# N/D method to finite-volume analysis of H-dibaryon including left-hand cut

Lin Qiu

Old Dominion University

Collaborators: Arkaitz Rodas, César Fernández-Ramírez, Vincent Mathieu, Glòria Montaña, Alessandro Pilloni and Adam P.Szczepaniak

The 43rd International Symposium on Lattice Field Theory

University of Maryland, College Park      July 26 to August 1, 2026

Based on:      A. Rodas, L. Qiu and JPAC members, arXiv:2605.22957

# Outline

Finite-volume quantization condition from the N/D representation

Jul 29, 2026, 9:00 AM
20m
Margaret Brent B (Adele H. Stamp Student Union)

Contributed talk | Hadronic and nucle...
Hadronic and nuclear s...

Speaker

Sebastian Dawid (Indiana University Bloomington)

Review on Lüscher's and N/D quantization conditions

Parameterizations on K-matrix and N/D formula

Finite-volume analysis of H-dibaryon including left-hand cut

Arkaitz Rodas Bilbao
Old Dominion University / Jefferson Lab

César Fernández Ramírez
Universidad Nacional de Educación a Distancia

Vincent Mathieu
University of Barcelona

Gloria Montaña
University of Barcelona

Alessandro Pilloni
Università di Messina

Adam Szczepaniak
Indiana University / Jefferson Lab

# Introduction

✦ Most hadronic states are associated with two-hadron thresholds

↪ molecular candidates mediated by $\pi, K, \eta, \rho, \omega, \phi$

✦ Hard scale $\Lambda \sim \Lambda_\chi \approx M_{\rho,\omega,\phi}$ → ERE, Lüscher QC

$$T_L = \frac{1}{\rho(s)\cot\delta(s) - i\rho(s)} = \frac{\sqrt{s}}{2}[\underbrace{\frac{1}{a} + \frac{r}{2}p^2 + vp^4 + \cdots}_{K^{-1}(p) = p\cot\delta} - ip]^{-1}$$

✦ Soft scale $\mathcal{Q} \sim m_\pi$ → new singularities in $p\cot\delta$

s-channel three-body cut, left-hand cut from t/u-channel exchange

↪ constrain convergence radius of ERE & Lüscher

↪ relevant to LQCD study like $DD^*$, NN, H-dibaryon…

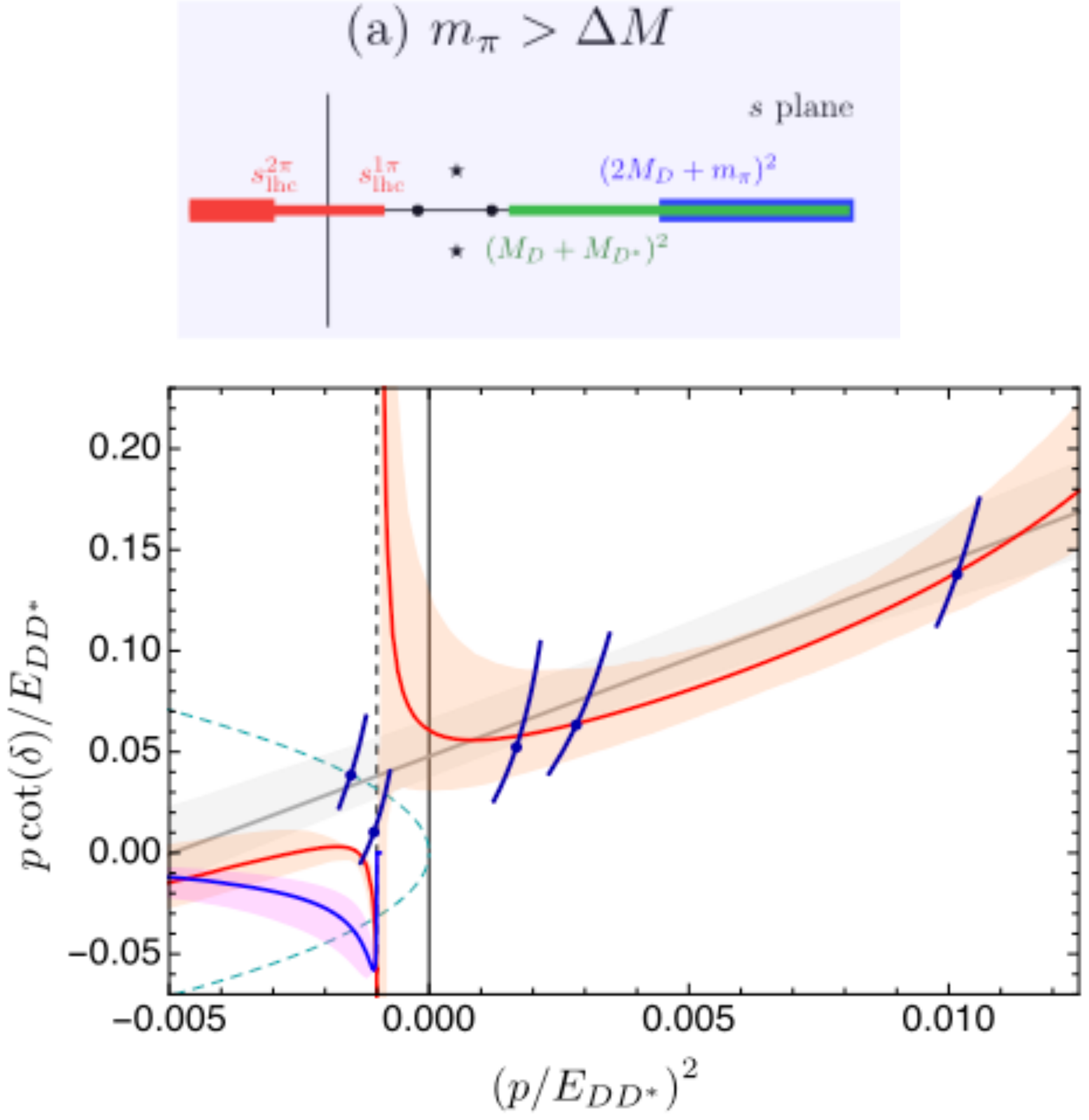
✦ Methods to accommodate left-hand cut

· $\chi$PT+LSE+plane wave basis L. Meng et al., JHEP10(2021)051, PRD.109.L071506

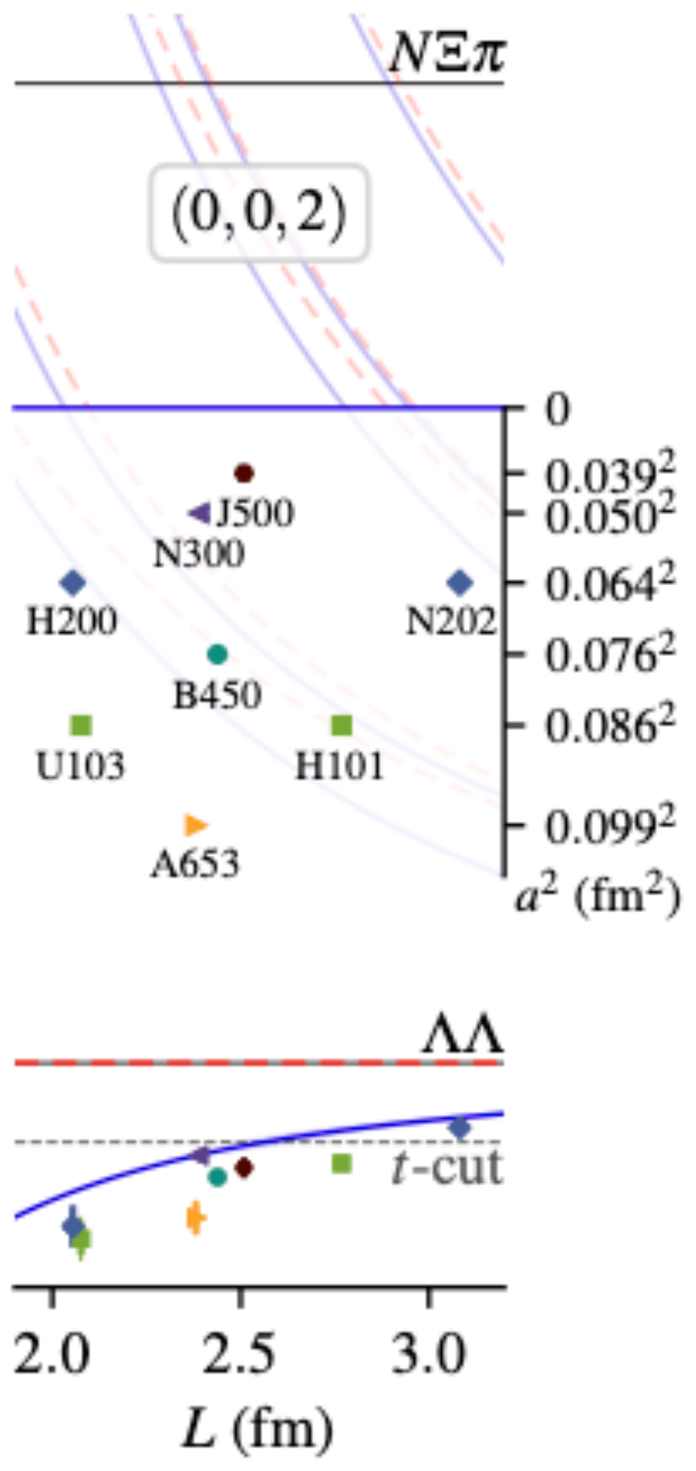
· Modified ERE, Lüscher QCs A.Baião Raposo et al., JHEP08(2024)075; *JHEP* 06 (2025) 186 R.Bubna et al., JHEP05(2024)168; JHEP10(2025)197

· Three-body QCs M.T.Hansen et al., JHEP06(2024)051; JHEP01(2025)060

M.-L. Du et al., PRL 131 (2023) 131903




Green et al., PRL127.242003




## And N/D representation!

• Finite volume:
S.M.Dawid, PLB.2025.139442

• Infinite volume:
M.L.Du et al., PRL135 (2025) 1, 011903
W.J.Wang et al., arXiv:2601.04989

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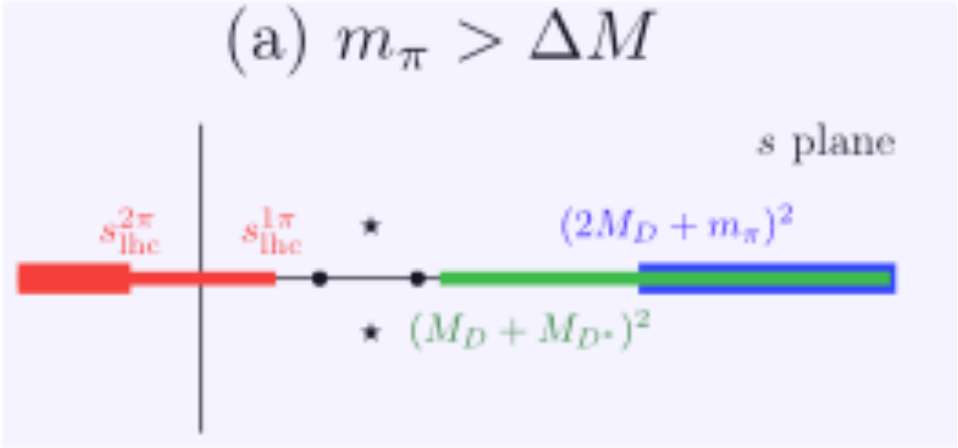



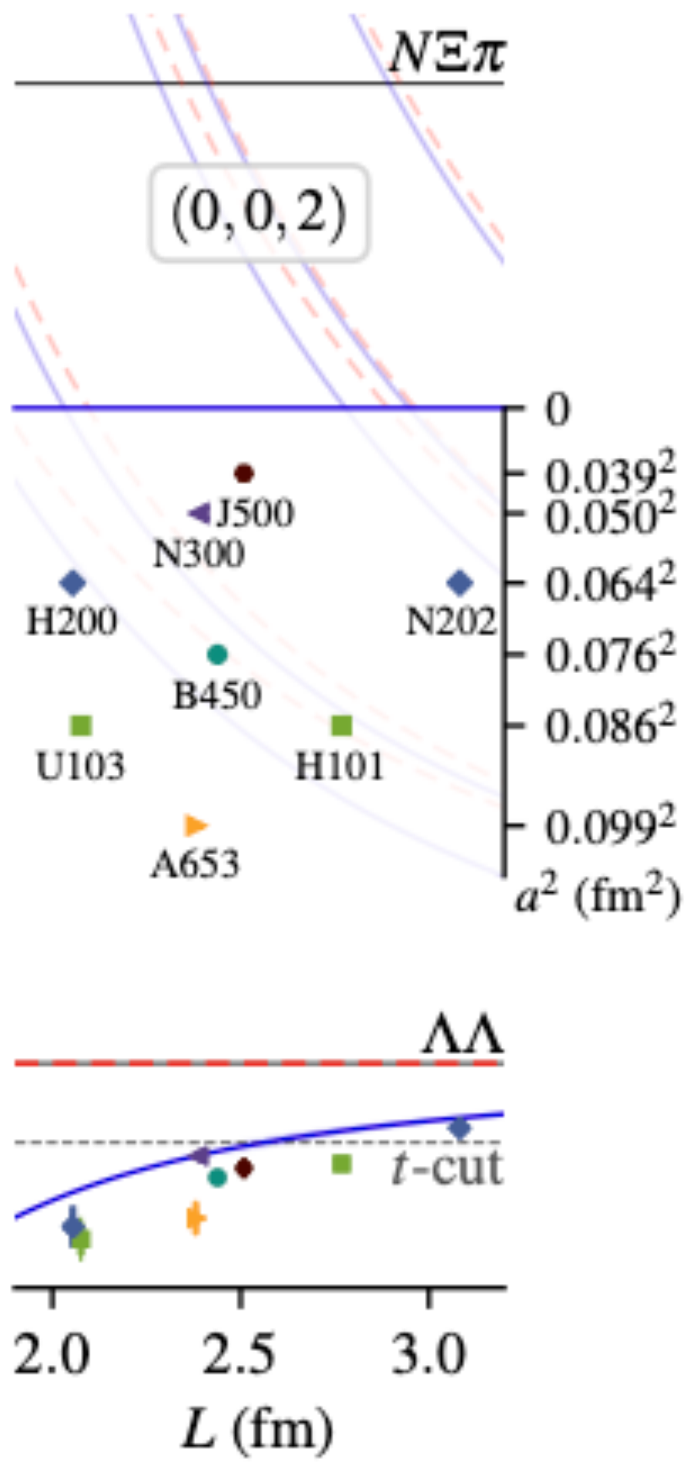


## Baryon-baryon interactions using the plane-wave quantization condition

Not scheduled

20m

University of Maryland, College Park

Poster presentation

Hadronic and nucle...

### Speaker

Jorge Baeza Ballesteros (DESY Zeuthen)

...presentation!

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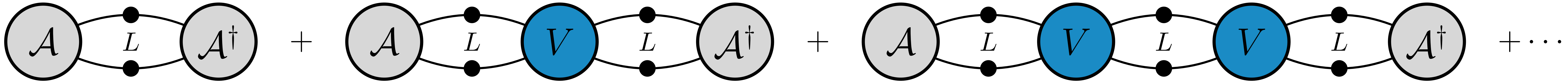
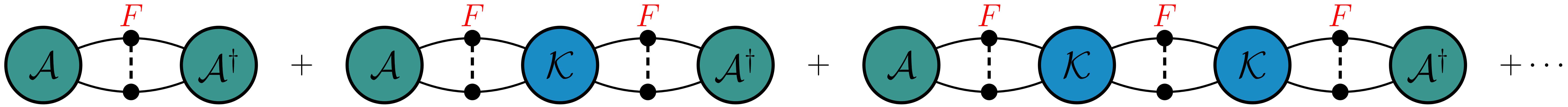
W.J.Wang et al., arXiv:2601.04989

# Recap Lüscher quantization condition

Lüscher, NPB354(1991)
C.h.Kim et al., NPB727(2005)
R.A.Briceno et al., RevModPhys.90.025001

$C_L(P) =$ 


$= \; C_\infty(P) \; +$ 


$$= \; C_\infty(P) \; + \; \boldsymbol{\mathcal{A}}(P) i [F(E,P,L)^{-1} + \mathcal{K}(E)]^{-1} \boldsymbol{\mathcal{A}}^\dagger(P)$$

$$\boxed{\det_{l,m} \left[ F(E, \boldsymbol{P}, L) + K^{-1}(E) \right] = 0}$$

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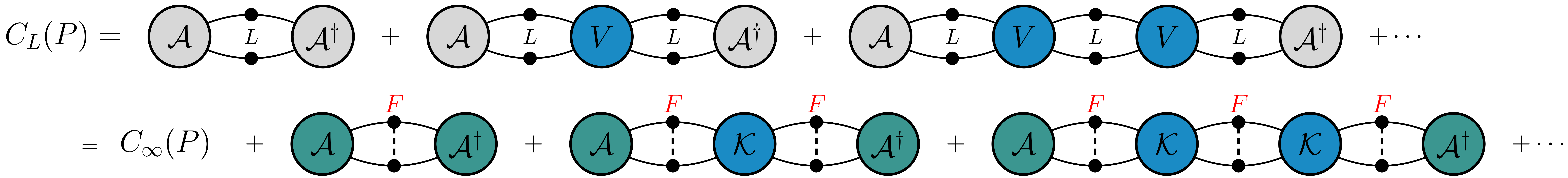
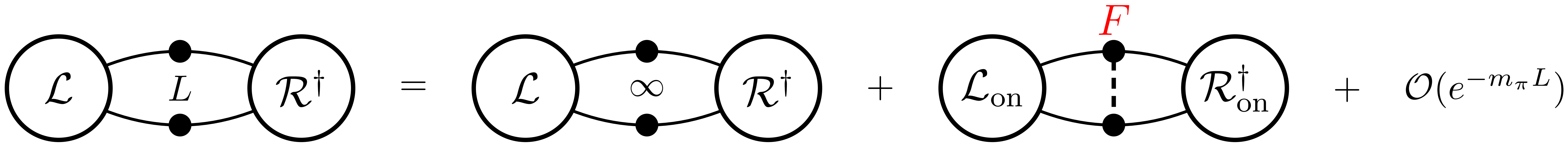



$$= C_\infty(P) + \boldsymbol{\mathcal{A}}(P)i[F(E,P,L)^{-1} + \mathcal{K}(E)]^{-1}\boldsymbol{\mathcal{A}}^\dagger(P)$$

$$\boxed{\det_{l,m}\,[F(E,\boldsymbol{P},L) + K^{-1}(E)] = 0}$$

Valid only when:

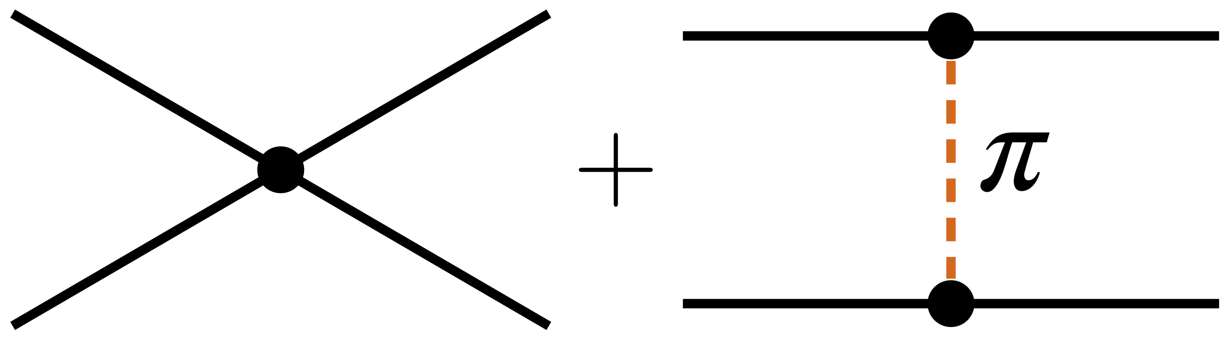



- Above lhc, on-shell factorization works

  ↪ K-matrix, integral ↔ sum

$$\frac{\mathscr{L}(P-k,k)\mathscr{R}^\dagger(P-k,k) - \mathscr{L}(P,\boldsymbol{p}^*)\mathscr{R}^\dagger(P,\boldsymbol{p}^*)}{E-\omega_k-\omega_{Pk}+i\epsilon}$$

- Below lhc, integrand is still singular




$$\propto \log\left(\frac{(k-k')^2+m_\pi^2}{(k+k')^2+m_\pi^2}\right)$$

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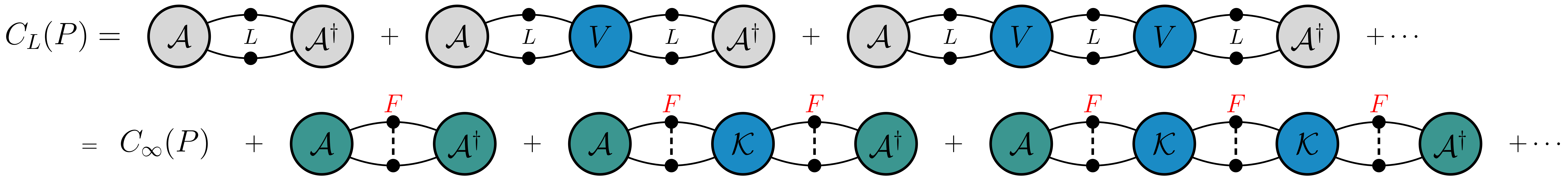
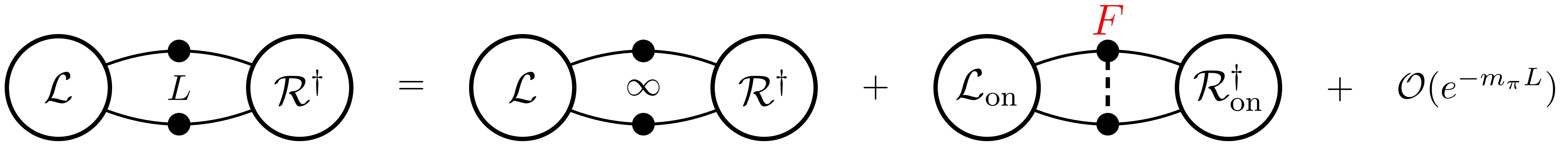



$$= C_\infty(P) \quad + \quad \boldsymbol{\mathcal{A}}(P) i[F(E,P,L)^{-1} + \mathcal{K}(E)]^{-1} \boldsymbol{\mathcal{A}}^\dagger(P)$$

$$\boxed{\det_{l,m}\ [F(E,\boldsymbol{P},L) + K^{-1}(E)] = 0}$$

Valid only when:

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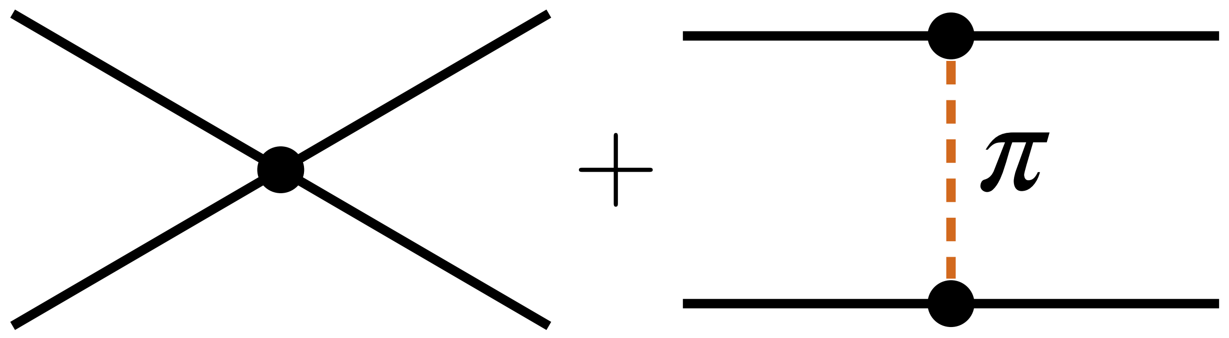



$$\propto \log\left(\frac{(k-k')^2 + m_\pi^2}{(k+k')^2 + m_\pi^2}\right)$$

What is the correct factorization?

