Domain-wall-seeded bubble nucleation on the lattice

Jaakko Hällfors
University of Helsinki

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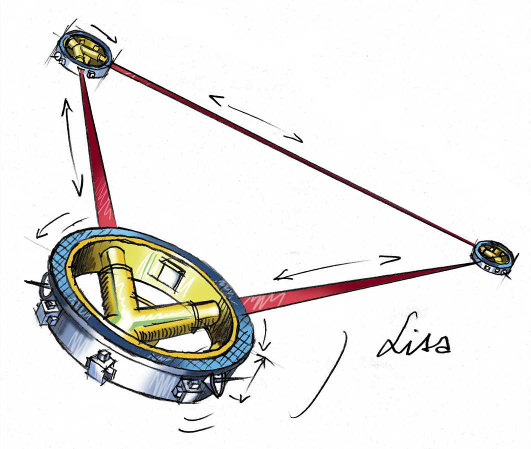
[arXiv:2605.31194]

With Simone Blasi, Andreas Ekstedt and Kari Rummukainen

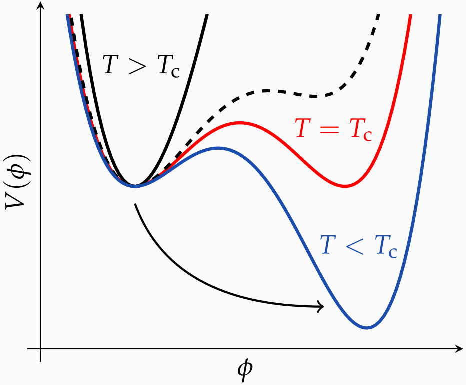
Cosmological phase transitions

- Many extensions of the standard model can have first order phase transitions
- These are expected to produce a model-dependent gravitational wave background
- Future GW observatories might be able to rule out models based on the detected background



[European Space Agency]

First-order phase transitions



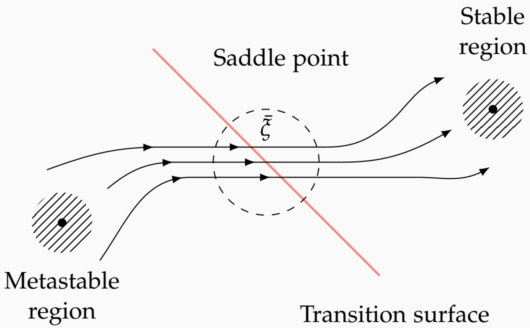
- First-order phase transitions happen through bubble nucleation from the metastable to the stable phase
- The nucleation rate tells us how often bubbles of the stable phase appear

Computing the nucleation rate

- For supressed transitions the rate Γ is mostly determined by the probability of critical configurations
- A saddle-point approximation gives

$$\Gamma \approx A \exp\{-B\},$$

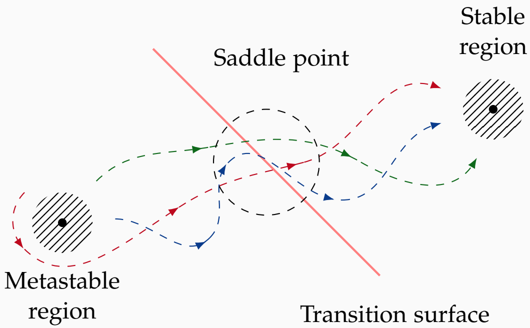
where B is the energy cost of the saddle-point configurations and A contains dynamical information



[Langer 1969; Coleman 1977; Callan and Coleman 1977]

They easy way out

- Using lattice to investigate the saddle-point is quite involved
- If rate Γ is large, there is a simpler option: just simulate the system[a] until transition happens
- Repeated simulations of the metastable phase yield the average lifetime τ



[a] If you know how to do it, of course

Seeded nucleation

- Almost all observed nucleation is seeded by impurities
- Inhomogeneities such as topological defects could also enhance nucleation rates in the early universe
    - Cosmic strings [Preskill and Vilenkin 1993]
    - Monopoles [Steinhardt 1981]
    - Domain walls

