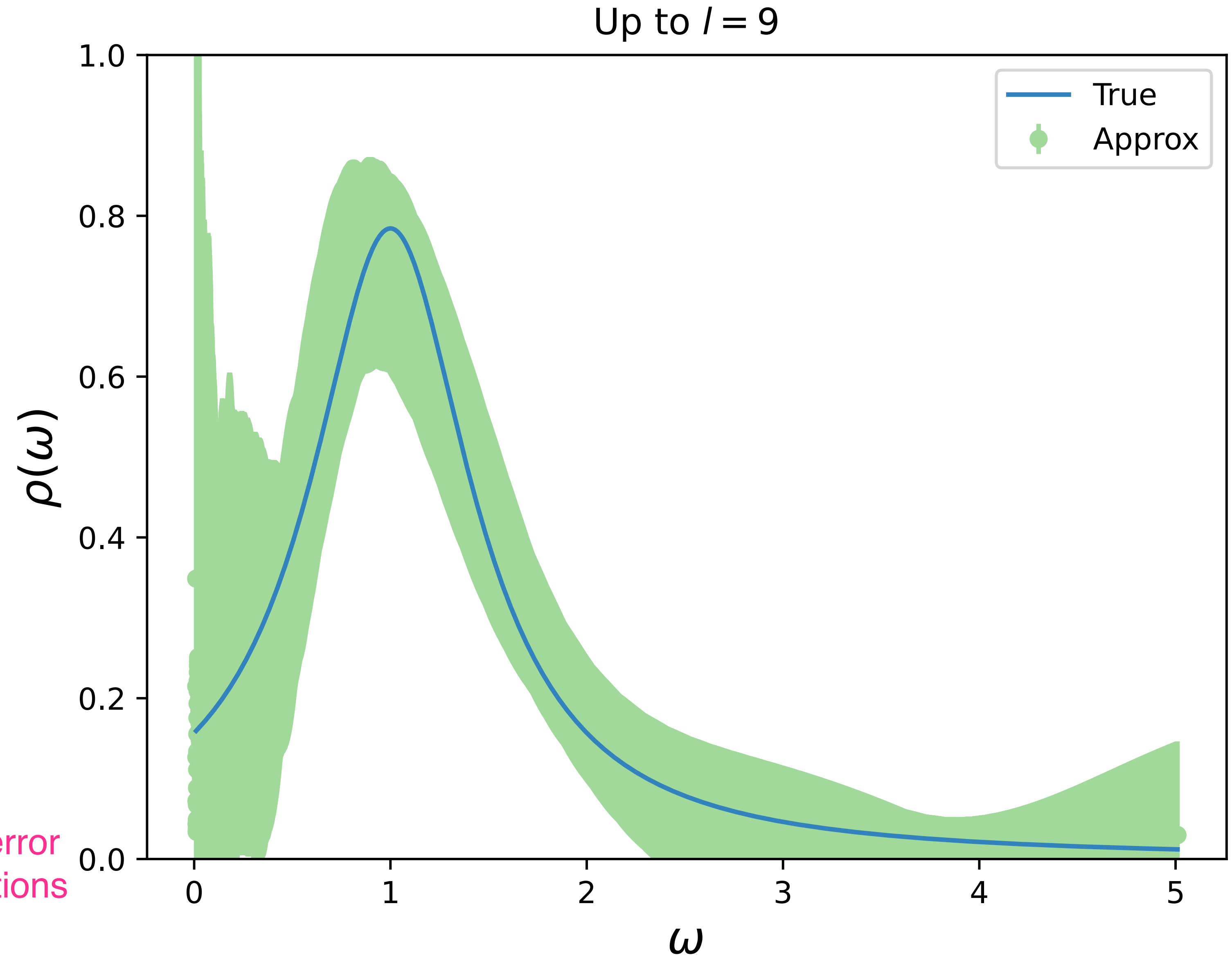
Reproduce True with

$$\rho^{(N_{\rm tr})}(\omega) \approx \sum_{l=0}^{N_{\rm tr}} V_l(\omega) \cdot \frac{1}{\sigma_l} \int_t U_l(t) C(t)$$

Reconstruction of $\rho(\omega)$

R. Tsuji, and S. Hashimoto, arXiv:2605.14652.





This approach provides a transparent basis for estimating systematic error.

→ Any method cannot improve the error without introducing extra assumptions