# N/D method to left-hand cut problem

M.L.Du et al., PRL135 (2025) 1, 011903
W.J.Wang et al., arXiv:2601.04989
A. Baião Raposo and M. T. Hansen, 2303.18793

$$T_L(s) = \frac{N(s)}{D(s)}$$

left-hand cut

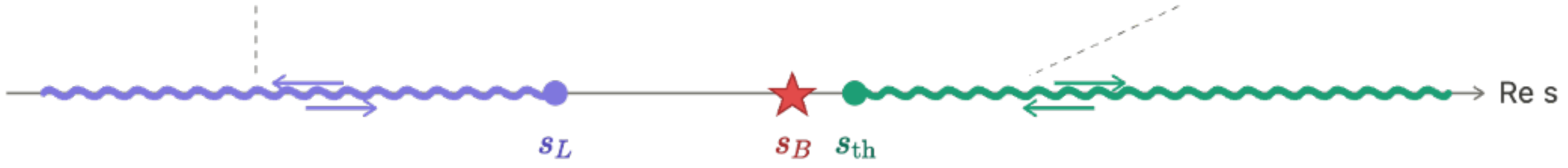
right-hand cut; pole of $T_L(s)$ is zero of $D(s)$

**Left-hand cut**

$\mathrm{Im}\,N(s) = D(s)\,\mathrm{Im}\,T_L(s), \quad \mathrm{Im}\,D(s) = 0$

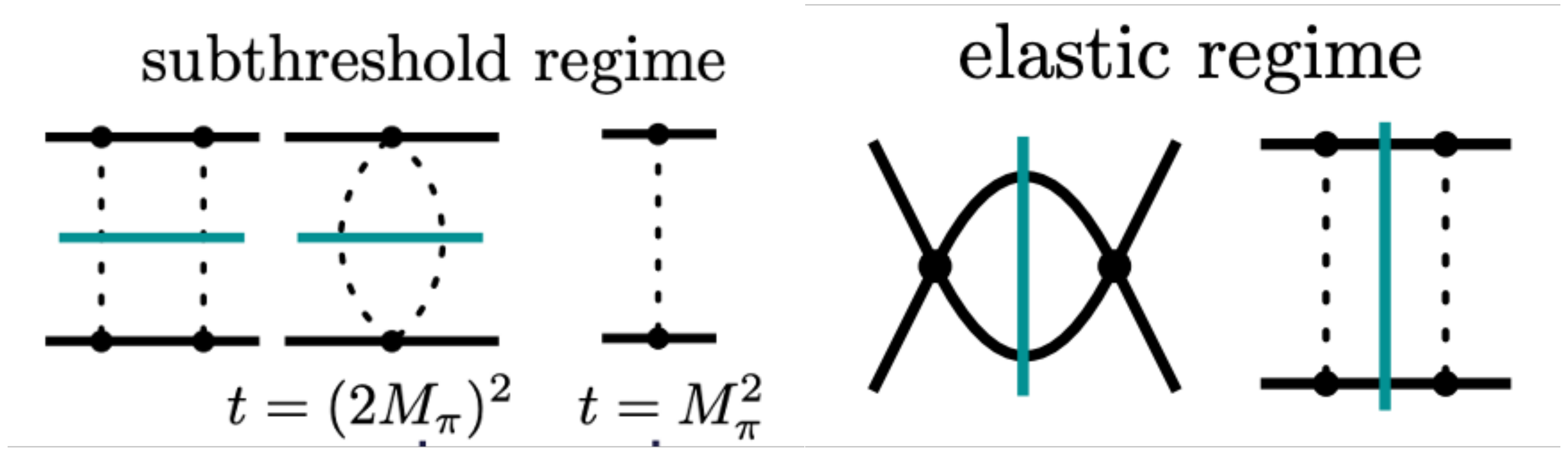
**Right-hand cut**

$\mathrm{Im}\,D(s) = -\rho(s)\,N(s), \quad \mathrm{Im}\,N(s) = 0$




Input:

$$\mathrm{Im}T_L(s) = \mathrm{Im}V(s)$$



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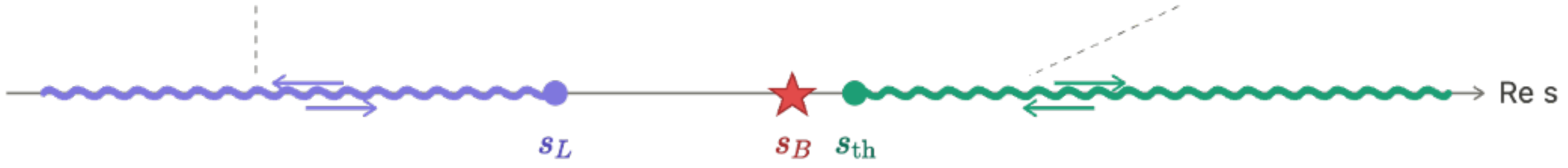


- One-pion exchange

$$V^{\mathrm{OPE}}(s) \quad \propto \quad L_t(s) = -\frac{1}{s - s_{\mathrm{thr}}} \log \frac{s - s_{\mathrm{thr}} + m_\pi^2}{m_\pi^2}$$

Input:

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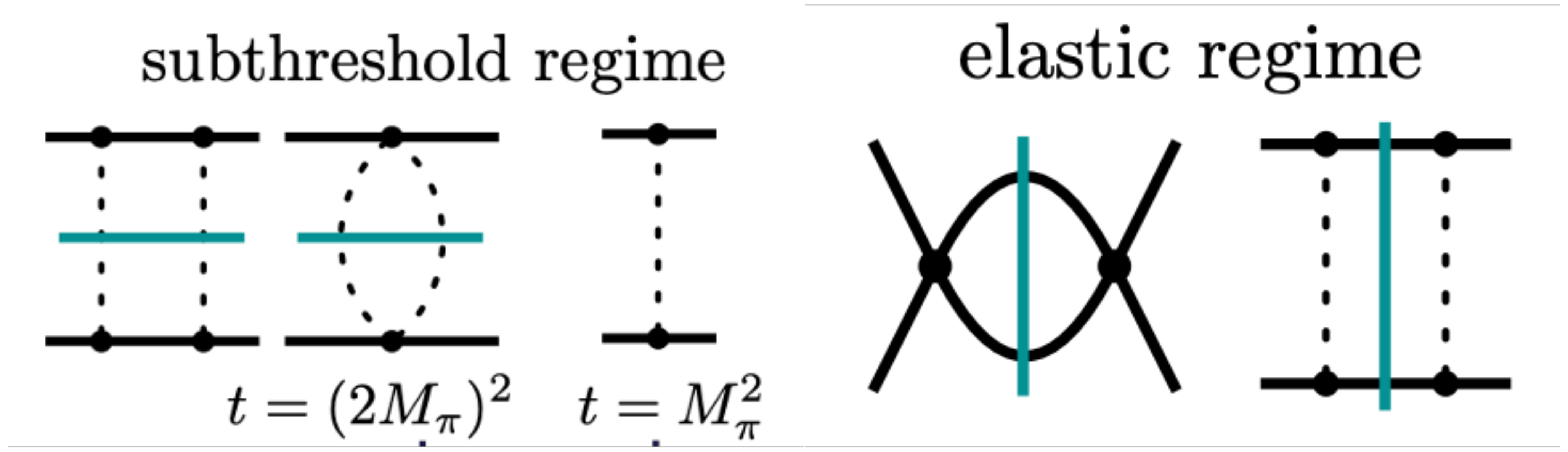

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$N(s)$: left-hand cut

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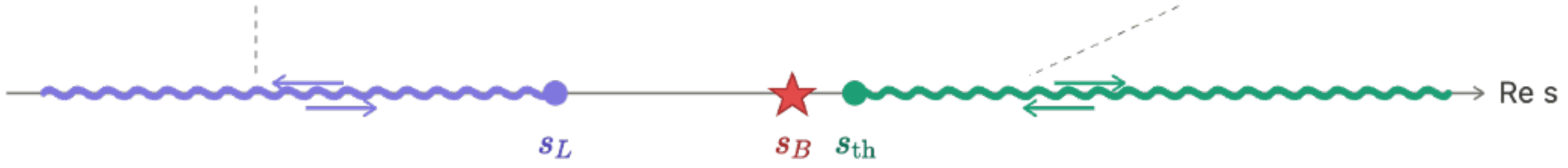


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- Expand $N$ then $D$: $[m, n]$ series

$$N(s) = n_0 + n_1 s + \cdots + n_m s^m + g_0^2 L_t(s)$$

$$D(s) = 1 + d_0 s + d_1 s^2 + \cdots + d_n s^n - \frac{s}{\pi} \int_{s_{\mathrm{thr}}}^{\infty} ds' \frac{\rho(s') N(s')}{s'(s' - s)}$$

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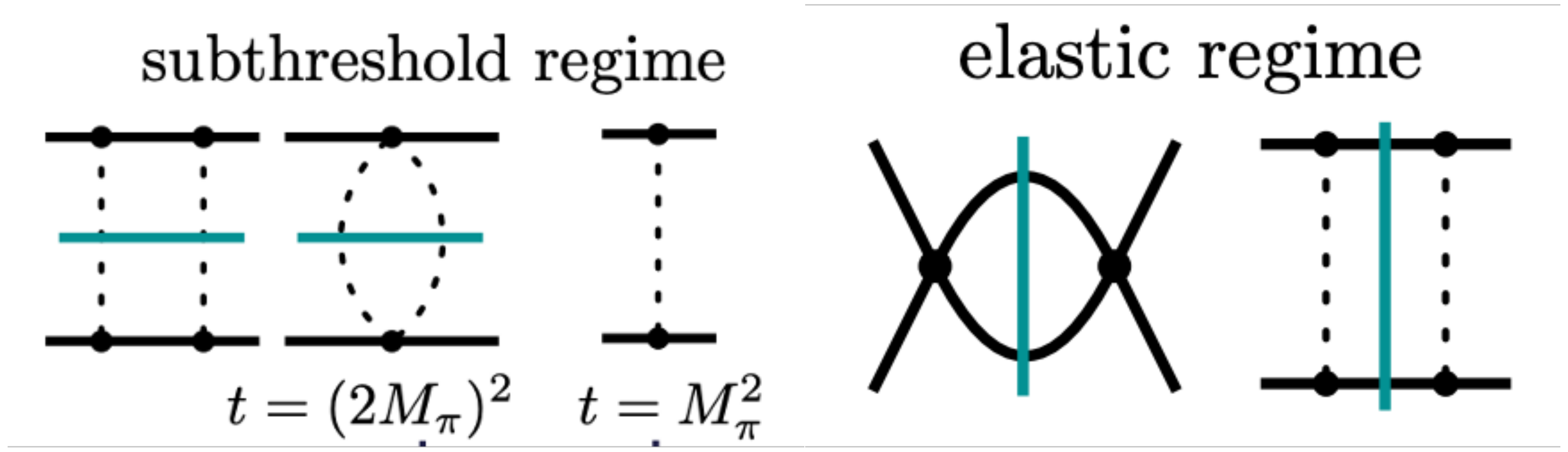

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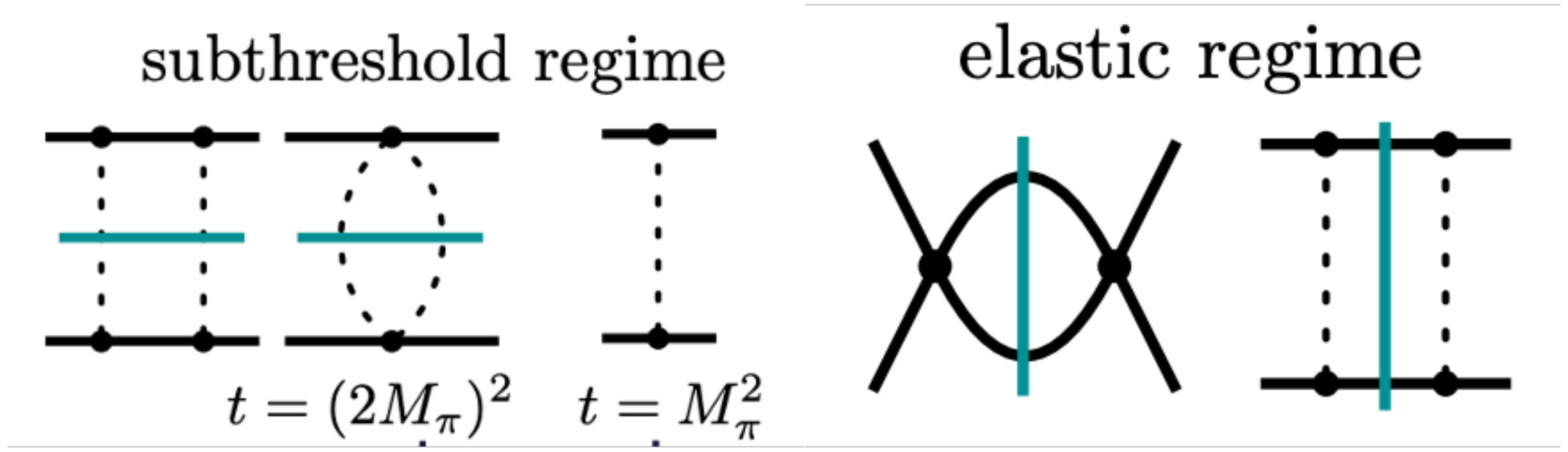


- When $g_0 = 0$, $[0,n]$ reduces to ERE-CM type

$$-i\rho(s) \to -\frac{s - s_{\rm thr}}{\pi} \int_{s_{\rm thr}}^{\infty} ds' \frac{\rho(s')}{(s' - s_{\rm thr})(s' - s)}$$

- [0,1] for virtual-, shallow bound states (small $g$)

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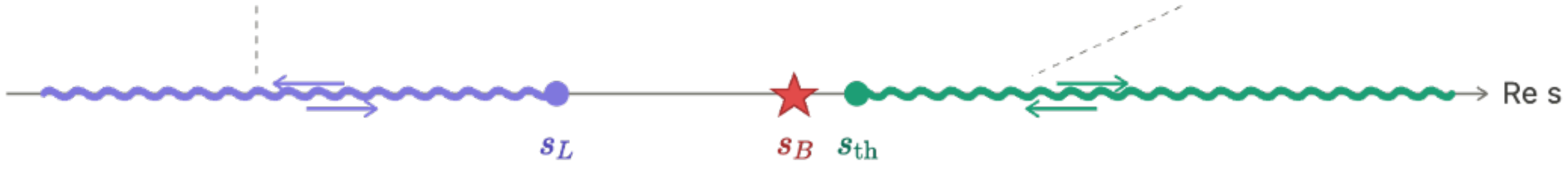


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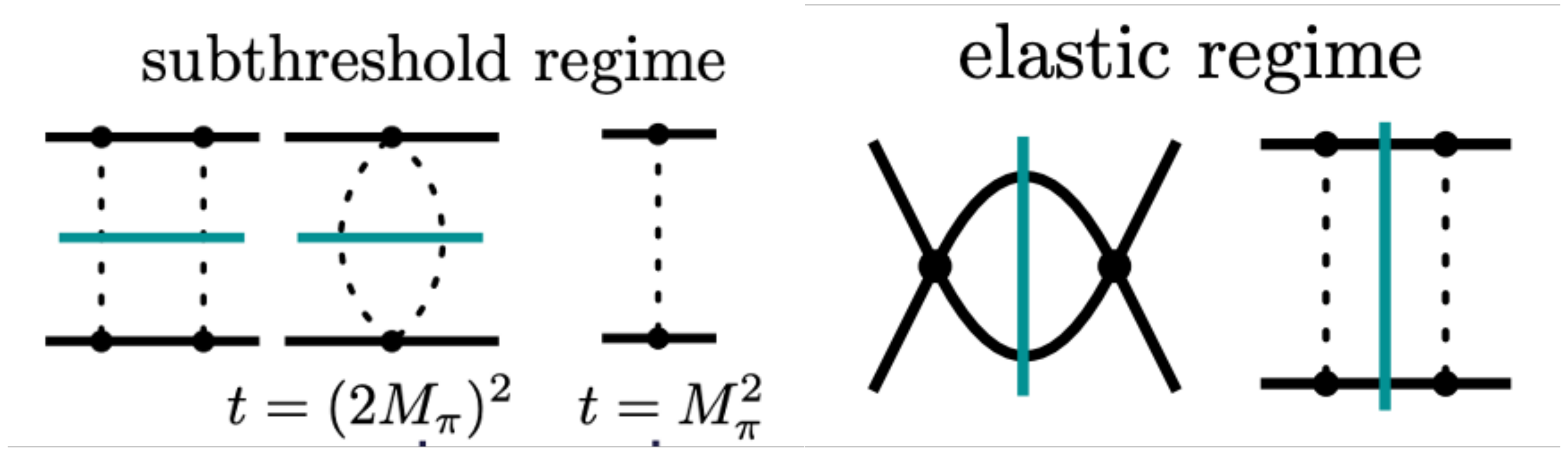


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- $[0,1]$ for virtual-, shallow bound states (small $g$)
- For bound states with large $g$: $\quad g_0^2 \to g_0^2 + g_1 s + \cdots$

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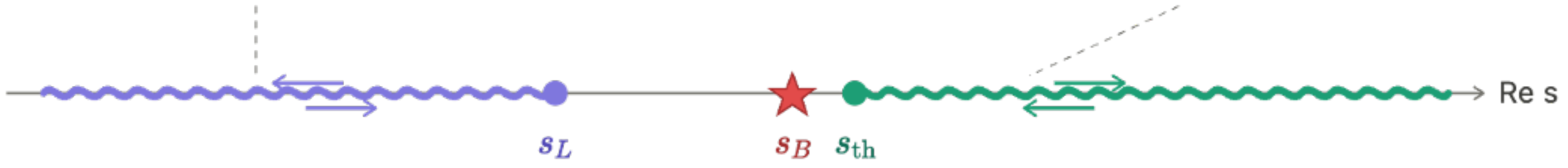


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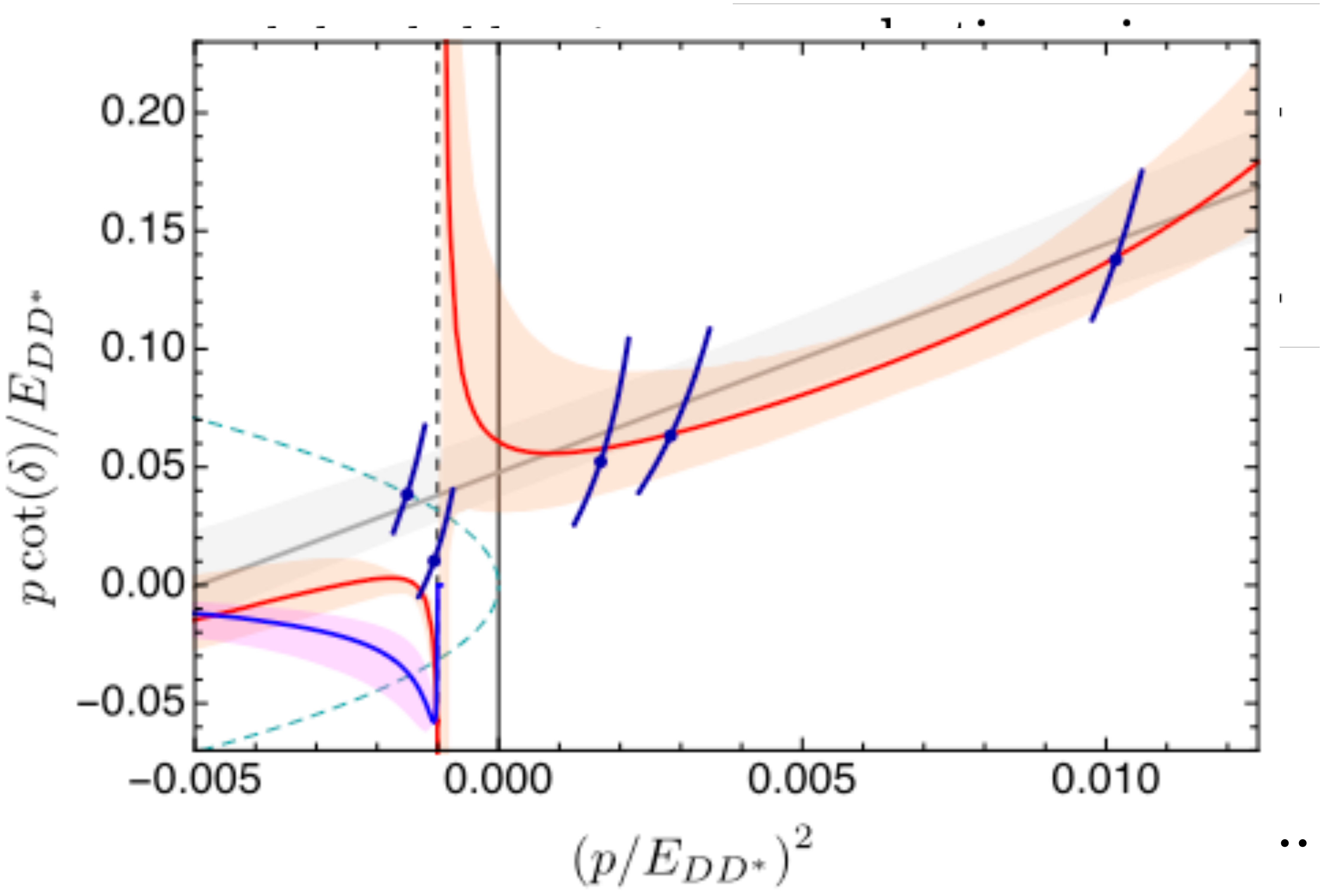
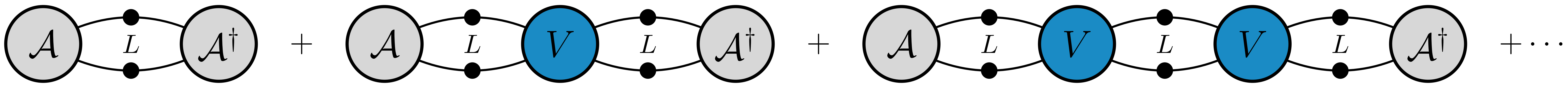