$\vdots$

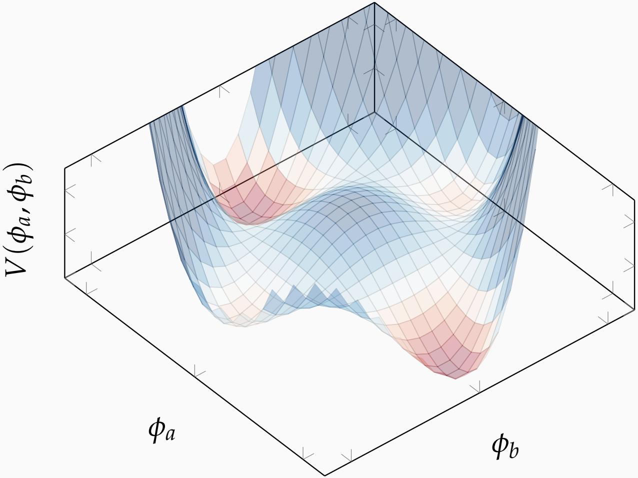
[Wikimedia Commons]

The setup: two scalar fields in 2 + 1 D

$$\mathcal{L} = \sum_{\alpha=a,b} \left[\frac{1}{2}(\partial_\mu \phi_\alpha)^2 - \frac{1}{2} m_\alpha^2 \phi_\alpha^2\right] + \frac{\lambda_a}{24}\phi_a^4 + \frac{\lambda_b}{24}\phi_b^4 + \frac{\mu}{4}\phi_a^2\phi_b^2$$

$$m_a^2 = 0.09, \quad m_b^2 = 0.226,$$
$$\lambda_a = 0.006, \quad \lambda_b = 0.096,$$
$$\mu = 0.026\,(0.028)$$

(In units of some mass scale M^2)



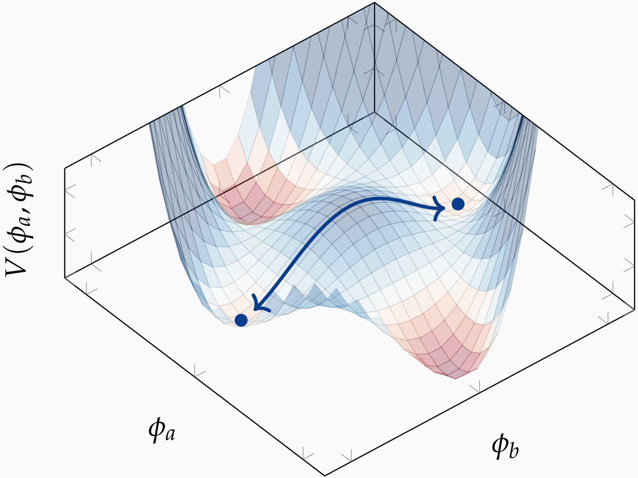
Domain-wall-seeded nucleation

- Two-step transition:

$$(\phi_a, \phi_b) : (0, 0)$$

$$\xrightarrow{\text{DWs form}} \quad (0, \pm v_b)$$

$$\xrightarrow{\text{1st order}} \quad (\pm v_a, 0)$$



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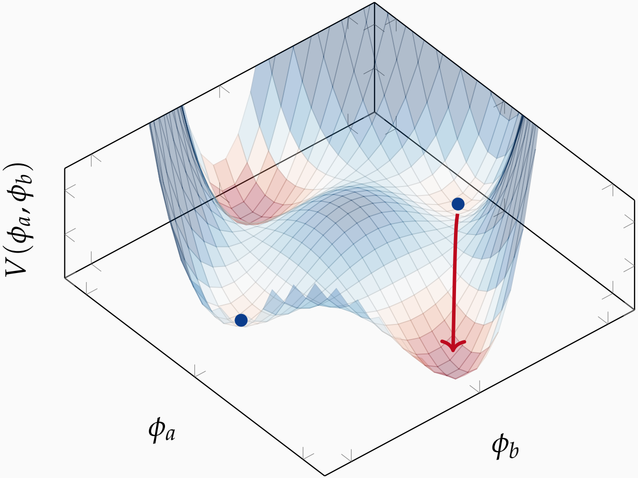