- Capture amplitude-zero in $N(s)$

$\hookrightarrow$ singularity in $p\cot\delta = \dfrac{\mathrm{Re}D}{N}$

# N/D quantization condition

S.M.Dawid, PLB.2025.139442

$C_L(P) =$ 

$$= \begin{pmatrix}\mathcal{A}_1 & \cdots & \mathcal{A}_N\end{pmatrix}\begin{pmatrix}L_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & L_N\end{pmatrix}\begin{pmatrix}\mathcal{A}_1^\dagger\\ \vdots\\ \mathcal{A}_N^\dagger\end{pmatrix} + \begin{pmatrix}\mathcal{A}_1 & \cdots & \mathcal{A}_N\end{pmatrix}\begin{pmatrix}L_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & L_N\end{pmatrix}\begin{pmatrix}V_{11} & \cdots & V_{1N}\\ \vdots & \ddots & \vdots\\ V_{N1} & \cdots & V_{NN}\end{pmatrix}\begin{pmatrix}L_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & L_N\end{pmatrix}\begin{pmatrix}\mathcal{A}_1^\dagger\\ \vdots\\ \mathcal{A}_N^\dagger\end{pmatrix} + \cdots$$

in complete momentum set

$$= \mathcal{A}^T(1 - \mathcal{G}_L V_L)^{-1}\mathcal{G}_L\mathcal{A}^\dagger$$

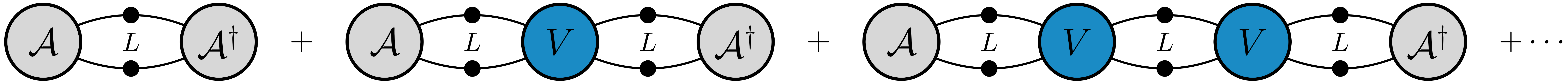
$$= \frac{\mathcal{A}^T \mathrm{Adj}(1-\mathcal{G}_L V_L)\mathcal{G}_L\mathcal{A}^\dagger}{|1-\mathcal{G}_L V_L|} \quad \rightsquigarrow \quad \mathcal{M}_L(p^\star, \boldsymbol{P}) = \mathcal{N}_L(p^\star, \boldsymbol{P})\mathcal{D}_L(p^\star, \boldsymbol{P})^{-1}$$

$$\boxed{\det_{l,m} \mathscr{D}_L(p^*, \boldsymbol{P}) = 0}$$

# N/D quantization condition

S.M.Dawid, PLB.2025.139442

$C_L(P) =$




$$= (\mathcal{A}_1 \ \cdots \ \mathcal{A}_N)\begin{pmatrix} L_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & L_N \end{pmatrix}\begin{pmatrix} \mathcal{A}_1^\dagger \\ \vdots \\ \mathcal{A}_N^\dagger \end{pmatrix} + (\mathcal{A}_1 \ \cdots \ \mathcal{A}_N)\begin{pmatrix} L_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & L_N \end{pmatrix}\begin{pmatrix} V_{11} & \cdots & V_{1N} \\ \vdots & \ddots & \vdots \\ V_{N1} & \cdots & V_{NN} \end{pmatrix}\begin{pmatrix} L_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & L_N \end{pmatrix}\begin{pmatrix} \mathcal{A}_1^\dagger \\ \vdots \\ \mathcal{A}_N^\dagger \end{pmatrix} + \cdots$$

in complete momentum set

$$= \mathcal{A}^T(1-\mathcal{G}_L V_L)^{-1}\mathcal{G}_L\mathcal{A}^\dagger$$

$$= \frac{\mathcal{A}^T \mathrm{Adj}(1-\mathcal{G}_L V_L)\mathcal{G}_L\mathcal{A}^\dagger}{|1-\mathcal{G}_L V_L|} \quad \rightsquigarrow \quad \mathcal{M}_L(p^\star, \boldsymbol{P}) = \mathcal{N}_L(p^\star, \boldsymbol{P})\mathcal{D}_L(p^\star, \boldsymbol{P})^{-1}$$

$$\boxed{\det_{l,m} \mathscr{D}_L(p^*, \boldsymbol{P}) = 0}$$

☞ S wave: $D_{\mathrm{FV}}(s, \boldsymbol{P}) = 1 + p_n(s) - \dfrac{16\pi}{L^3}\sum_{\boldsymbol{k}} \dfrac{2E(\boldsymbol{k})}{2\omega_1(\boldsymbol{k})2\omega_2(\boldsymbol{P}-\boldsymbol{k})}\dfrac{s}{E^*(\boldsymbol{k})^2}\dfrac{N(\boldsymbol{k}^*, \boldsymbol{P})}{E^*(\boldsymbol{k})^2 - s}$

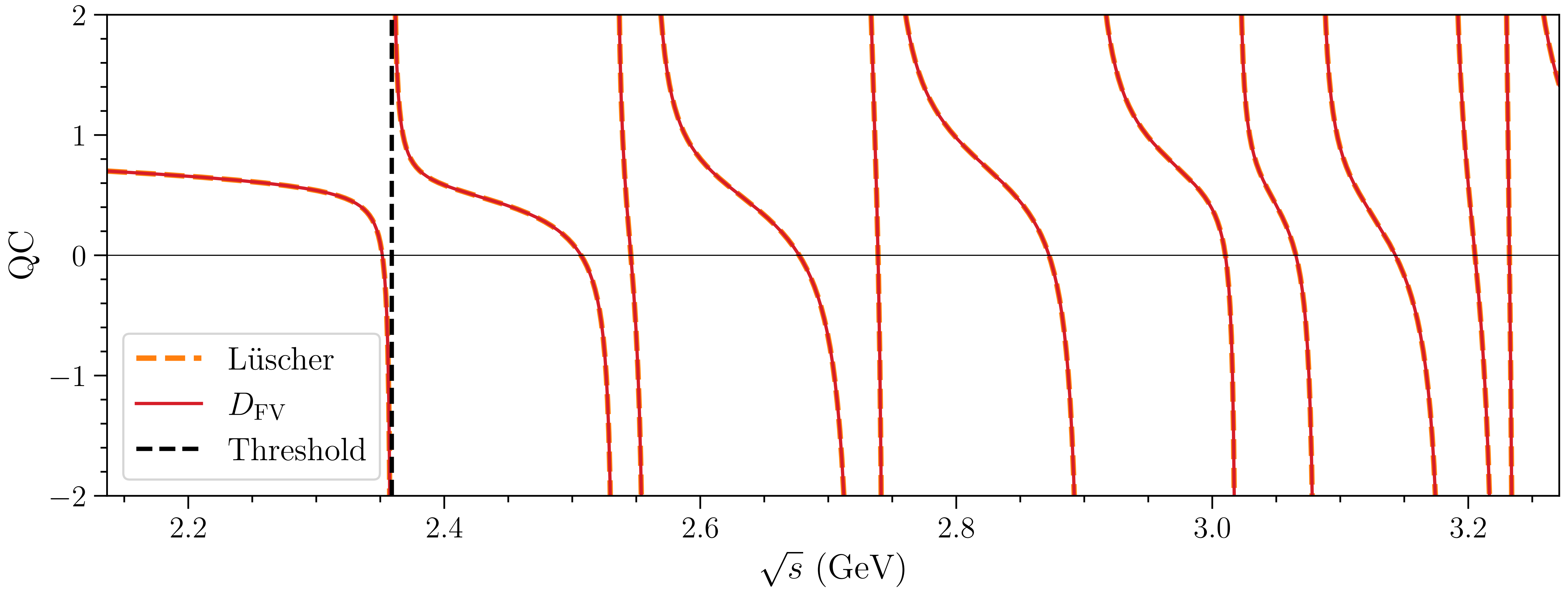
- Cut in $D_{\mathrm{IFV}}$ $\rightarrow$ a series of poles in $D_{\mathrm{FV}}$
- On the right-hand cut, $\mathcal{N} = \mathcal{N}_L + \mathcal{O}(e^{-\Delta L})$
- CDD poles in $D_{\mathrm{FV}}$ $\leftrightarrow$ Zeros in $N$

# Comparison between QCs

The two quantization conditions are equivalent above lhc.

$$\frac{N(s)}{D(s)} \leftrightsquigarrow \frac{1}{K^{-1}(s) - i\rho(s)}$$

$$N(s) = 1$$



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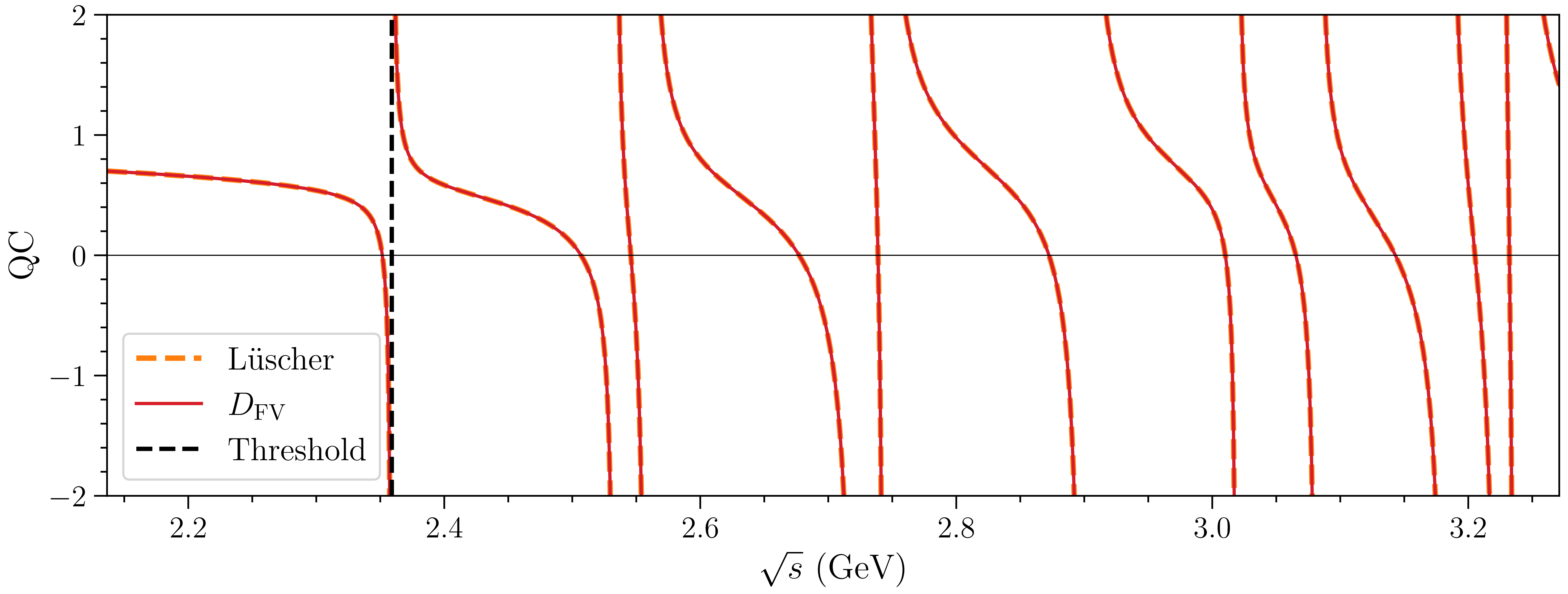
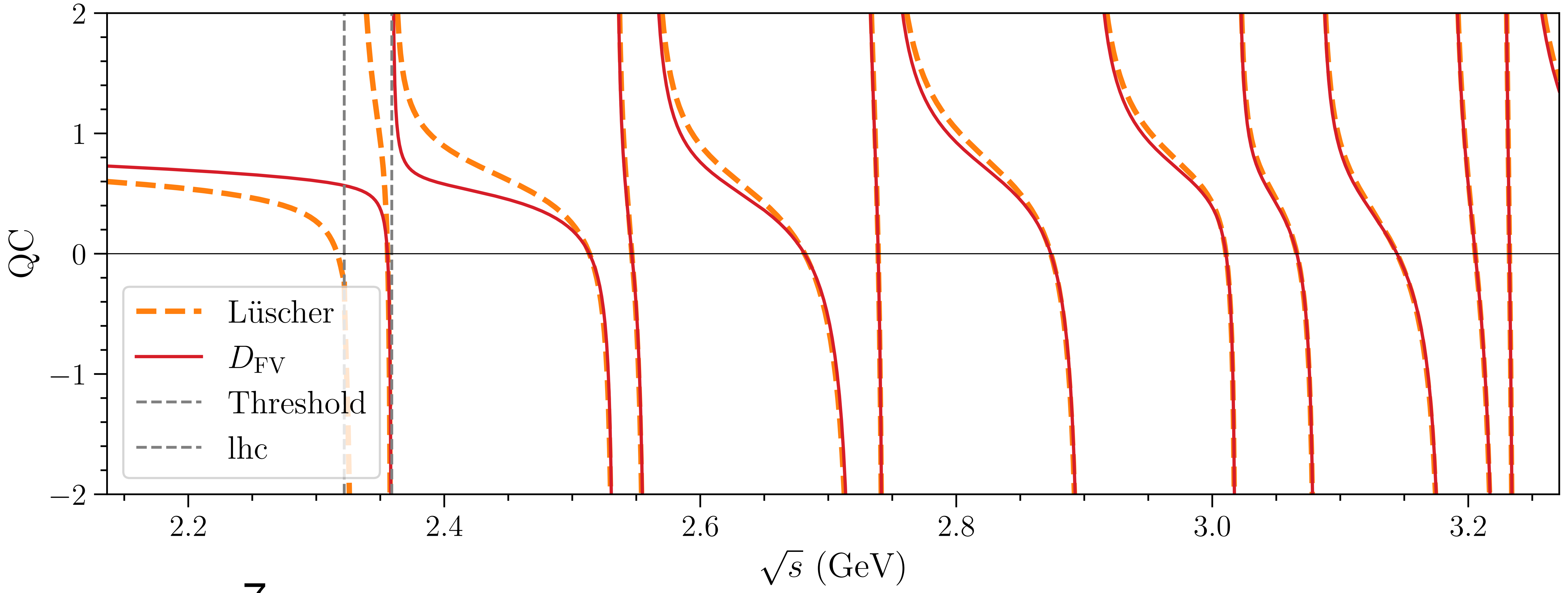



$$N(s) = 1 - \frac{(0.3\mathrm{GeV})^2}{s - s_{\mathrm{thr}}} \log \frac{s - s_{\mathrm{thr}} + m_\pi^2}{m_\pi^2}$$

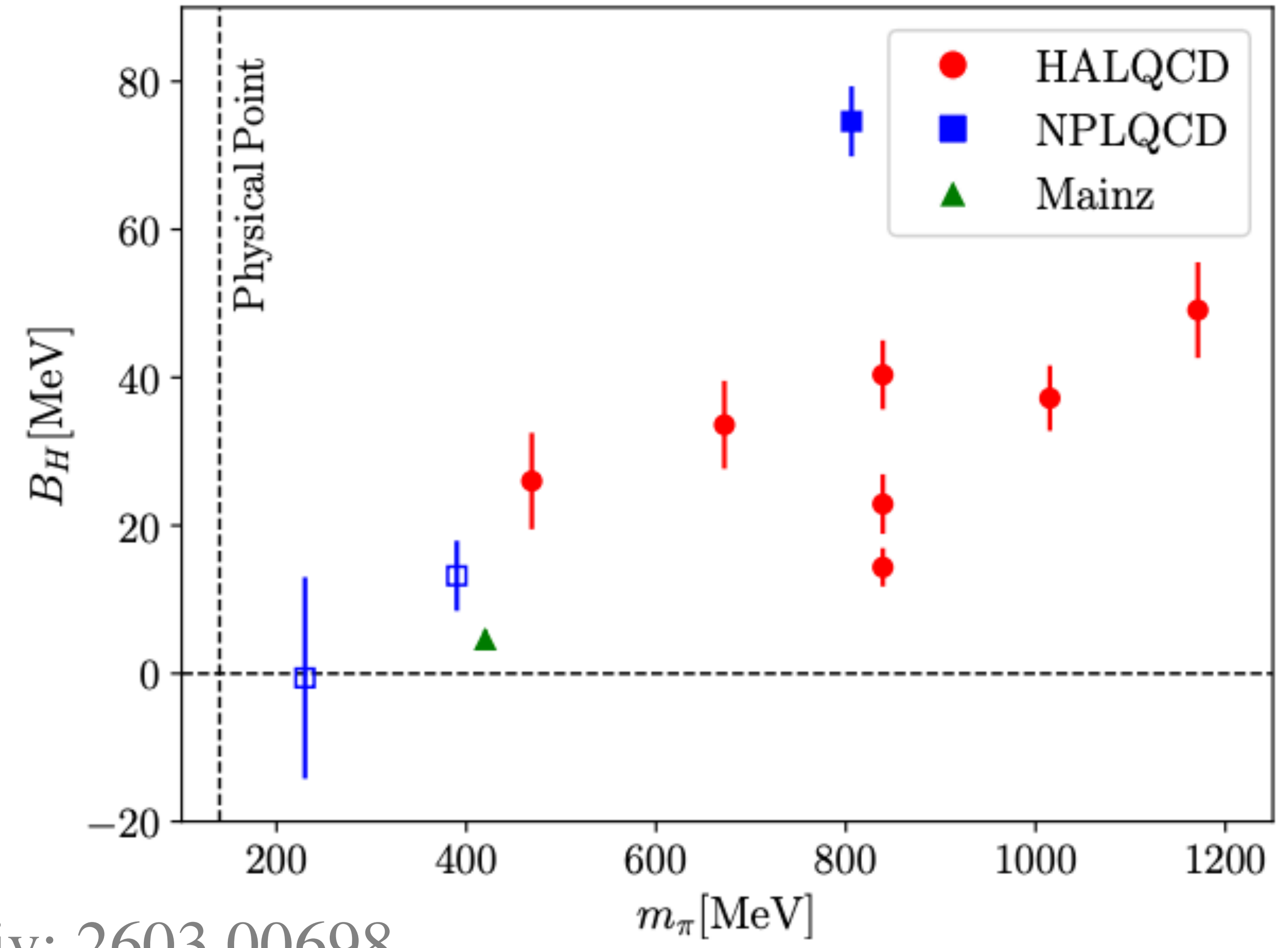


# LQCD study on H-dibaryon

H-dibaryon, a hypothetical six-quark (*uuddss*) bound state:

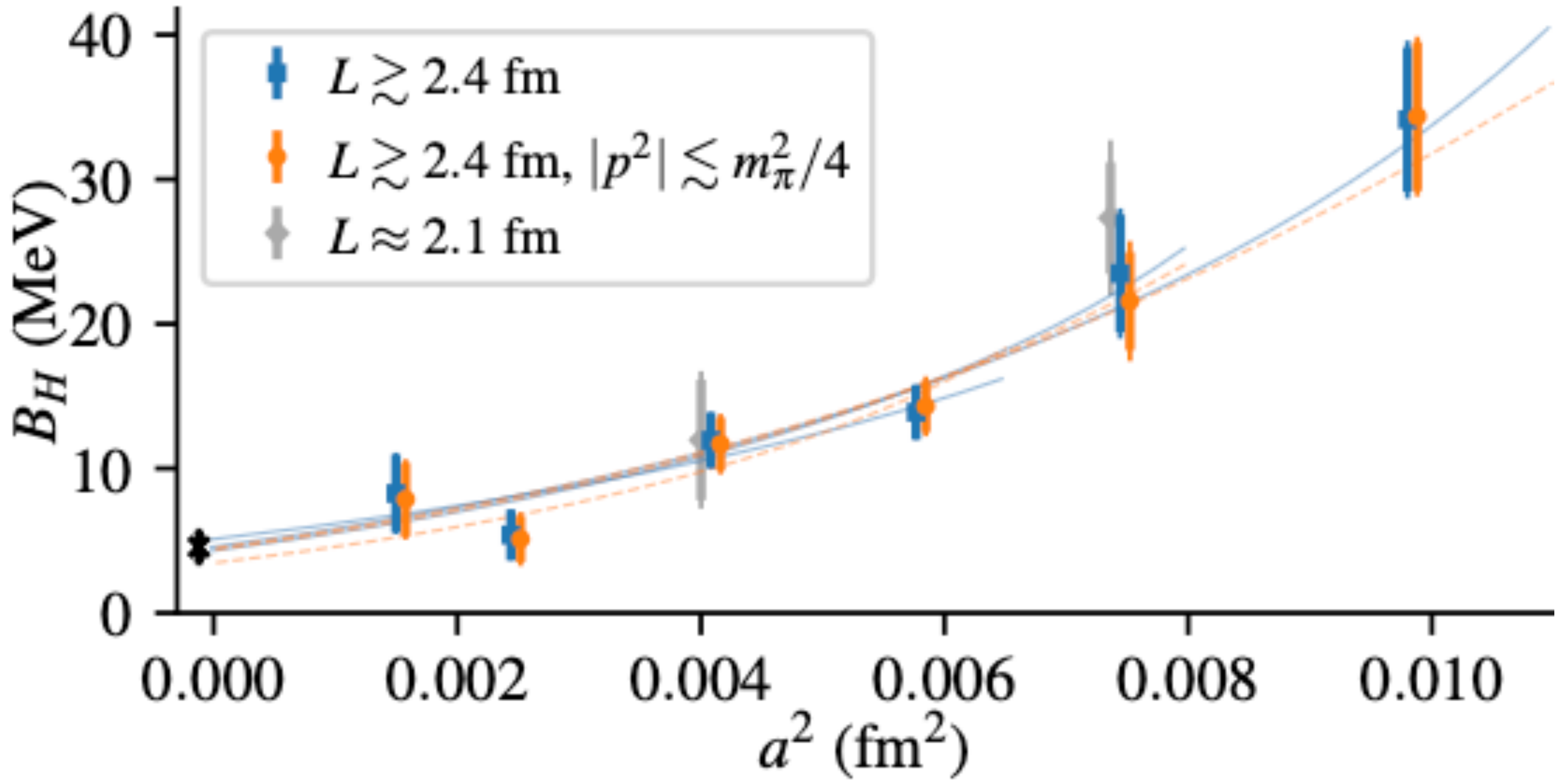
Jaffe 1977, PRL 38,195

- $J^P = 0^+,\ I = 0,\ S = -2$, flavor-singlet $-\sqrt{\frac{1}{8}}|\Lambda\Lambda\rangle + \sqrt{\frac{3}{8}}|\Sigma\Sigma\rangle + \sqrt{\frac{4}{8}}|N\Xi\rangle$
- Inconclusive in experiments → deeply bound, weakly bound, virtual state/resonance
- Weakly $m_\pi$ dependence ☞ contrast to deuteron & ideal to study left-hand-cut effect $(K, \eta)$ at heavier $m_\pi$
- No agreement on binding energy ← discretization errors / spacing dependence