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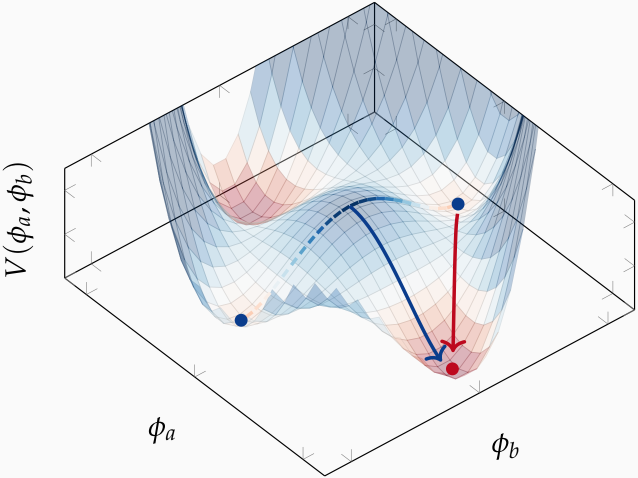
- At the domain wall $\phi_b \approx 0$ and only gradient terms prevent the transition



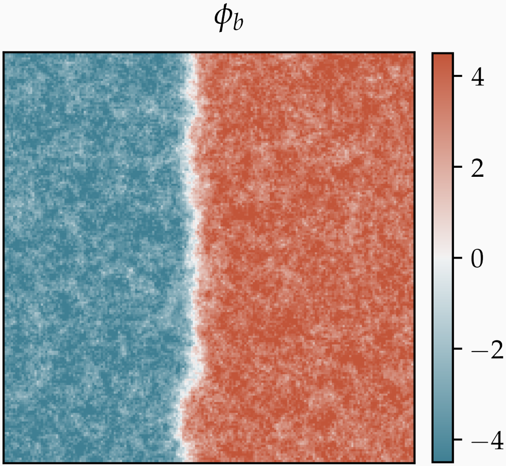
Estimating the rate analytically

[Rajaraman 1982; Blasi and Mariotti 2022]

- Decompose both fields perpendicular to the domain wall

$$\phi_b = DW(x) + \sum_n \phi_{b,n}(y)\psi_n(x)$$
$$\phi_a = \sum_n \phi_{a,n}(y)\xi_n(x),$$

- Perform integrals numerically to obtain a 1D-theory with infinitely many fields



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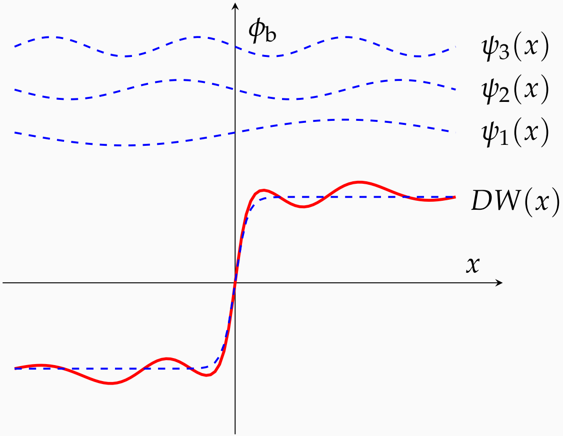


On the lattice:

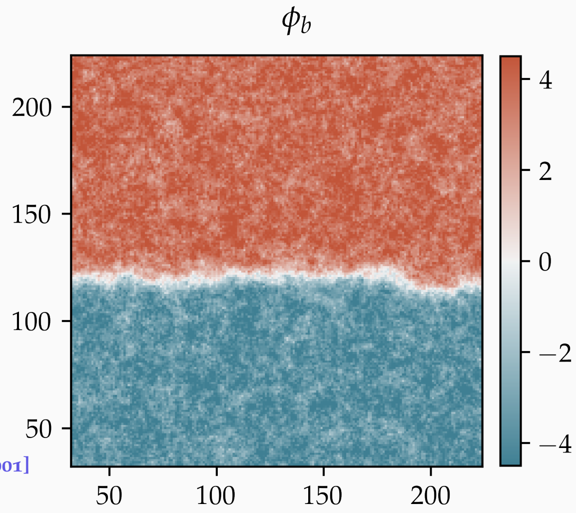
- Antiperiodic boundary conditions perpendicular to the wall
- Stochastic Hamiltonian equations

$$\frac{\partial \phi_\alpha}{\partial t} = \frac{\delta H}{\delta \pi_\alpha}, \; \frac{\partial \pi_\alpha}{\partial t} = -\frac{\delta H}{\delta \phi_\alpha} - \gamma \frac{\delta H}{\delta \pi_\alpha} + \xi$$

- Integrate numerically and take ξ into account by updating every $\Delta t \ll a$

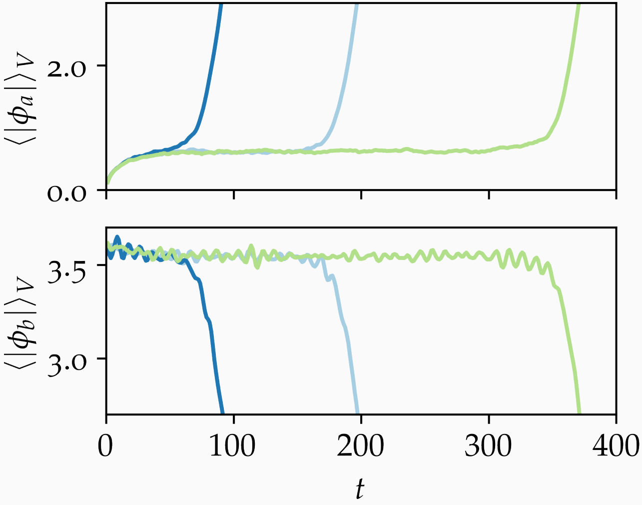
$$\pi_\alpha(x,t) = (1-\epsilon^2)^{1/2}\pi_\alpha(x,t) + \epsilon\eta_\alpha(x,t),$$

where $\epsilon^2 = 1 - \exp(-2\gamma\Delta t)$ [Moore et al. 2001]

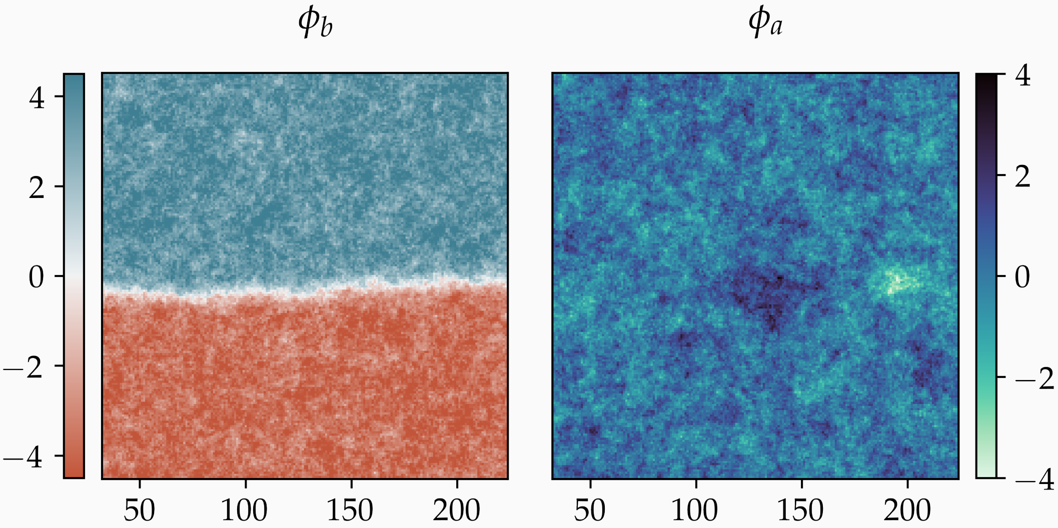


Simulation procedure

1. Initialize fields with anti-periodic boundary conditions to one direction
2. Evolve only ϕ_b to form the wall
3. Evolve both fields until $\langle|\phi_a|\rangle_V$ reaches some semi-arbitrary threshold value
4. Record t_i
5. `goto 1.`