BaSc, arXiv: 2603.00698

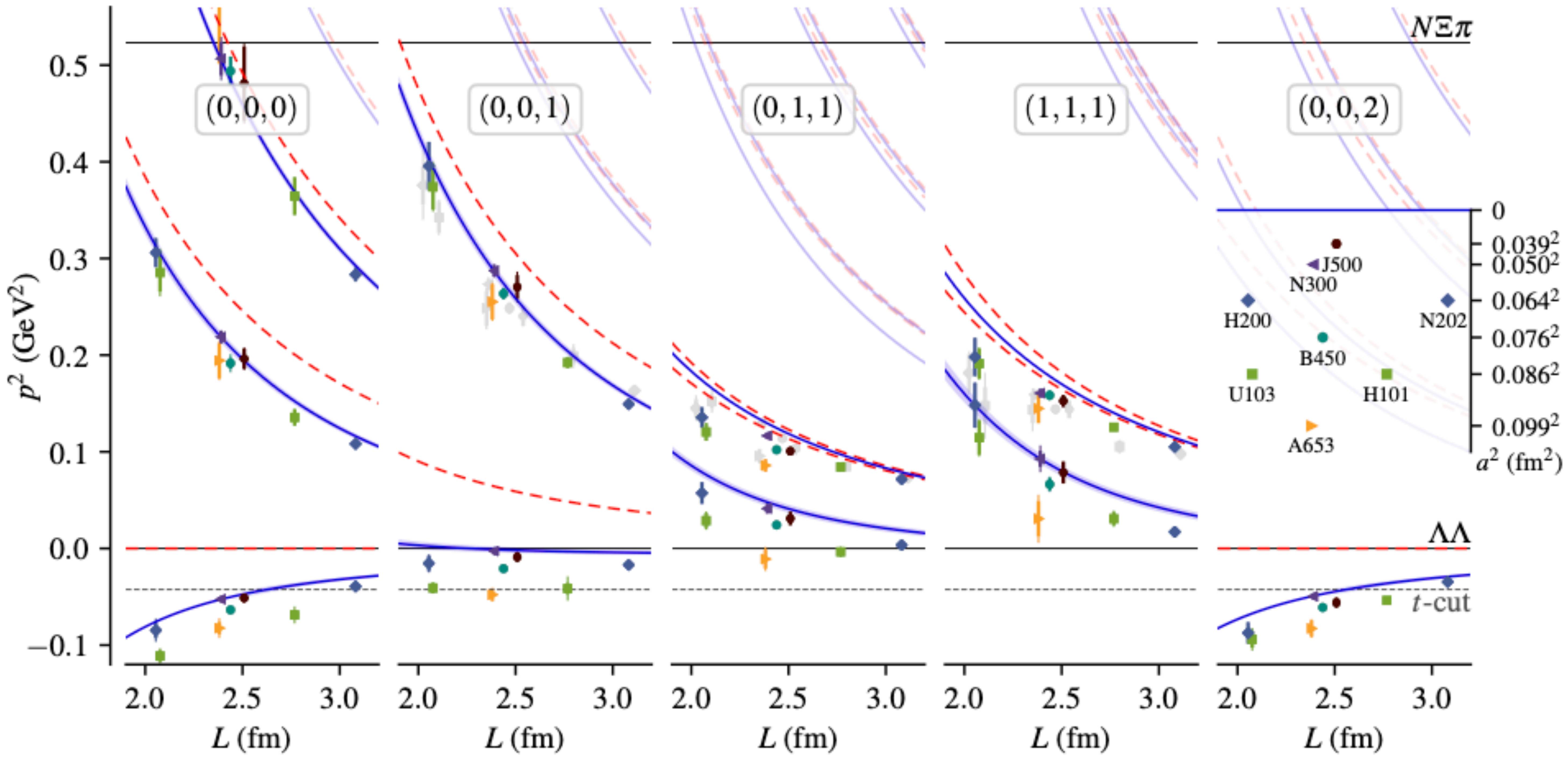
Green et al., PRL 127.242003




More works/talks are available!

# H-dibaryon at $\mathrm{SU}(3)_{\mathrm{F}}$-symmetric point

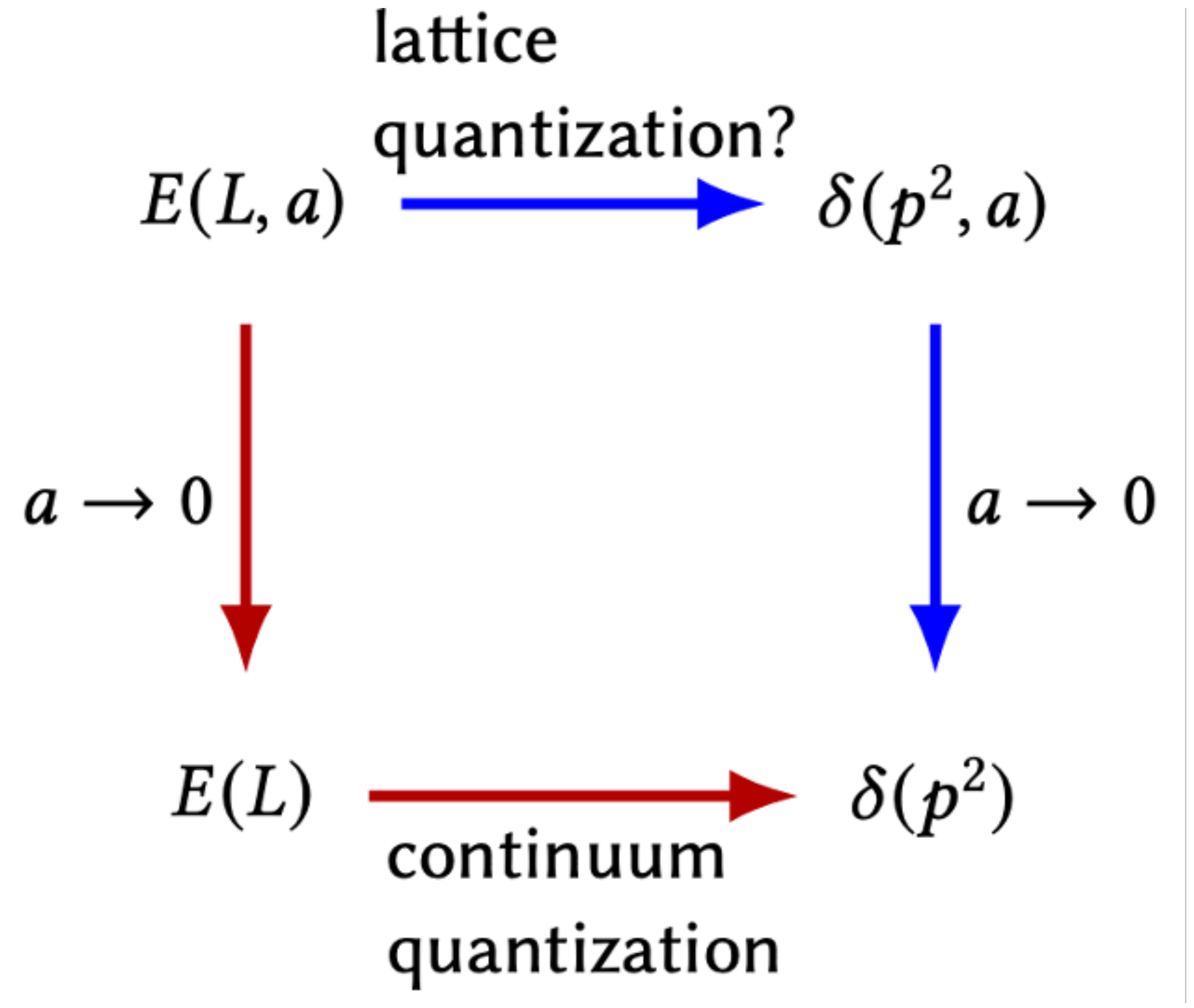
Green et al., PRL 127.242003




- $\mathcal{O}(a)$-improved Wilson fermions → 6 $a$
- $m_\pi = m_K = m_\eta \approx$ 420MeV
- 8 ensembles $(L, a)$ + 5 frames

$$p\cot\delta(p) = \sum_{i=0}^{N-1} c_i p^{2i}, \quad c_i = c_{i0} + c_{i1}a^2$$

From J.R.Green's talk




. Near-threshold fitting $(|p^2| \leq \frac{m_\pi^2}{4})$: $N = 2$ (×2)

☞ $B_H^{\mathrm{SU(3)_f}} = 4.56 \pm 1.13 \pm 0.63\mathrm{MeV}$

. Full-range fitting $(p^2 \geq -\frac{m_\pi^2}{4})$: $N = 3$ (×2)

Levels below t-cut not covered

# Fitting schemes including left-hand cut

- Fitting range: lowest energy level $\rightarrow$ $N\Xi\pi$ threshold (covering lhc)

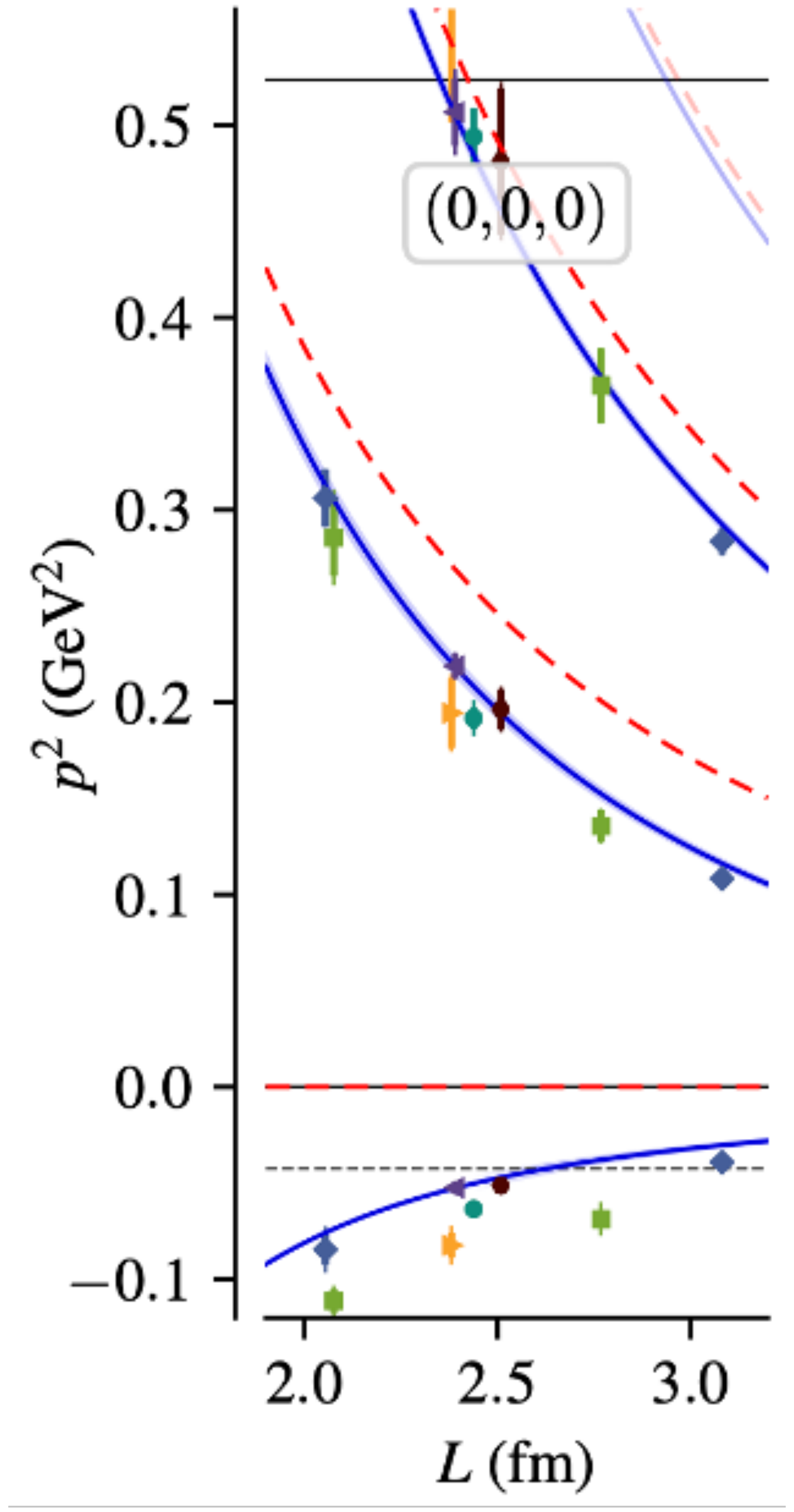
•Lüscher QC

$$\begin{cases} \text{ERE}: t(s) = [\frac{2}{\sqrt{s}}(\frac{1}{c_0}+c_1 p(s)^2+c_2 p(s)^4) - i\rho(s)]^{-1} \\ \text{ERE CM}: t(s) = [\frac{2}{\sqrt{s}}(\frac{1}{c_0}+c_1 p(s)^2+c_2 p(s)^4) + I(s)]^{-1} \end{cases}$$

$g_0 = 0$

•N/D QC

$$\begin{cases} N(s) = n_0+g_0{}^2 L_t(s) \\ D_{\mathrm{FV}}(s,\boldsymbol{P}) = 1+d_0 s+d_1 s^2 - \frac{16\pi}{L^3}\sum_{\boldsymbol{k}} L(\boldsymbol{k},\boldsymbol{P})\frac{s}{E^\star(\boldsymbol{k})^2}\frac{N(\boldsymbol{k}^\star,\boldsymbol{P})}{E^\star(\boldsymbol{k})^2 - s} \end{cases}$$




$\times 5$

- Spacing effect

$$\begin{cases} \text{w/o spacing dependence}: \quad \xi_i \\ \text{w/ spacing dependence}: \quad \xi_i = \xi_{i0} + \xi_{ia}a^2 \end{cases}$$

- Data covariance (61 points)

$$\chi^2 = \sum_{i,j}(p_i^2 - p_{\mathrm{FV},i}^2)\Sigma_{ij}^{-1}(p_j^2 - p_{\mathrm{FV},j}^2)$$

$$\begin{cases} \Sigma_{\mathrm{stat,ij}}: \ \text{from } 10^3 \text{ boostrapping samples} \\ \Sigma_{\mathrm{syst,ij}} = (\delta p^2)_i(\delta p^2)_j \end{cases}$$

# Predicted spectrum w/o a-dependence

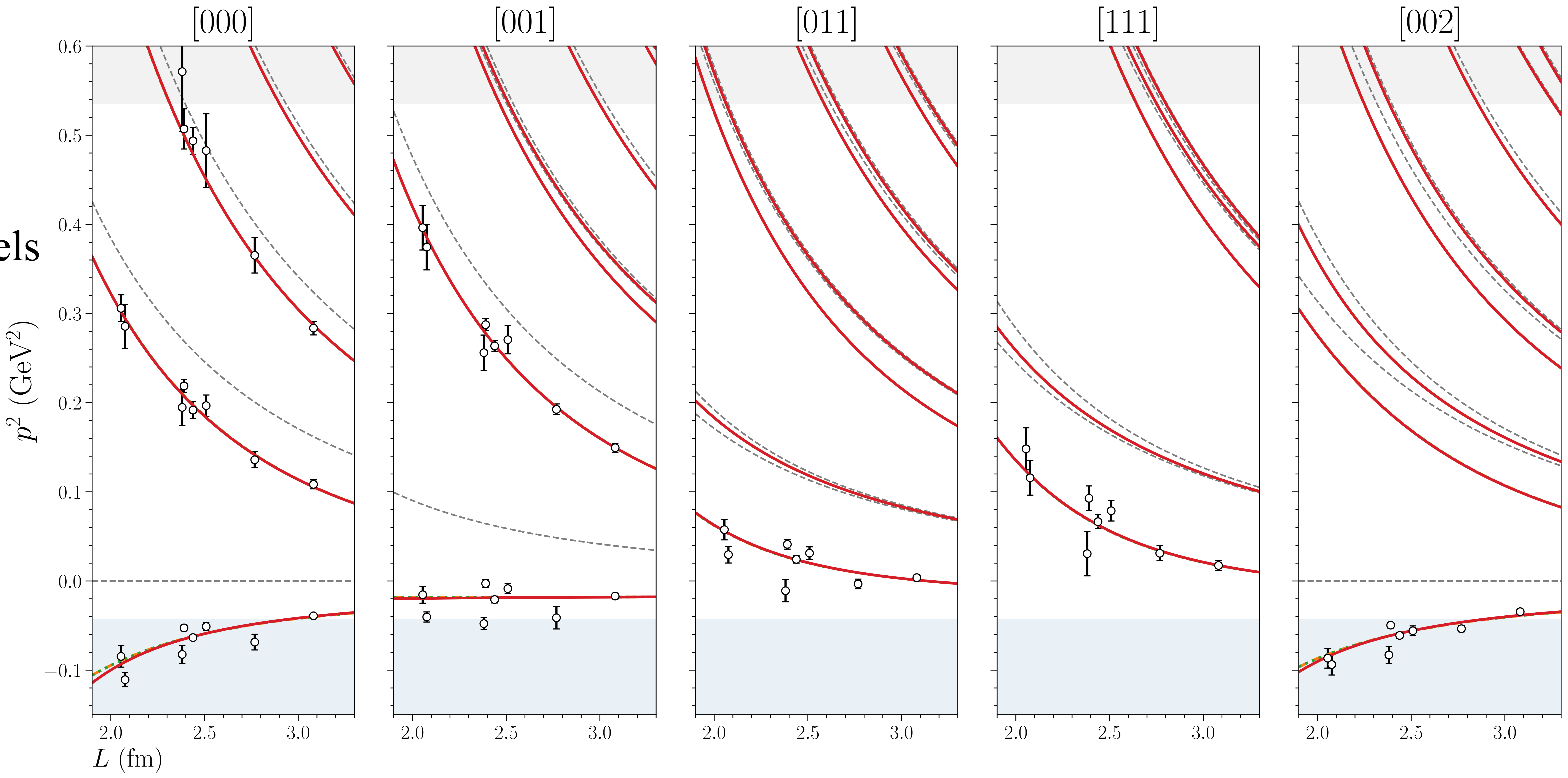
1.62 < 1.65

- Better $\chi^2/N_{\mathrm{dof}}$ in N/D

    ☞only differ in the lowest levels

    ☞Mild left-hand-cut effect

- Significant $L$ dependence
- Deeply bound $\leftarrow$ spacing effect




Orange dashed: ERE

Green dotted: ERE CM

Red solid: N/D

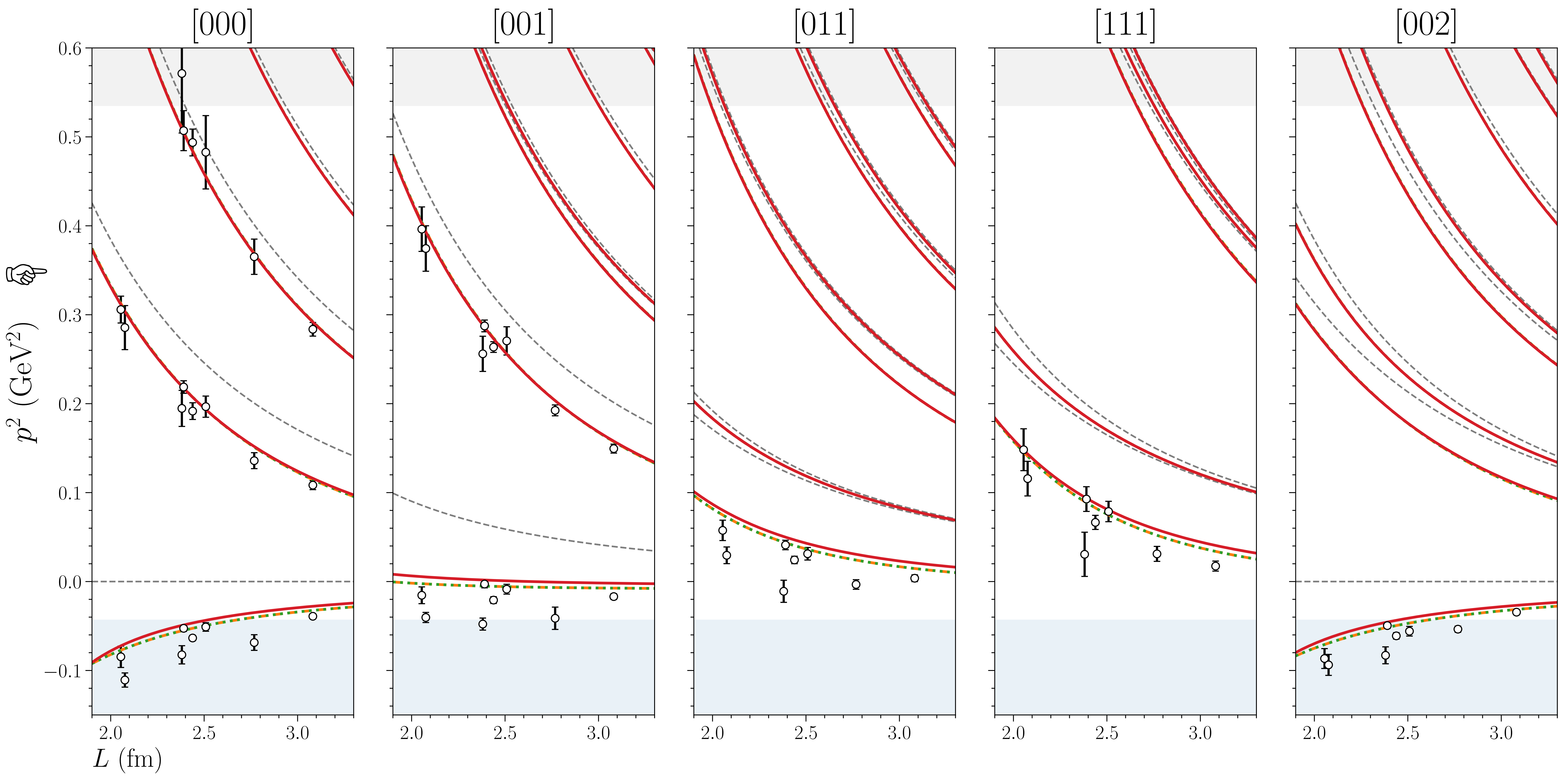
$\#(c_0, c_1, c_2)$

$\#(n_0, g_0, d_0, d_1)$

# Predicted spectrum w/ a-dependence

$0.7615 < 0.7622$

- Slightly better $\chi^2/N_{\rm dof}$ in N/D
- Spectrum in the continuum ($a = 0$) ☞
- Comparable parameters at $\mathcal{O}(a^0)$
  - Comparable observables
  - Shallow bound state




*Special thanks to Robert J.Perry!*

N/D model is rather stable in extrapolation

$$E(L,a) \xrightarrow{a\to 0} E(L) \xrightarrow{\rm QC} \delta(p^2) \text{ or } t(p^2)$$

Orange dashed: ERE $\#(c_{00}, c_{0a}, c_{10}, c_{1a}, c_{20}, c_{2a})$

Green dotted: ERE CM

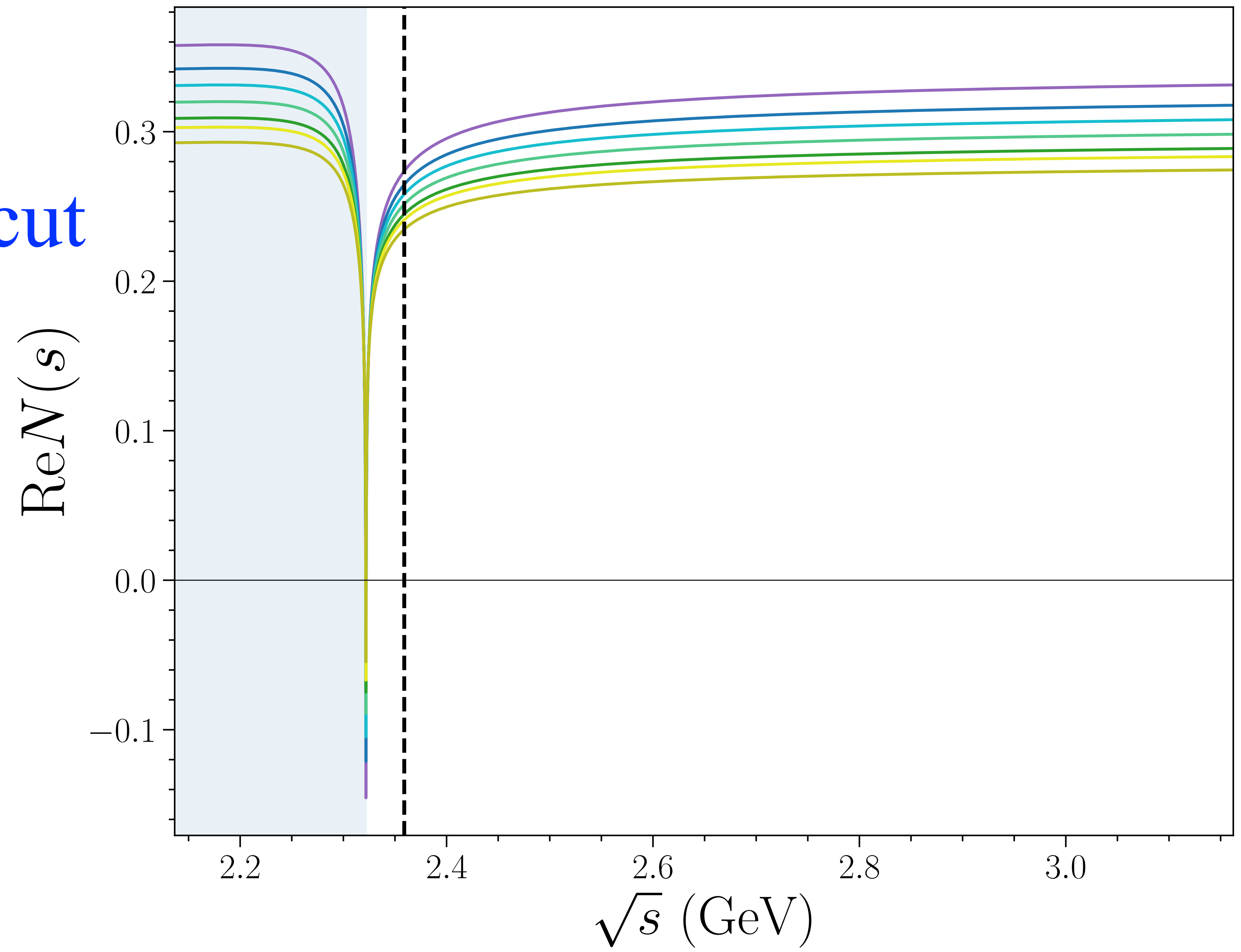
Red solid: N/D $\#(n_{00}, n_{0a}, g_{00}, g_{0a}, d_{00}, d_{0a}, d_{10}, d_{1a})$

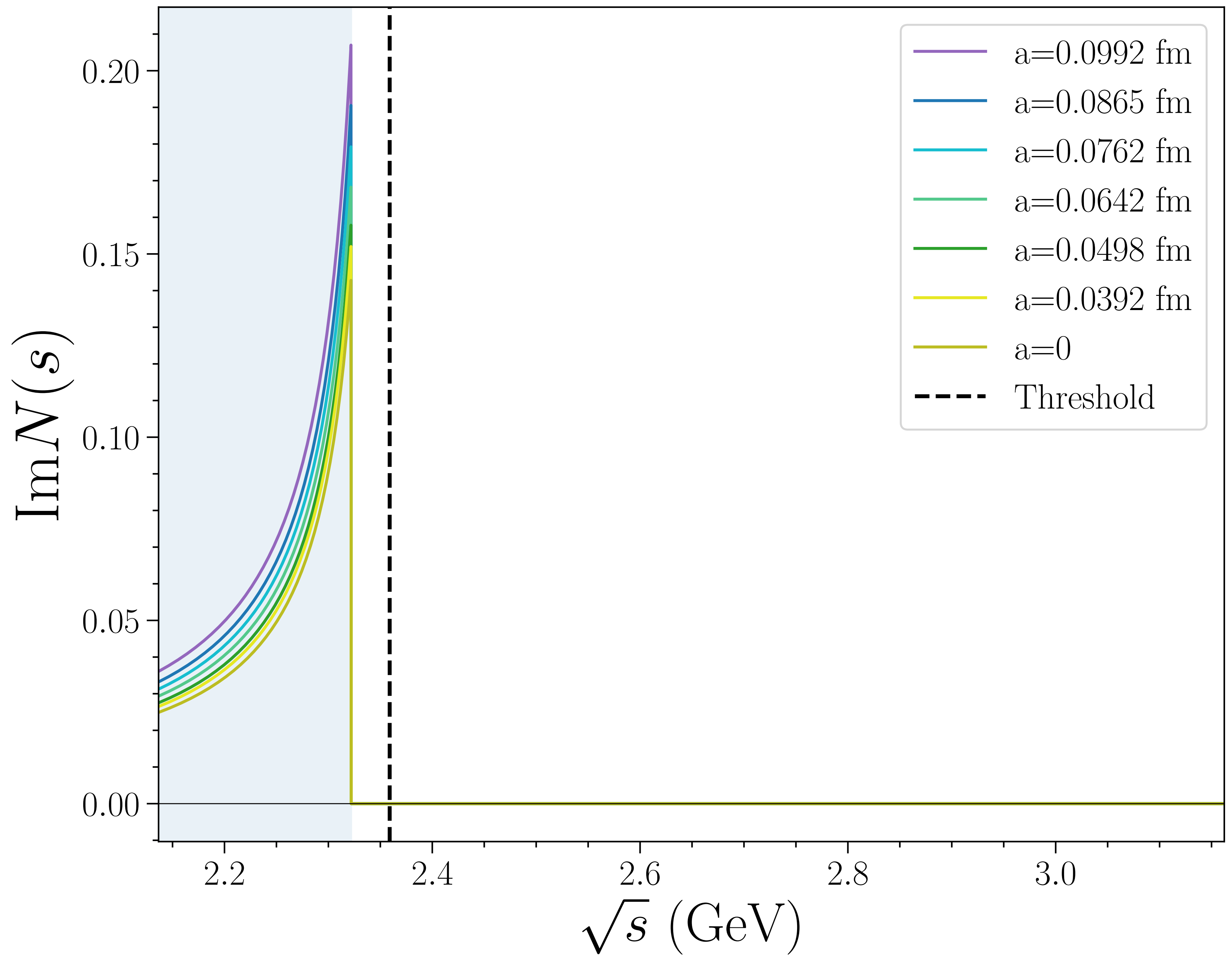
# N function and absence of CDD poles

$$t(s) = \frac{N(s)}{D(s)} = \frac{\tilde{N}(s)}{\frac{1}{s - s_z} + \cdots}$$

- Manifests around left-hand cut
- Polynomial above threshold

☞ slight change in ERE



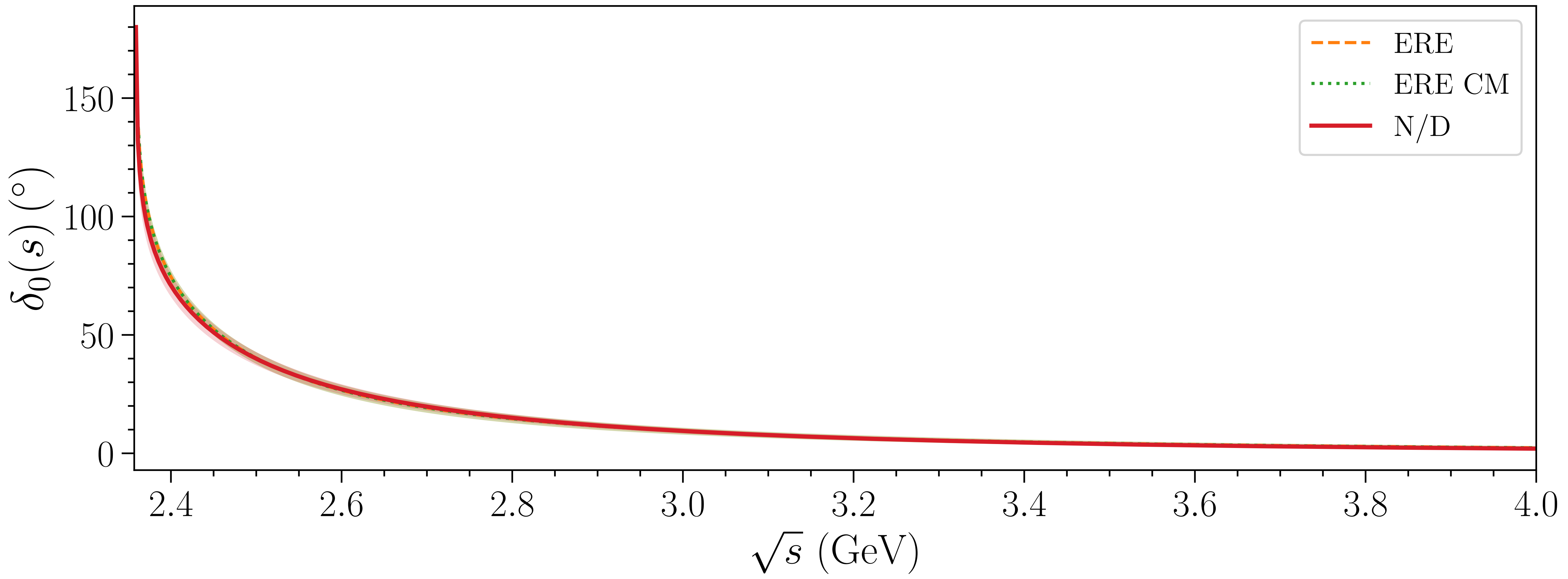



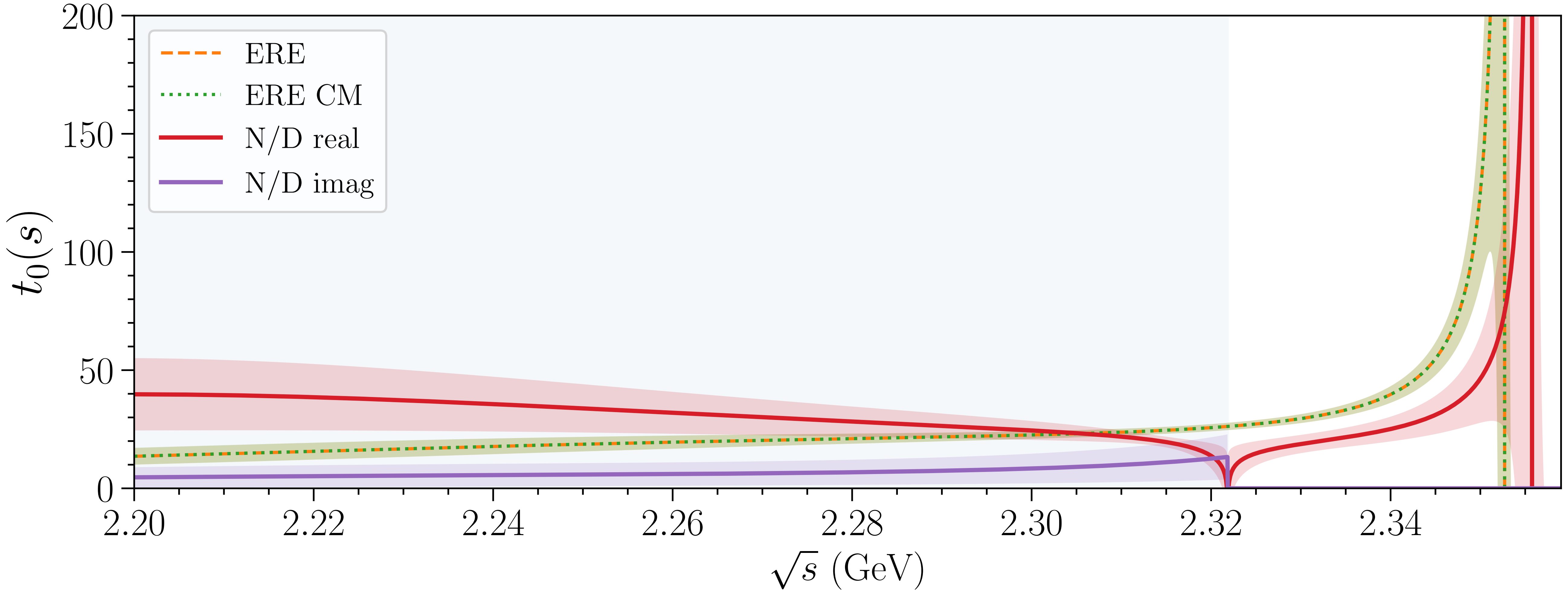
# Infinite-volume amplitudes

Results in the continuum ($a = 0$) from a-dependent fitting

$s > s_{\text{thr}}$:

- Overlapping energy levels ☞ phase shifts
- Drastic drop near threshold ☞ pole below




$s < s_{\text{thr}}$:

- Pole pushed upwards in N/D
- Larger uncertainty in N/D
- Nonzero imaginary part in N/D below lhc

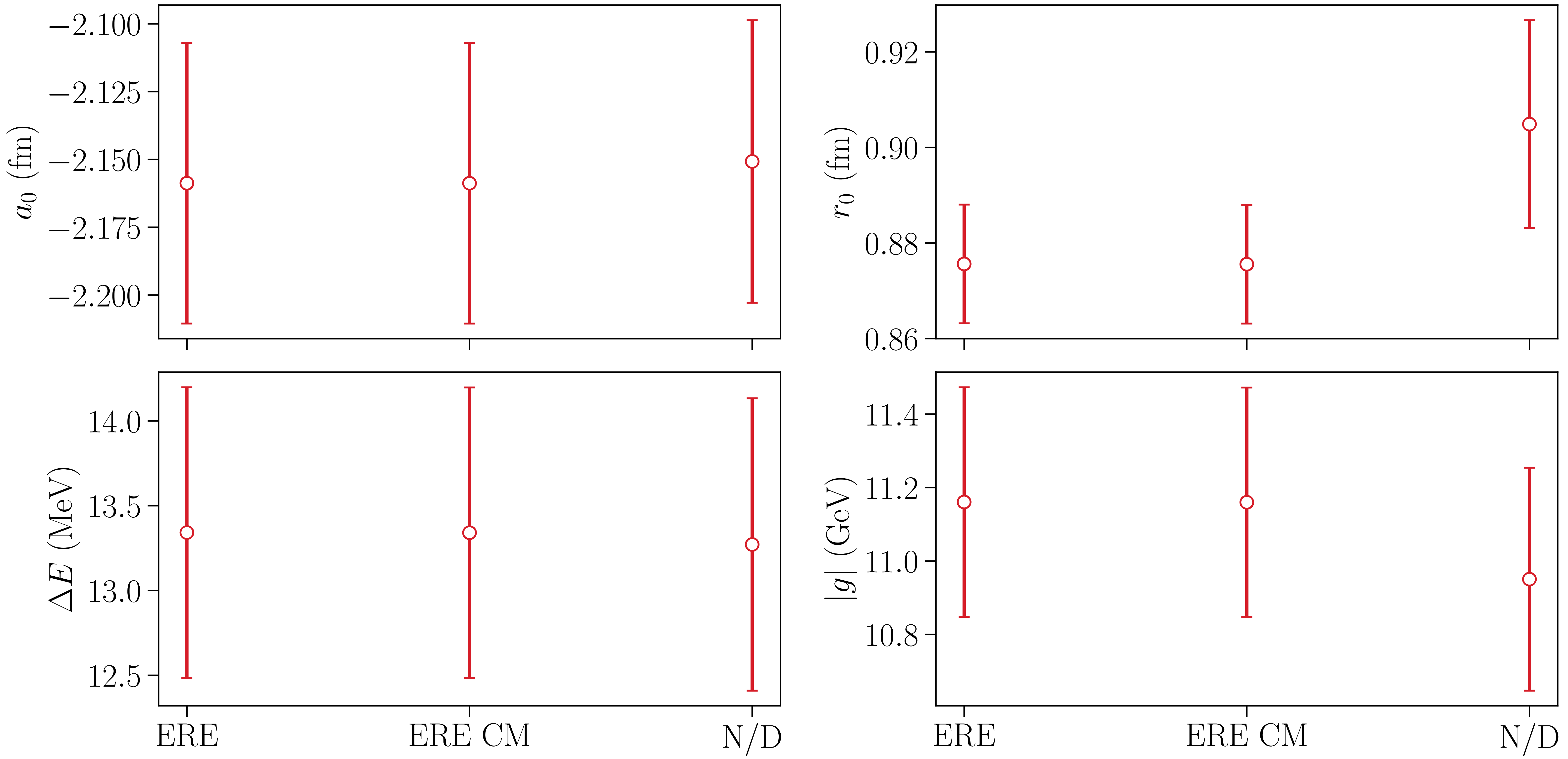
# Observables w/o a-dependence

Near threshold:

$$t_0(s) = \frac{\sqrt{s}}{2}[\frac{1}{a_0} + \frac{r_0}{2}p^2 - ip]^{-1}$$

Near pole:

$$t_0(s) \approx -\frac{1}{16\pi}\frac{|g|^2}{s - s_p}$$




Smaller $|a_0|$ and $r_0$ ☞ more bound

# Observables w/ a-dependence

In the continuum, N/D gives

Systematic uncertainty: $1\sigma\ m_B$

$$a_0 = (-3.737 \pm 0.787 \pm 0.001)\text{fm}, \quad r_0 = (0.969 \pm 0.041 \pm 0.001)\text{fm}$$

$$\Delta E = (3.26 \pm 1.60 \pm 0.04)\text{MeV}, \quad |g| = (6.44 \pm 1.05 \pm 0.04)\text{GeV}$$

$$\bar{X}_A = 0.8115 \pm 0.0243 \pm 0.0001$$

Green et al., PRL 127.242003

$$B_H^{\mathrm{SU(3)_f}} = (4.56 \pm 1.13 \pm 0.63)\text{MeV}$$

0.78 for ERE
0.778 for deuteron

Near threshold:

$$t_0(s) = \frac{\sqrt{s}}{2}\left[\frac{1}{a_0} + \frac{r_0}{2}p^2 - ip\right]^{-1}$$