What it looks like



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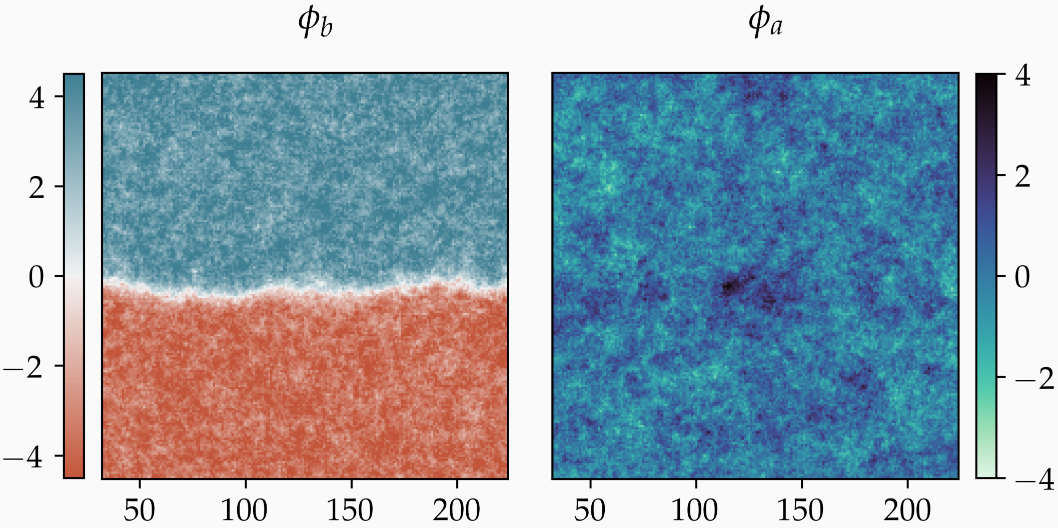

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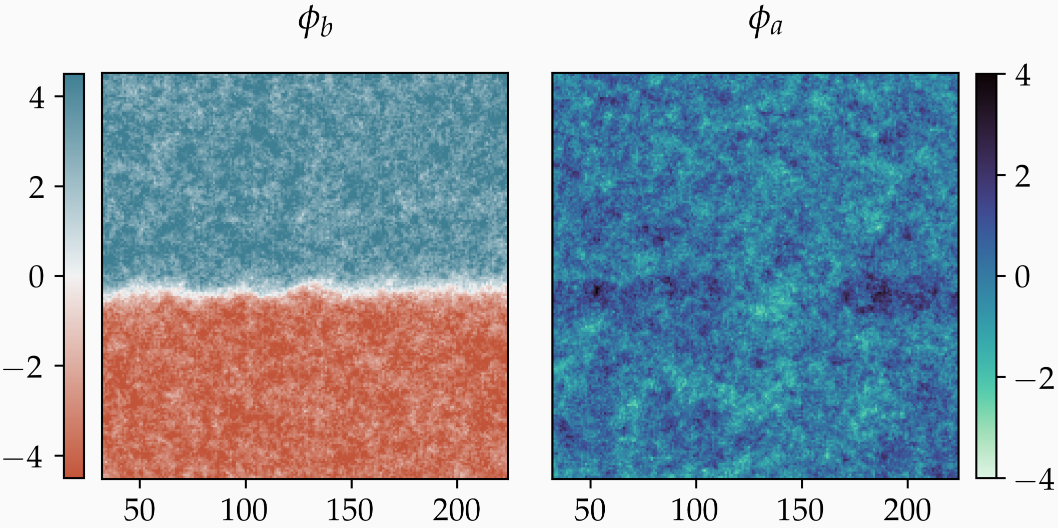

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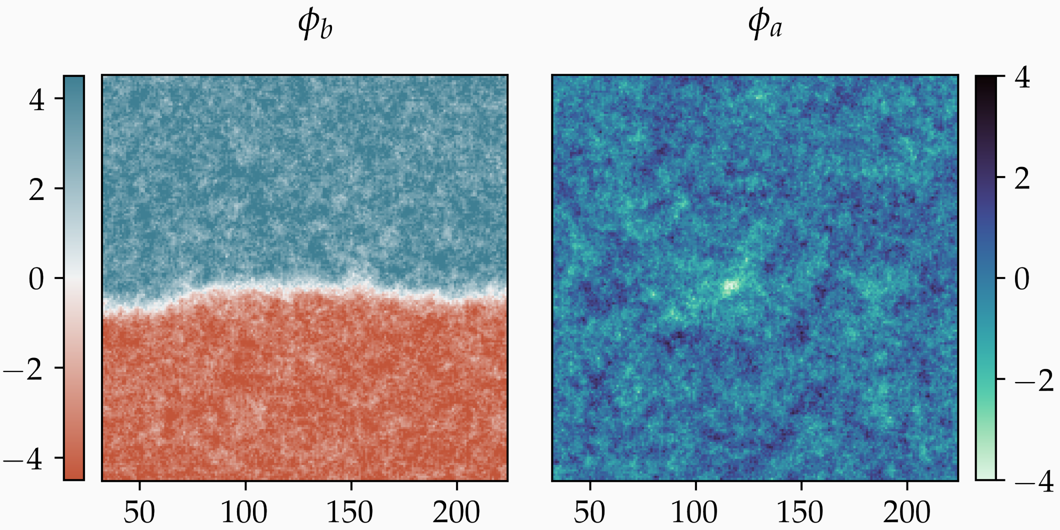

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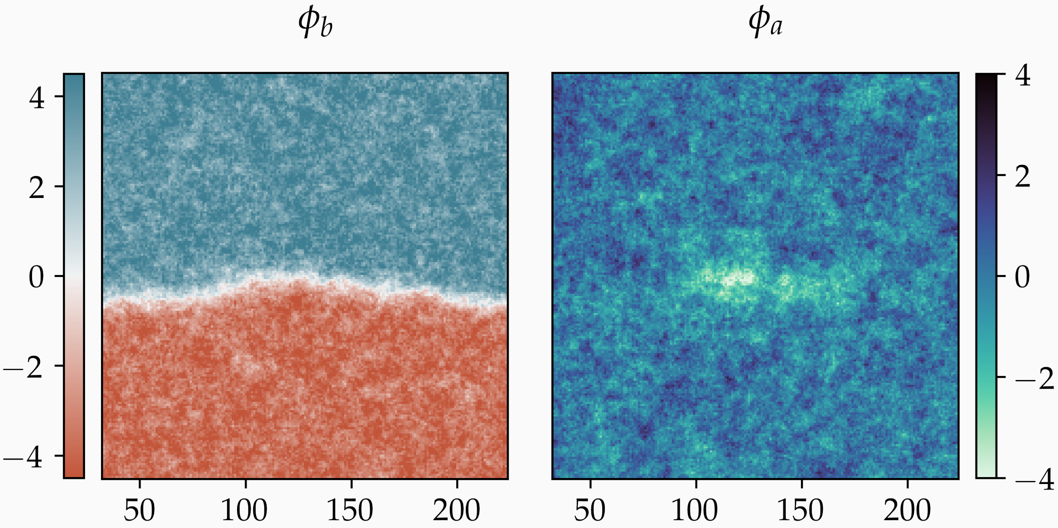

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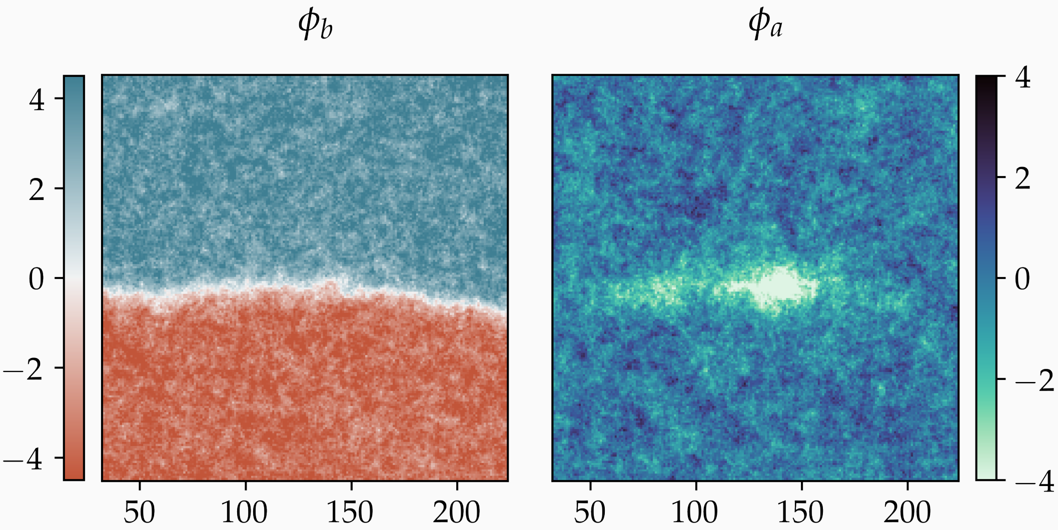