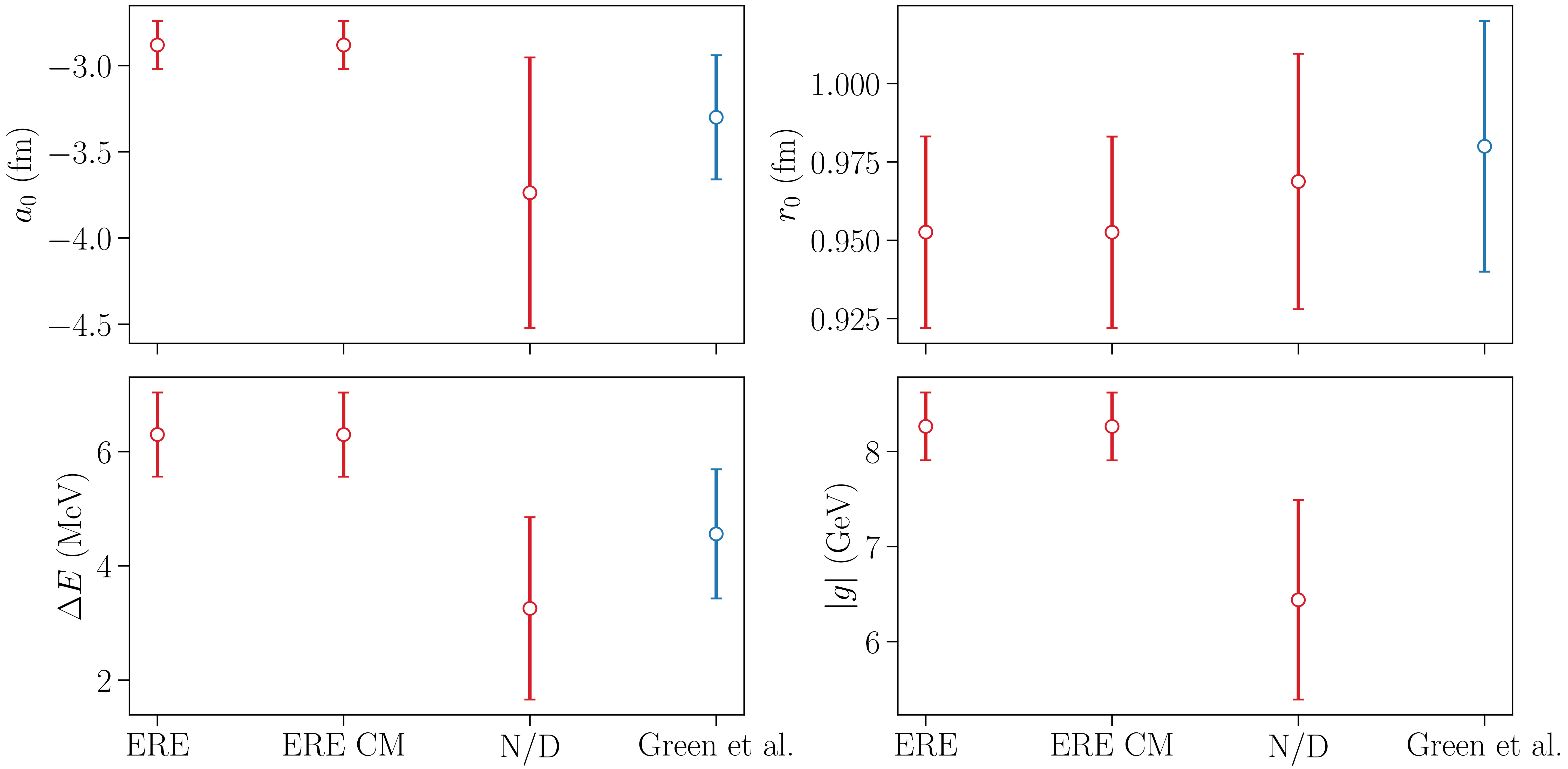
Near pole:

$$t_0(s) \approx -\frac{1}{16\pi}\frac{|g|^2}{s - s_p}$$

Inka Matuschek et al., EPJA57 (2021) 3, 101

Compositeness:

$$\bar{X}_A = 1 - \bar{Z}_A = \sqrt{\frac{1}{1 + |2r_0/a_0|}}$$



# Summary and outlook

- We implemented the first application of N/D representation to
  - finite-volume analysis of H-dibaryon at $SU(3)_F$-symmetric point
  - incorporate left-hand-cut effect by one-pion exchange & compare with Lüscher's
  - obtain reliable results in the continuum
    - ☞ shallow bound state, dominant molecular component
- We may extend to coupled-channel N/D case in the future
  - ☞ $\Lambda\Lambda - \Sigma\Sigma - N\Xi$, $NN$, $DD^* \cdots$
    - ↪ Juan's talk next

*Thank you!*

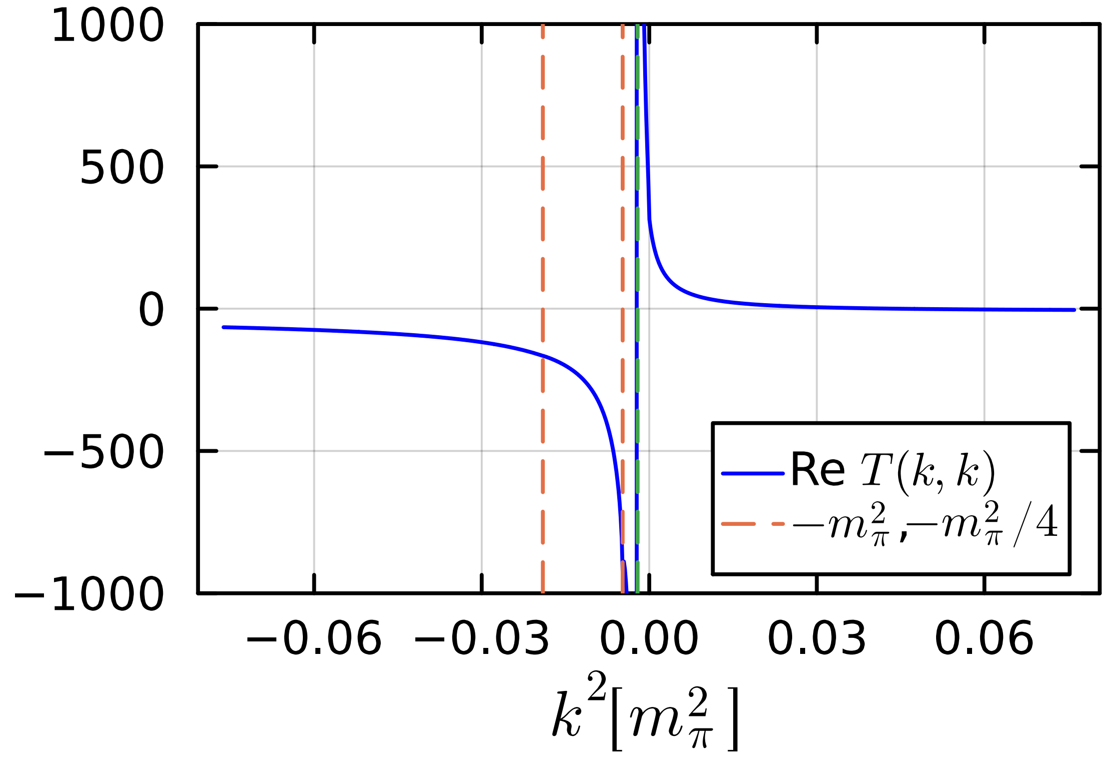
# Backup

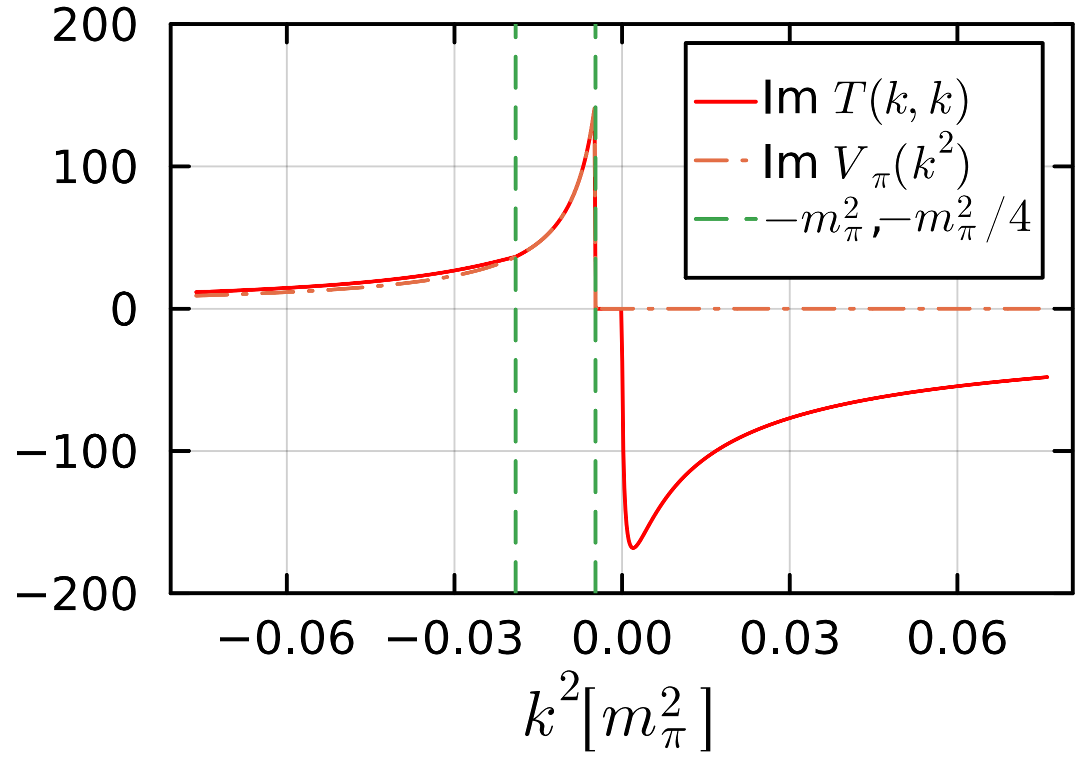
# N/D from LSE: bound deuteron

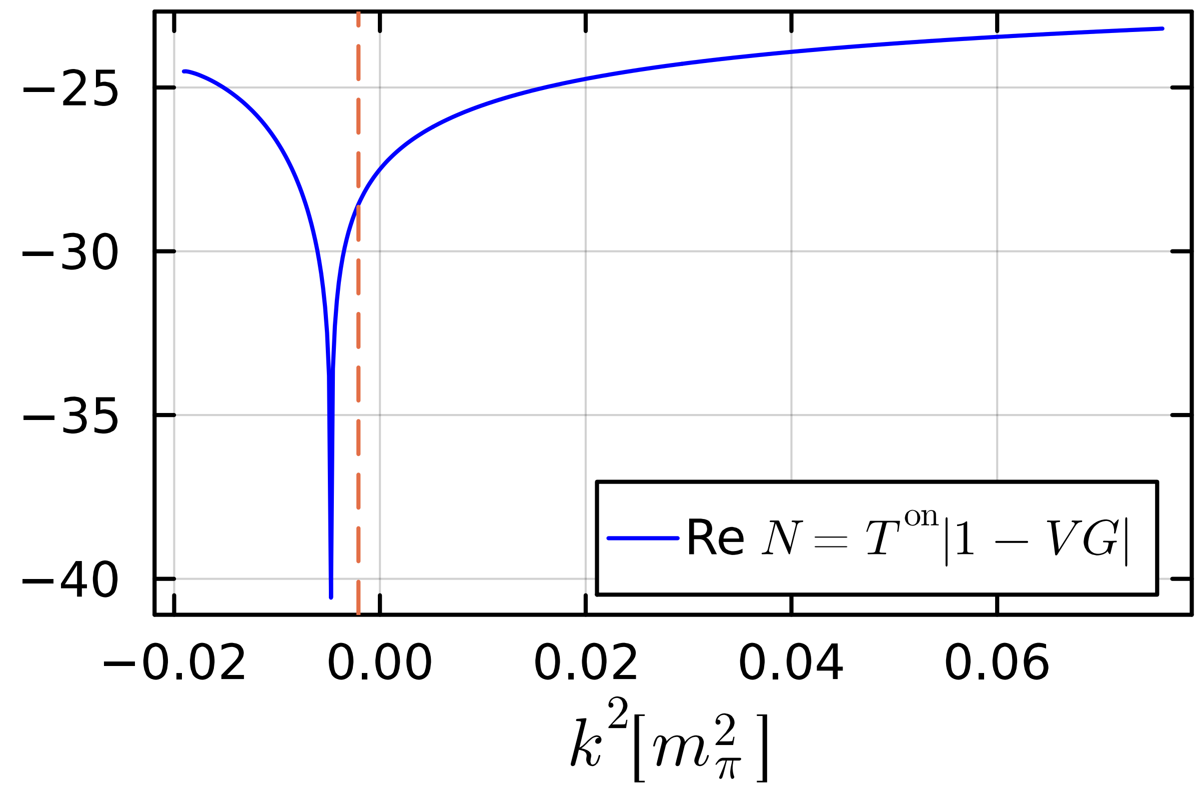
Solve the Lippman-Schwinger equation by matrix inversion in proper momentum set to avoid singularities,

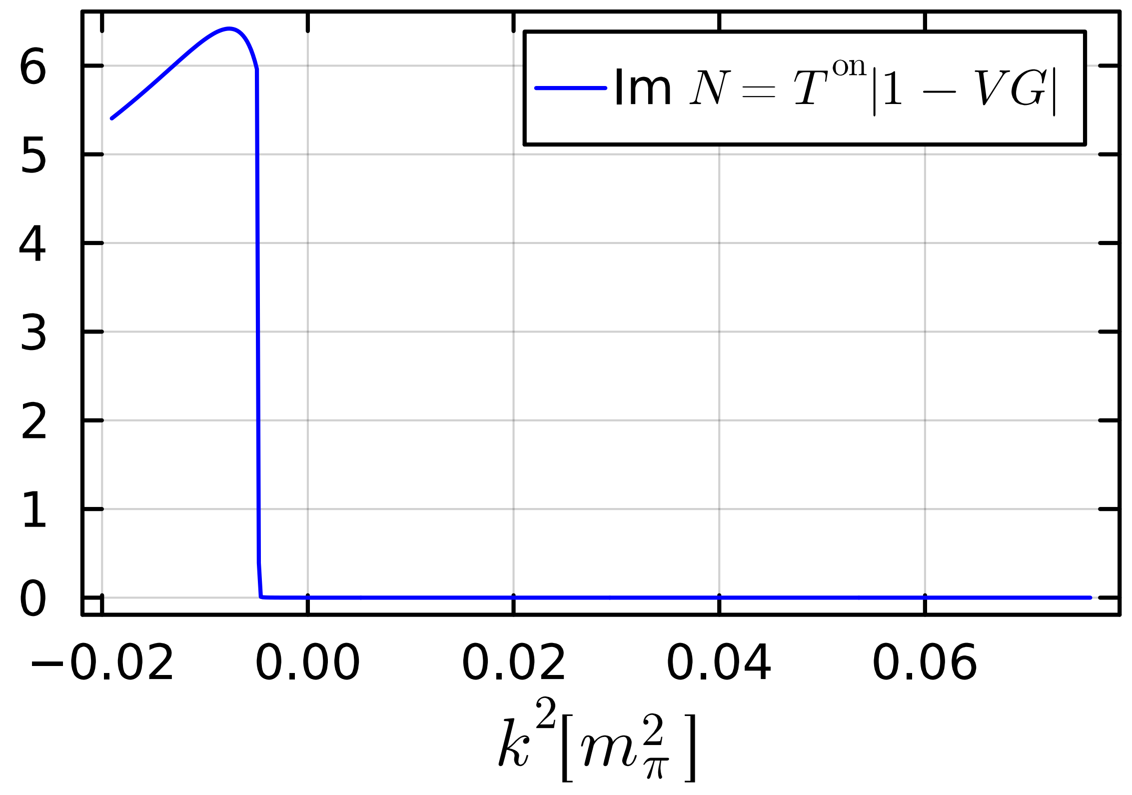
We have
$$T(p', p; E) = V(p', p; E) + \int \frac{d^3 l}{(2\pi)^3} \frac{V(p', l; E) T(l, p; E)}{E - m_1 - m_2 - l^2/(2\mu)}$$

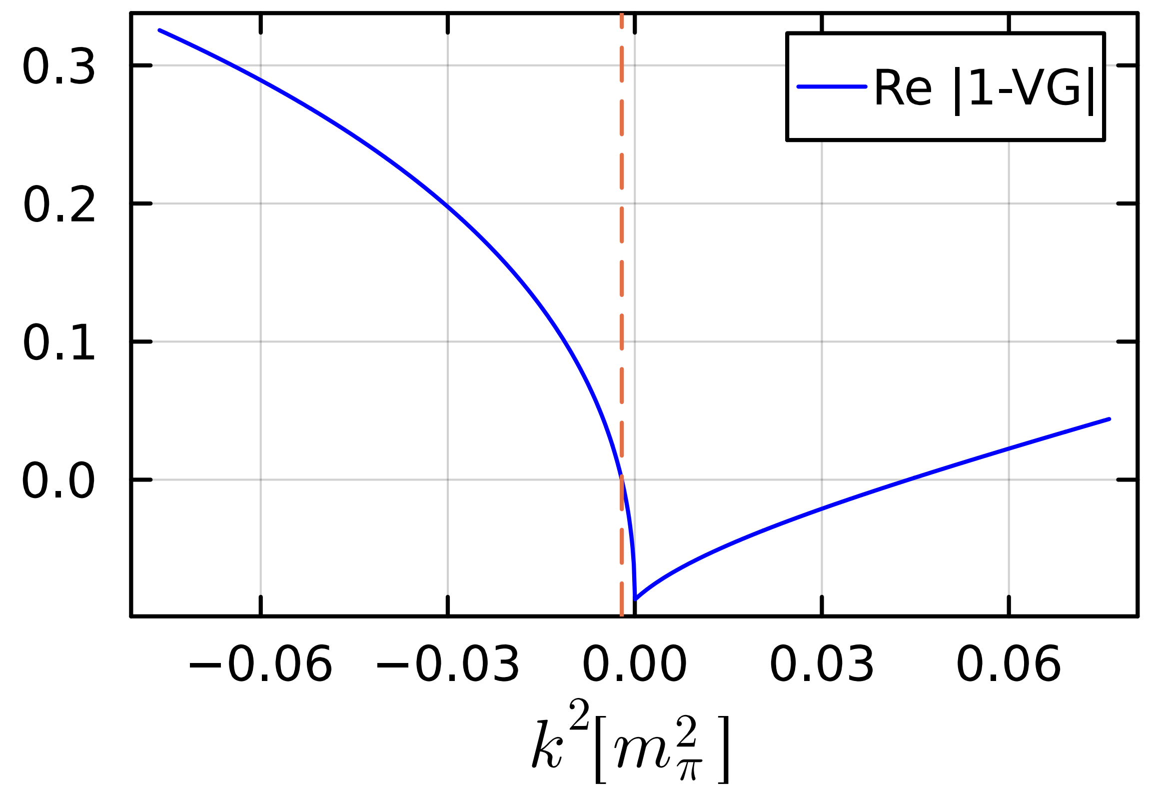
$$T(k^{\text{on}}, k^{\text{on}}; E) = \langle k^{\text{on}} | (1 - \boldsymbol{VG})^{-1} \boldsymbol{V} | k^{\text{on}} \rangle = \frac{N}{|1 - \boldsymbol{VG}|} = \frac{N}{D}$$

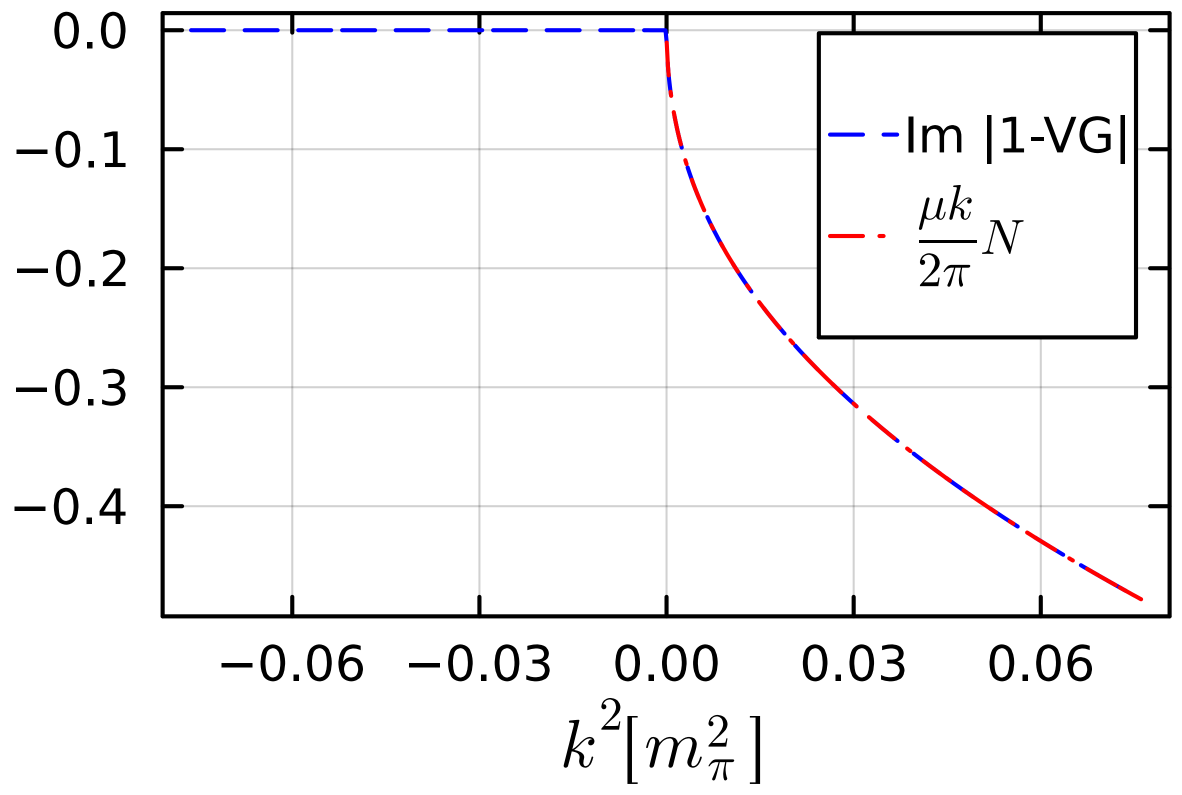


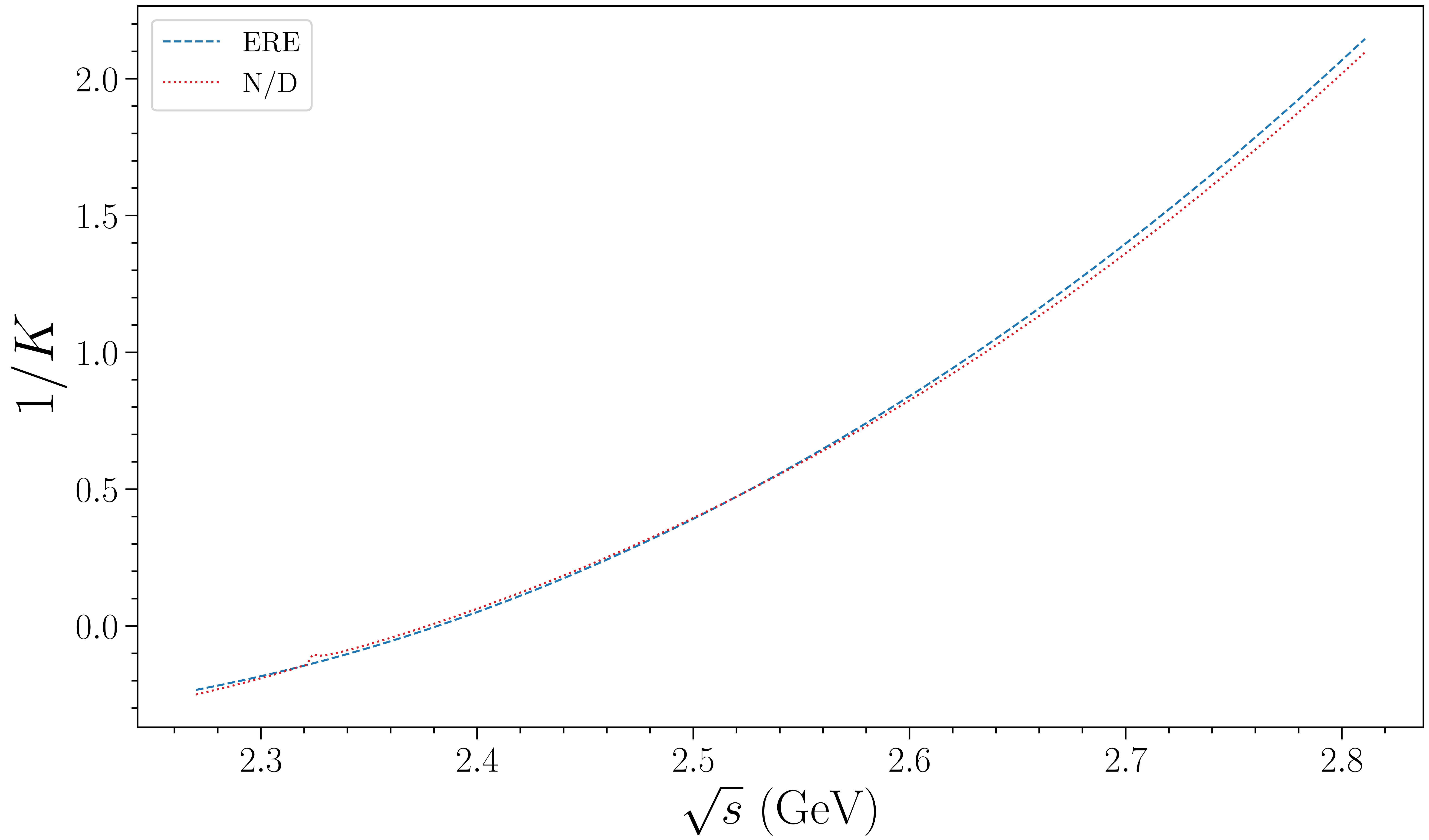



# Inverse of K matrix in the continuum

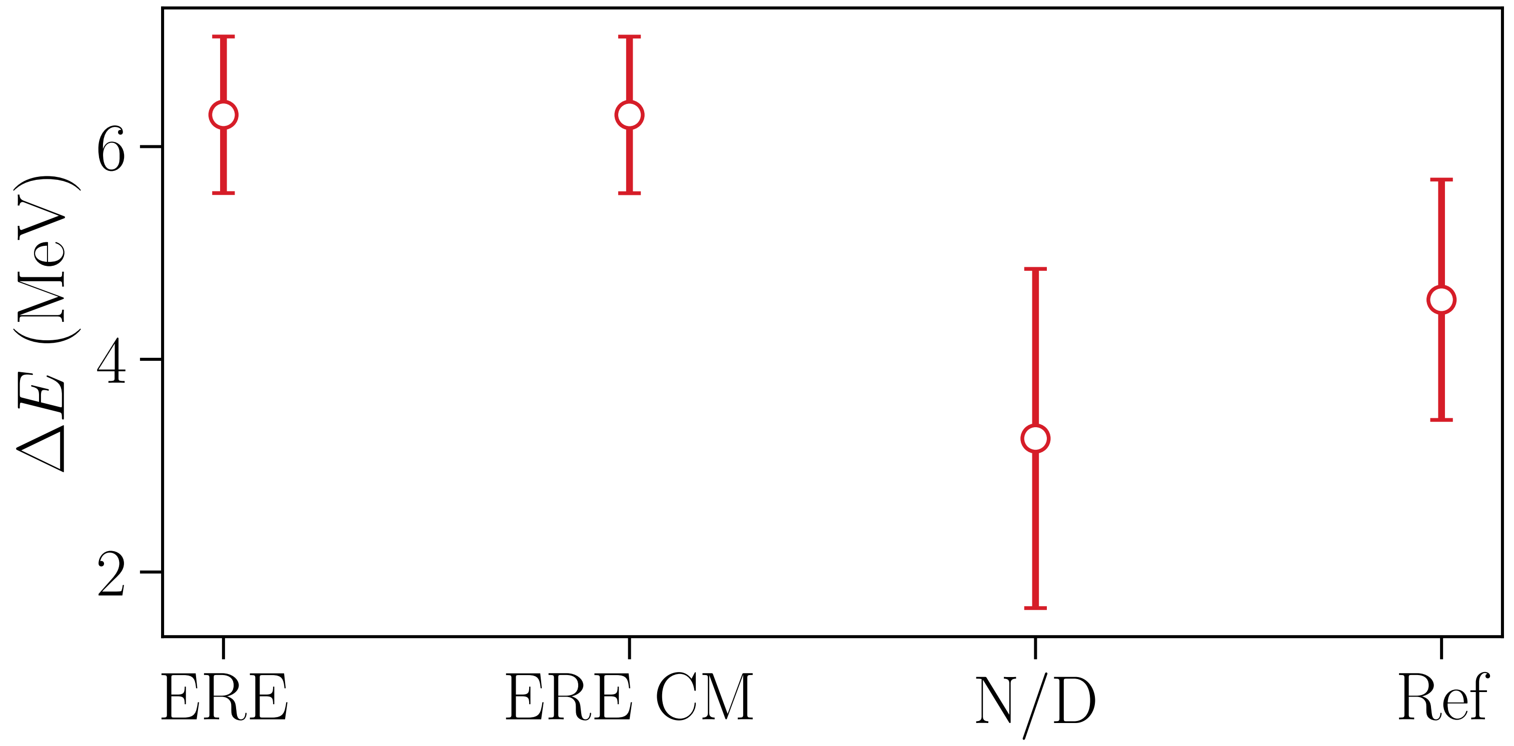
Above lhc,

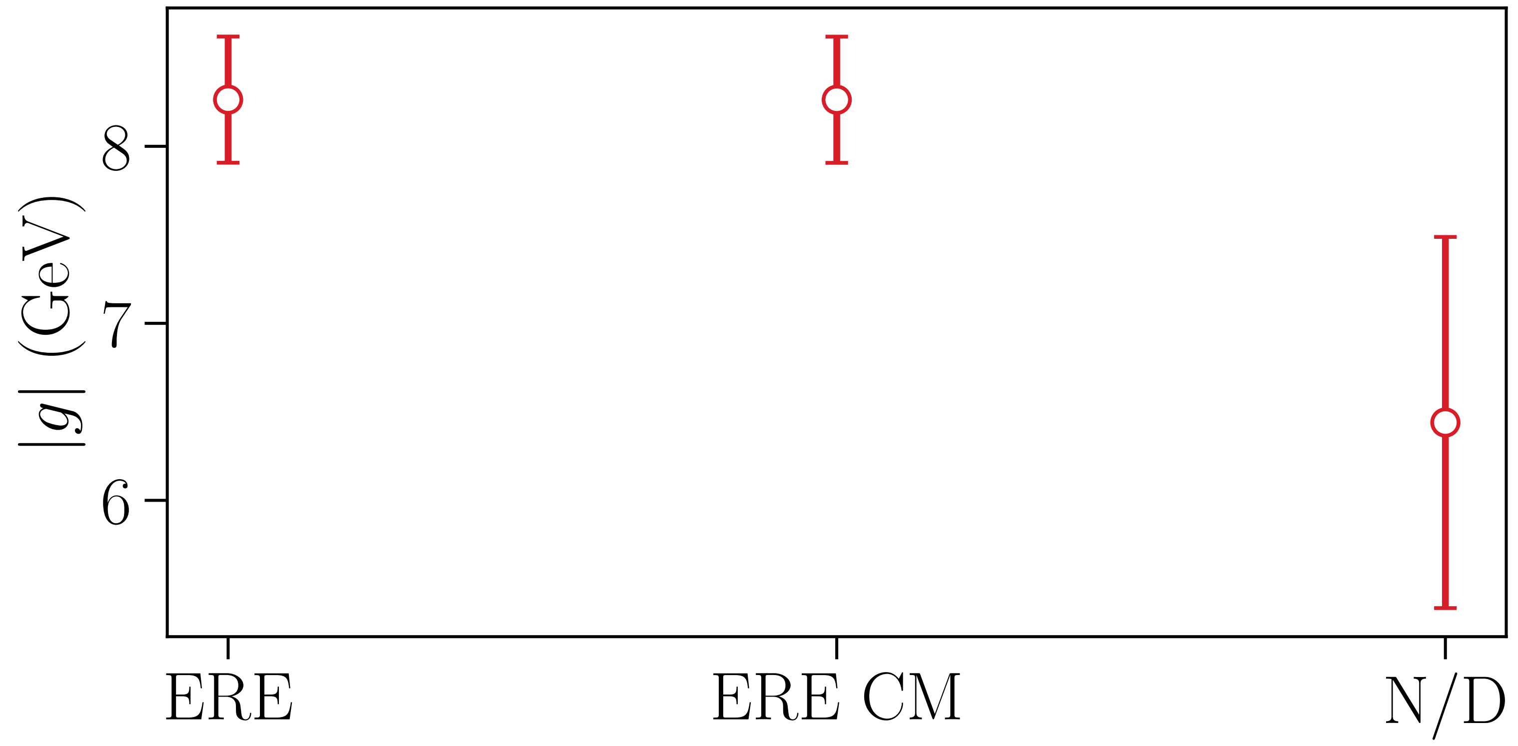
$$K^{-1}_{\mathrm{N/D}}(s) = \frac{\mathrm{Re}\ D(s)}{N(s)}$$



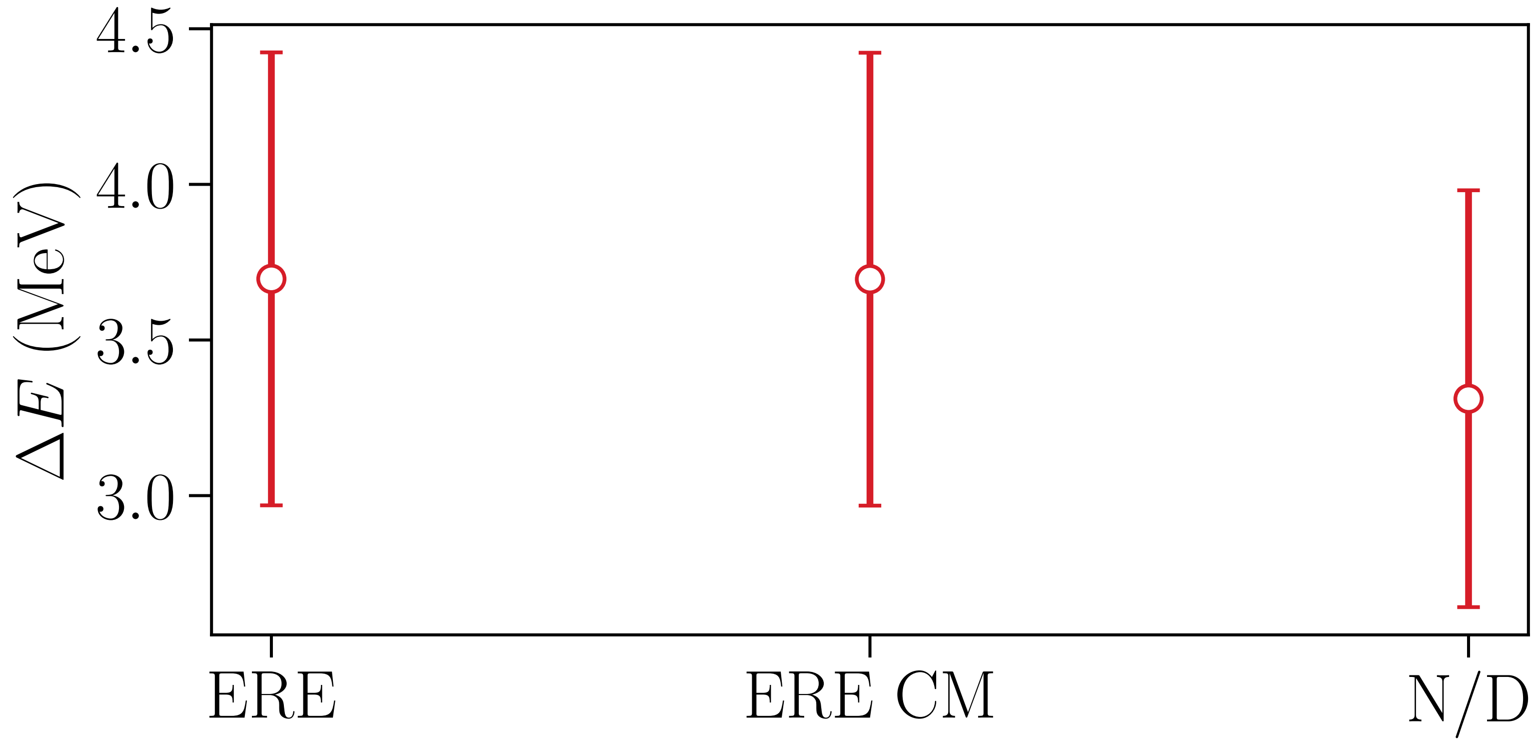
# Results in two extrapolations

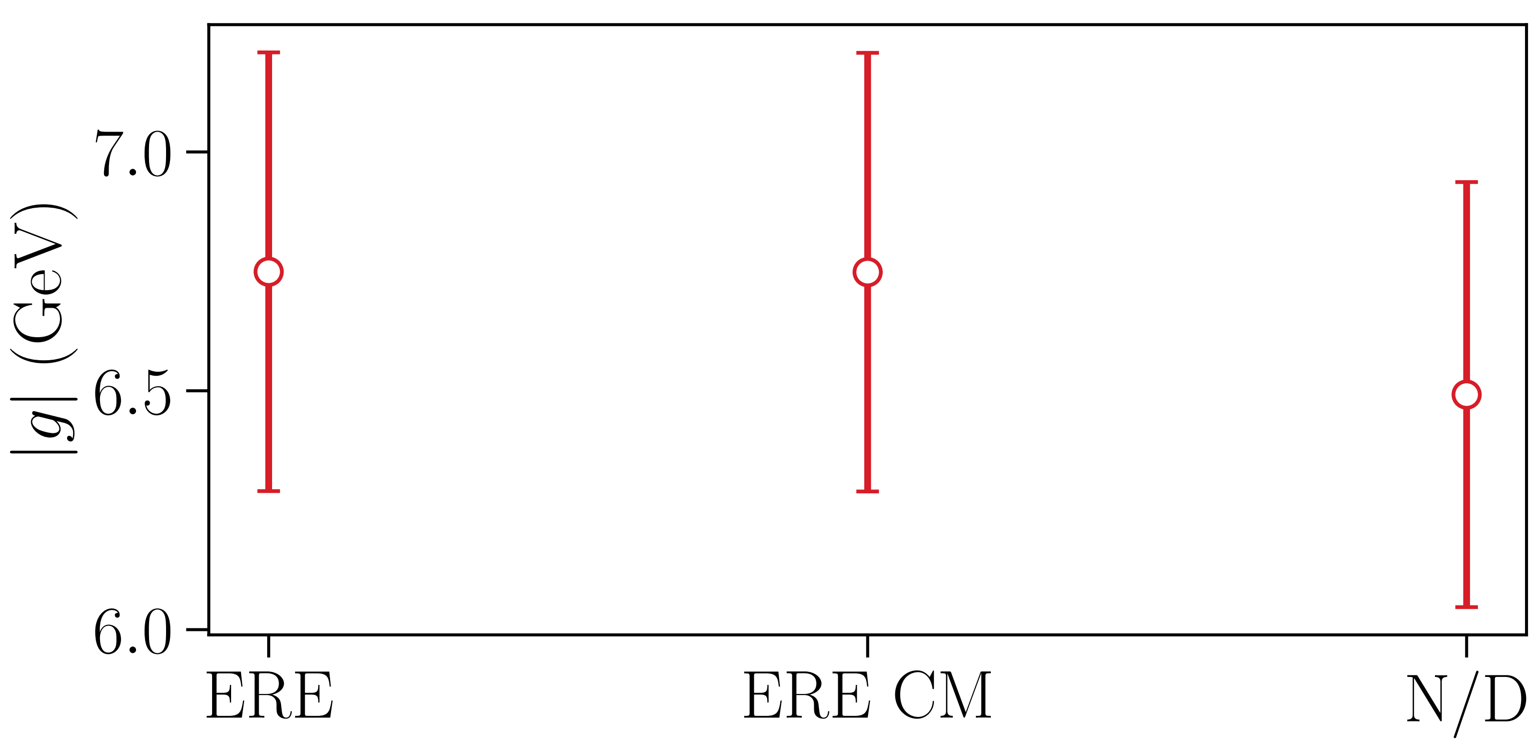
## Blue path:



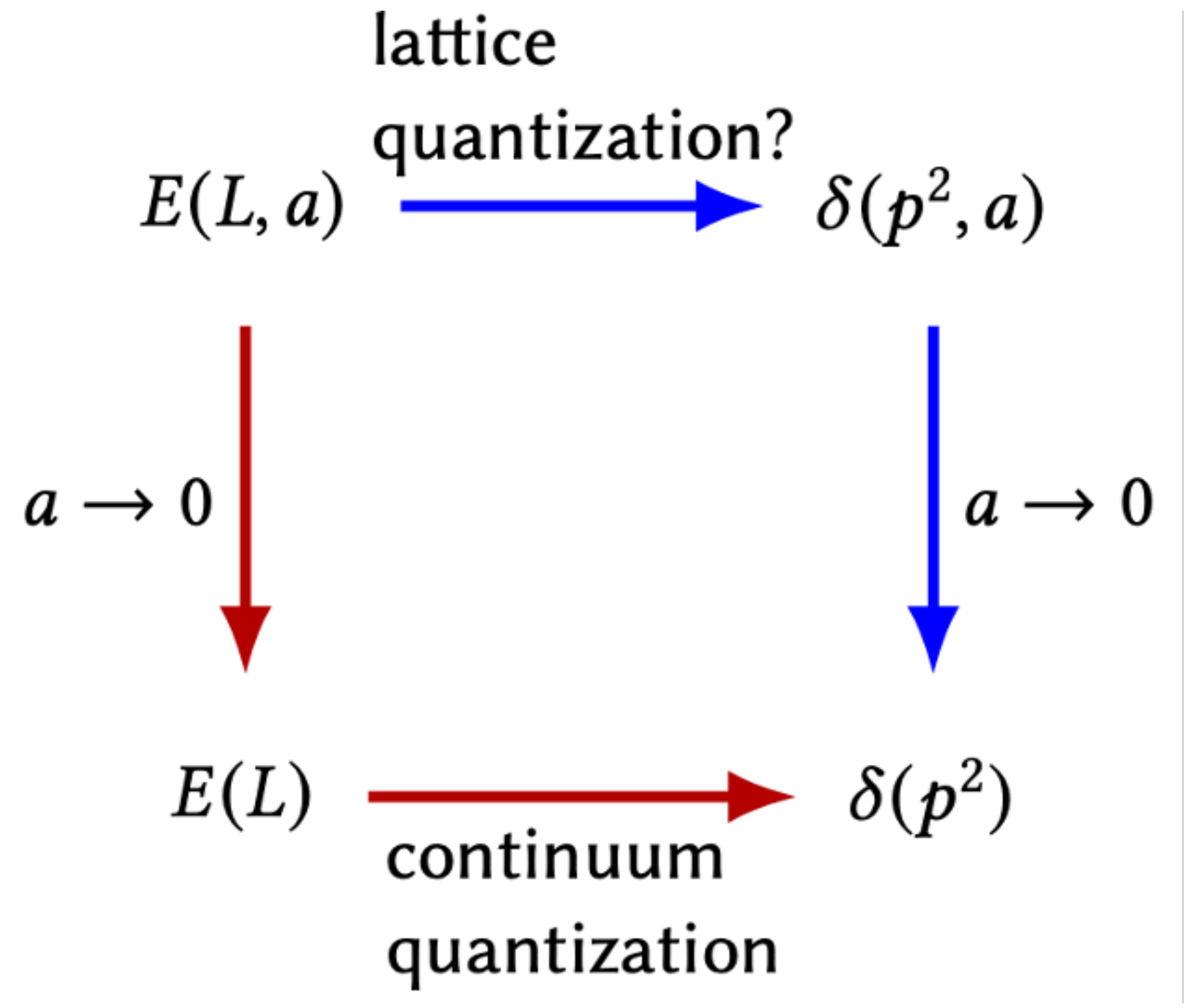




## Red path:

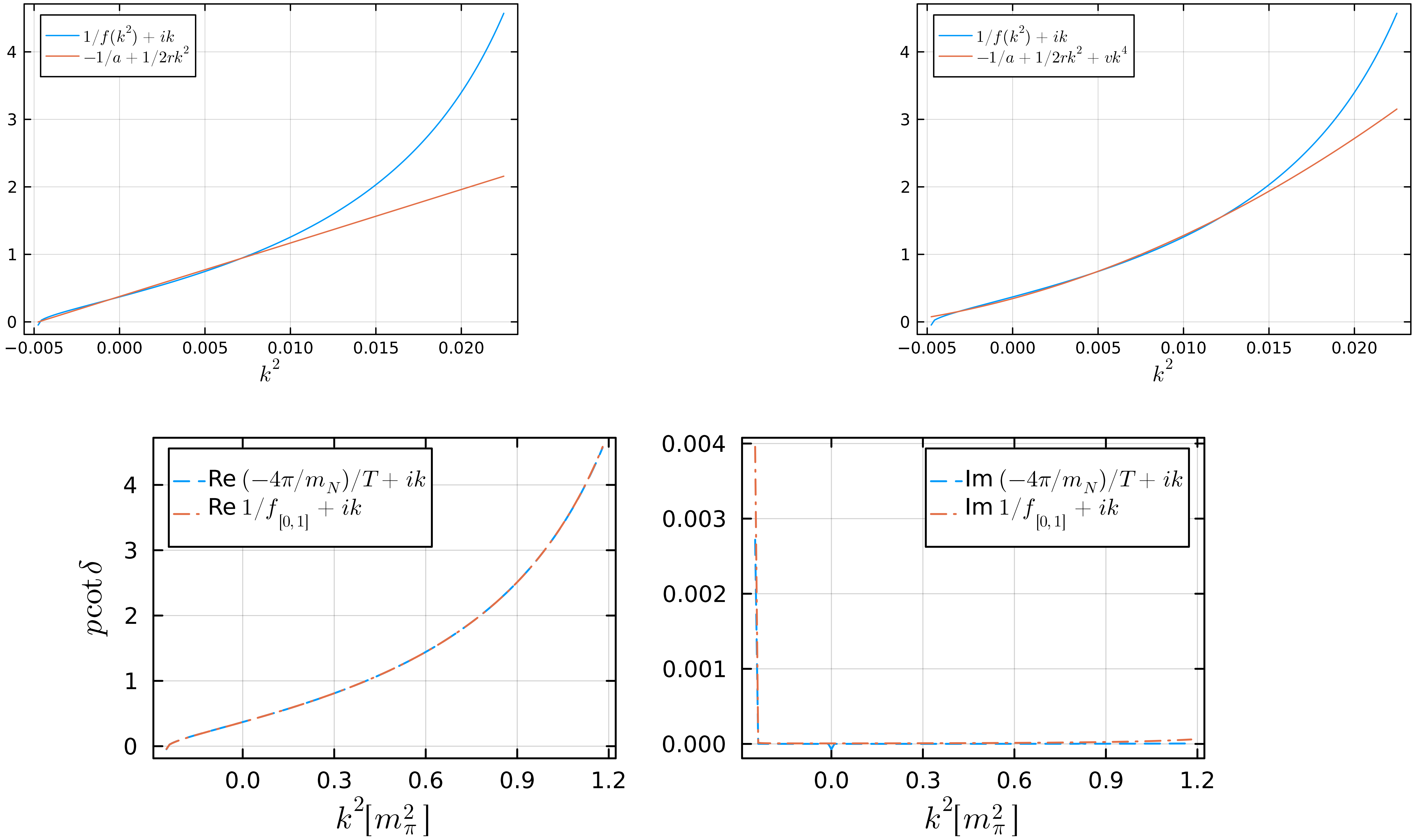



From J.R.Green's talk



# Demo of ERE with lhc: virtual state



# Glimpse of fitted parameters

## ERE w/ a-dependence

$$
\begin{aligned}
c_{00} &= -14.60(71) \\
c_{0a} &= 25.3(35) \\
c_{10} &= 2.414(77) \\
c_{1a} &= -1.34(53) \\
c_{20} &= 4.91(80) \\
c_{2a} &= -6.9(48)
\end{aligned}
\qquad
\begin{bmatrix}
1.00 & -0.95 & -0.61 & 0.45 & -0.17 & 0.11 \\
 & 1.00 & 0.65 & -0.58 & 0.16 & -0.13 \\
 & & 1.00 & -0.91 & 0.58 & -0.58 \\
 & & & 1.00 & -0.51 & 0.64 \\
 & & & & 1.00 & -0.91 \\
 & & & & & 1.00
\end{bmatrix}
$$

$$\chi^2/N_{\mathrm{dof}} = \frac{41.91}{61-6} = 0.76.$$

## N/D w/ a-dependence

$$
\begin{aligned}
n_{00} &= 0.280(25) \\
n_{0a} &= 0.24(18) \\
g_{00} &= 0.089(25) \\
g_{0a} &= 0.07(14) \\
d_{00} &= -0.4173(17) \\
d_{0a} &= 0.0342(93) \\
d_{10} &= 0.04780(43) \\
d_{1a} &= -0.0041(28)
\end{aligned}
\qquad
\begin{bmatrix}
1.00 & -0.85 & -0.03 & -0.03 & 0.35 & -0.49 & 0.73 & -0.61 \\
 & 1.00 & -0.08 & 0.09 & -0.52 & 0.34 & -0.39 & 0.84 \\
 & & 1.00 & -0.77 & -0.35 & 0.17 & -0.12 & 0.05 \\
 & & & 1.00 & 0.10 & -0.34 & 0.15 & 0.03 \\
 & & & & 1.00 & -0.37 & -0.30 & -0.38 \\
 & & & & & 1.00 & -0.26 & -0.16 \\
 & & & & & & 1.00 & -0.32 \\
 & & & & & & & 1.00
\end{bmatrix}
$$

$$\chi^2/N_{\mathrm{dof}} = \frac{40.35}{61-8} = 0.76.$$

## ERE w/o a-dependence

$$
\begin{aligned}
c_0 &= -10.94(26) \\
c_1 &= 2.219(31) \\
c_2 &= 3.85(33)
\end{aligned}
\qquad
\begin{bmatrix}
1.00 & -0.68 & -0.30 \\
 & 1.00 & 0.67 \\
 & & 1.00
\end{bmatrix}
$$

$$\chi^2/N_{\mathrm{dof}} = \frac{95.83}{61-3} = 1.65.$$

## N/D w/o a-dependence

$$
\begin{aligned}
n_0 &= 0.317(12) \\
g_0 &= 0.097(17) \\
d_0 &= -0.4127(13) \\
d_1 &= 0.04734(35)
\end{aligned}
\qquad
\begin{bmatrix}
1.00 & -0.40 & -0.18 & 0.83 \\
 & 1.00 & -0.57 & -0.11 \\
 & & 1.00 & -0.67 \\
 & & & 1.00
\end{bmatrix}
$$

$$\chi^2/N_{\mathrm{dof}} = \frac{94.30}{61-4} = 1.65.$$

# Amplitude zeros and CDD poles

A. Rodas, L. Qiu and JPAC members, arXiv:2605.22957

As explained in the main text, one crucial point of the discussion about the $N/D$ method is the presence or lack of Castillejo-Dalitz-Dyson (CDD) poles [67], i.e., poles of the denominator $D(s)$, which are not explicitly included in our parameterization. However, they can still be identified by zeros of the numerator above threshold. Without loss of generality, we suppose that there is only one simple root $s_z$ above threshold found in the numerator after fitting, i.e., $N(s_z) = 0$. As the amplitude does not change if one divides both $N(s)$ and $D(s)$ by a common factor, we can redefine the numerator and denominator as

$$\tilde{N}(s) = \frac{N(s)}{s - s_z}, \tag{S1}$$

$$\tilde{D}(s) = \frac{D(s)}{s - s_z} = \frac{\tilde{d}_{-1}}{s - s_z} + \sum_{j=0}^{n-1} \tilde{d}_j s^j + \frac{s}{\pi(s - s_z)} \int_{s_{\text{thr}}}^{\infty} ds' \frac{\rho(s')N(s')}{s'(s - s')}, \tag{S2}$$

where $\tilde{d}_j$ are polynomials in $s_z$ and the expansion coefficients $d_j$ of $D(s)$, and $\tilde{N}(s)$ is now free of zeros. Using the relation,

$$\frac{N(s')}{s - s_z} = \frac{N(s')}{s' - s_z} + \frac{N(s')(s' - s)}{(s' - s_z)(s - s_z)} = \tilde{N}(s') + \frac{s' - s}{s - s_z}\tilde{N}(s'), \tag{S3}$$

we have

$$\tilde{D}(s) = \frac{\tilde{d}_{-1}}{s - s_z} + \tilde{d}_0 + \sum_{j=1}^{n-1} \tilde{d}_j s^j + \frac{s}{\pi} \int_{s_{\text{thr}}}^{\infty} ds' \frac{\rho(s')\tilde{N}(s')}{s'(s - s')} - \frac{s}{\pi(s - s_z)} \int_{s_{\text{thr}}}^{\infty} ds' \frac{\rho(s')\tilde{N}(s')}{s'}, \tag{S4}$$

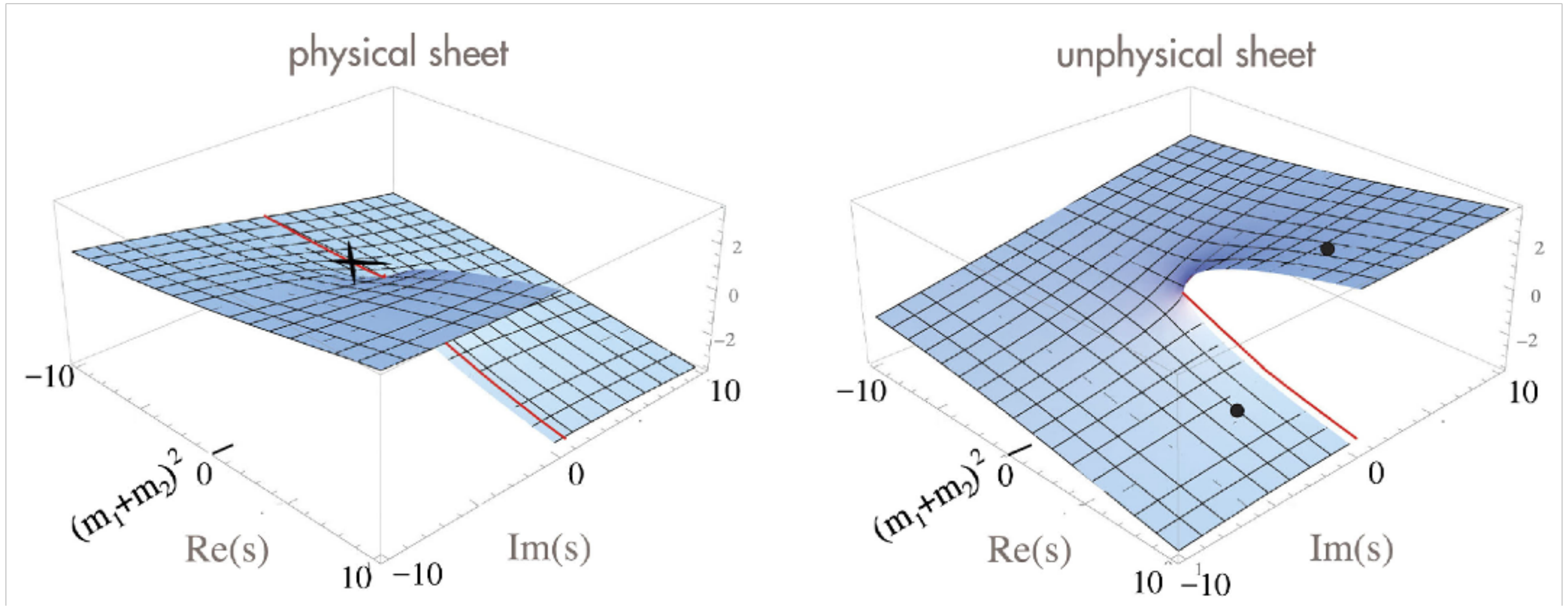
where the second integral is convergent and can be absorbed into the first two terms. Finally, we obtain

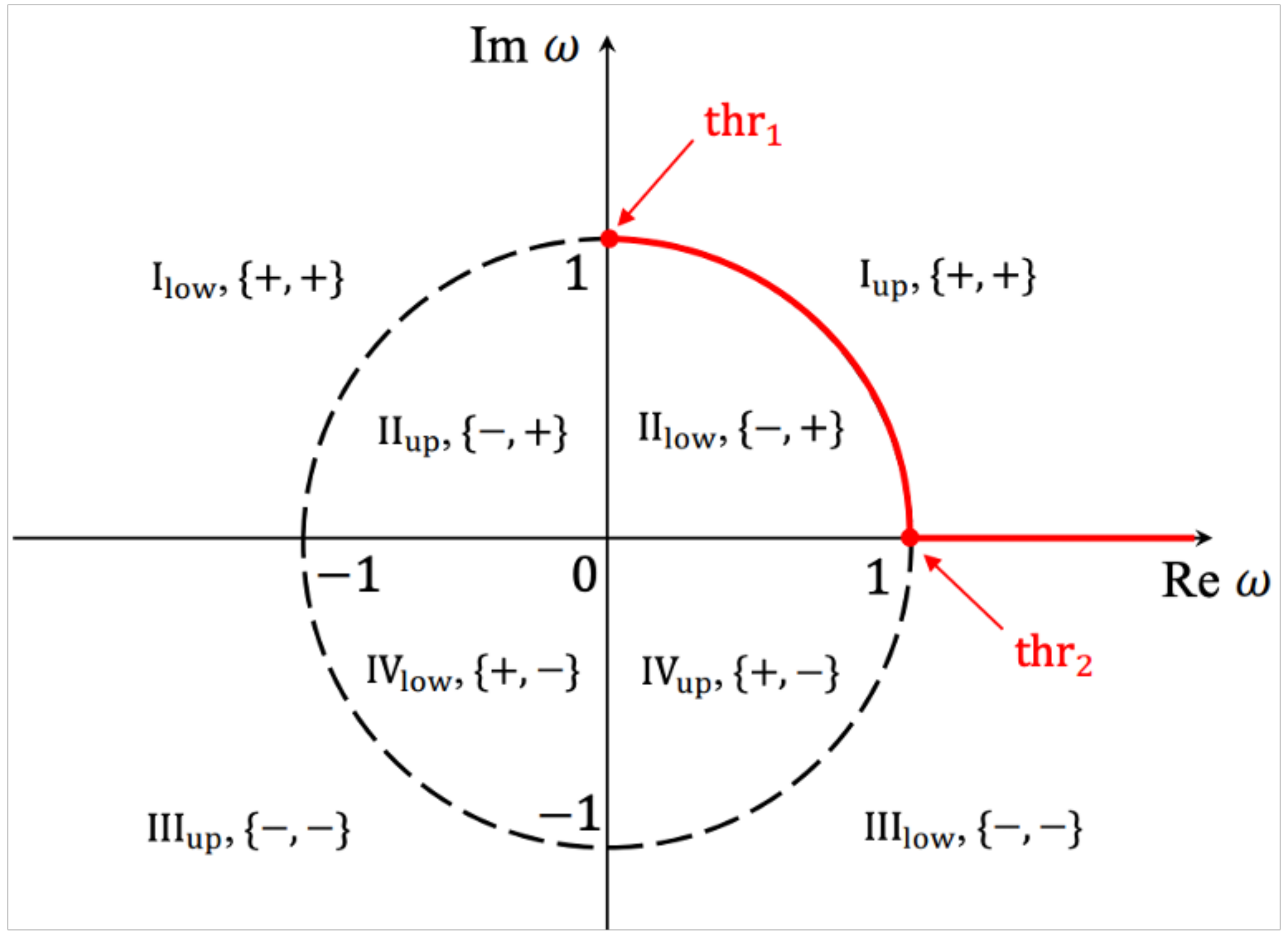
$$\tilde{D}(s) = \frac{\tilde{d}'_{-1}}{s - s_z} + \tilde{d}'_0 + \sum_{j=1}^{n-1} \tilde{d}_j s^j + \frac{s}{\pi} \int_{s_{\text{thr}}}^{\infty} ds' \frac{\rho(s')\tilde{N}(s')}{s'(s - s')}, \tag{S5}$$

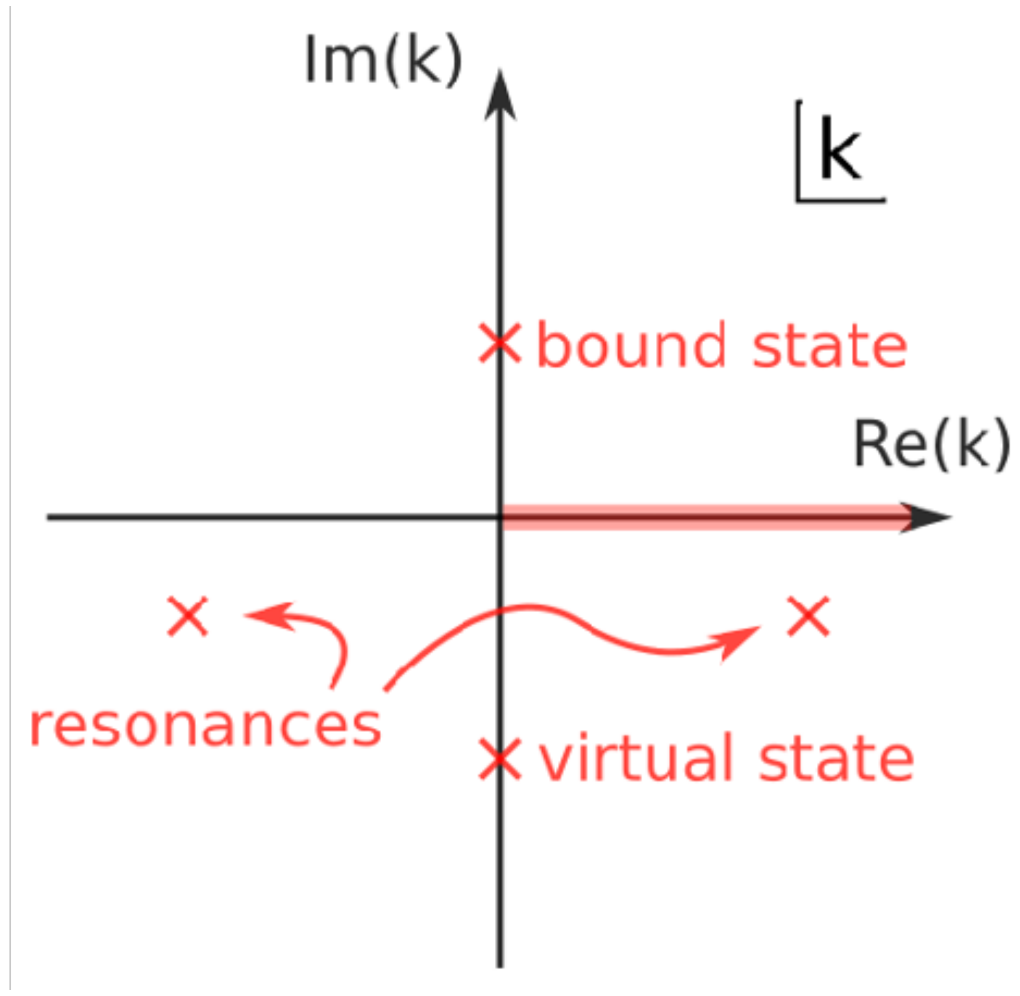
where we can further normalize $\tilde{D}(s)$ by dividing both the new numerator and denominator by $\tilde{d}'_0$, and thus we conclude that the need for additional CDD poles can be accommodated by the zeros of the numerator.

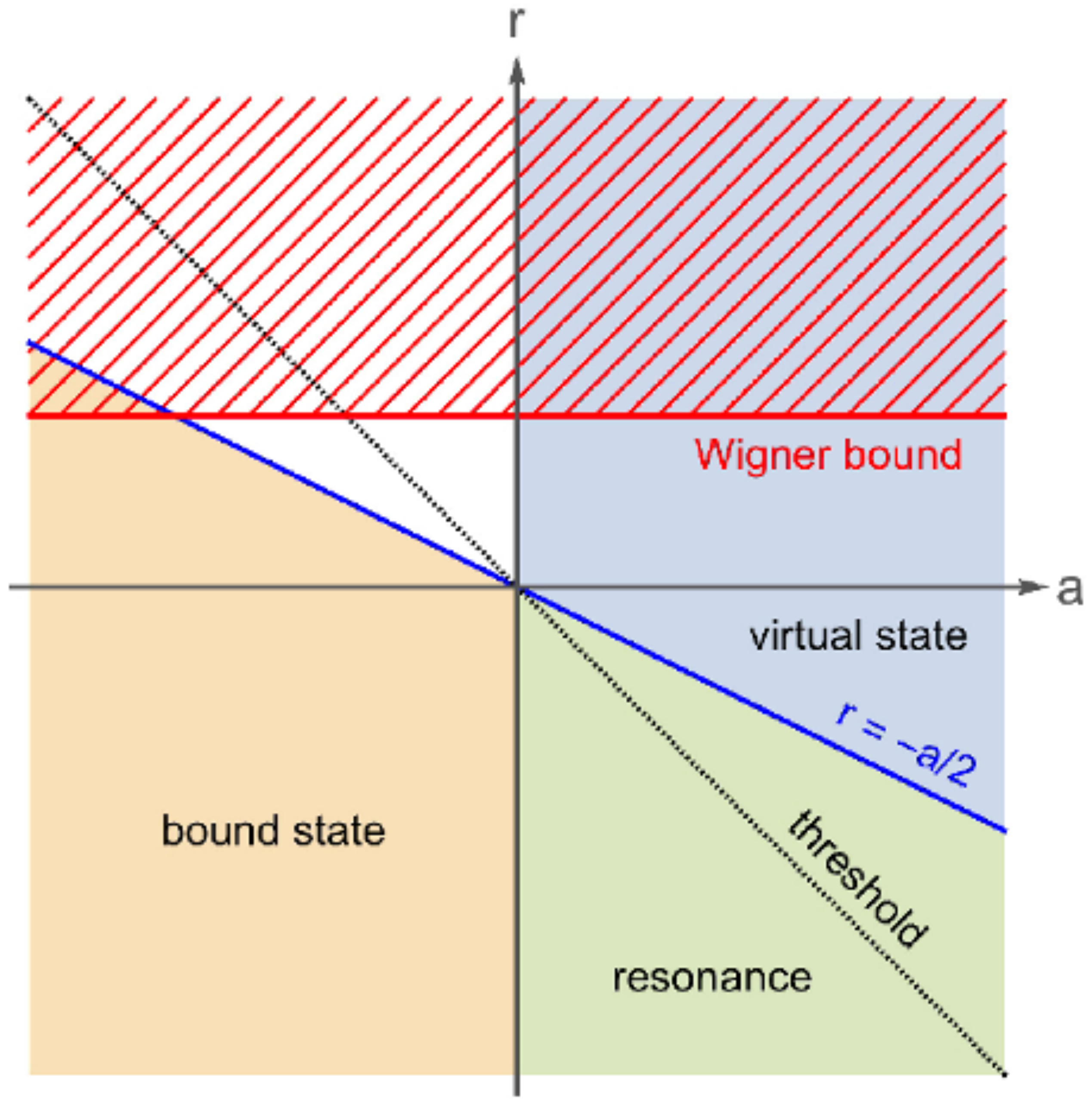
# Hadronic states…

F.K.Guo et al., RevModPhys.90.015004
Inka Matuschek et al., EPJA57 (2021) 3, 101
P.Y.Niu et al., Phys.Rev.D 110 (2024) 9, 094020




$$\bar{X}_A = 1 - \bar{Z}_A = \sqrt{\frac{1}{1 + |2r_0/a_0|}}$$