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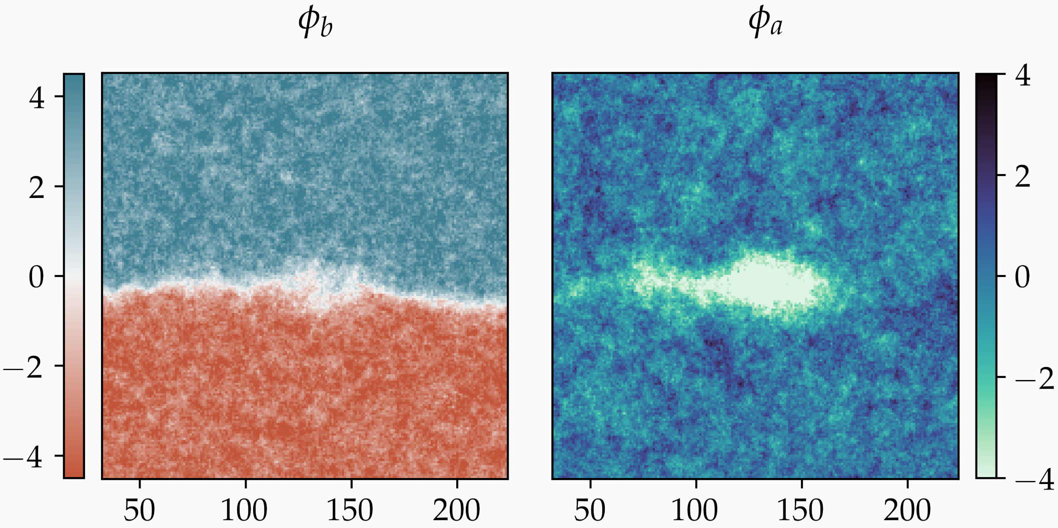

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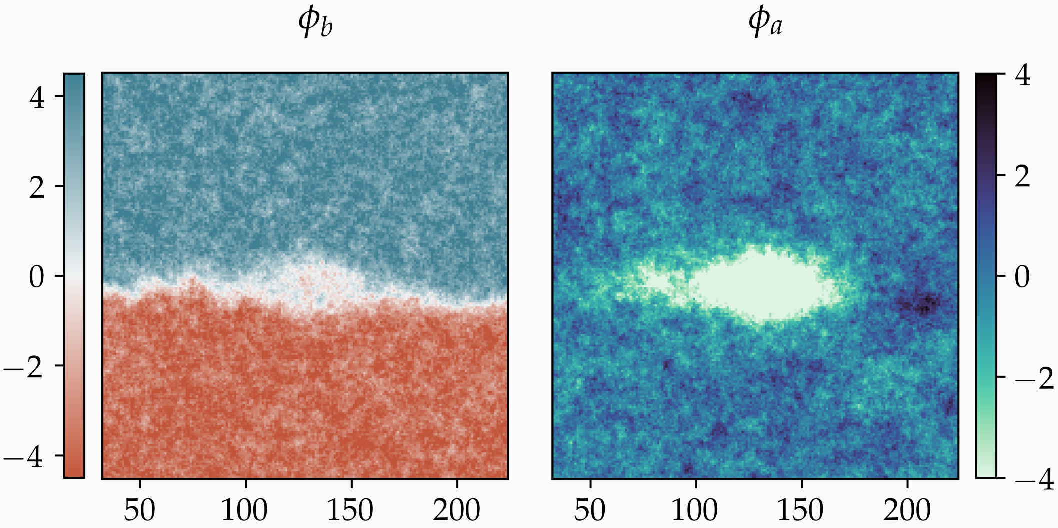

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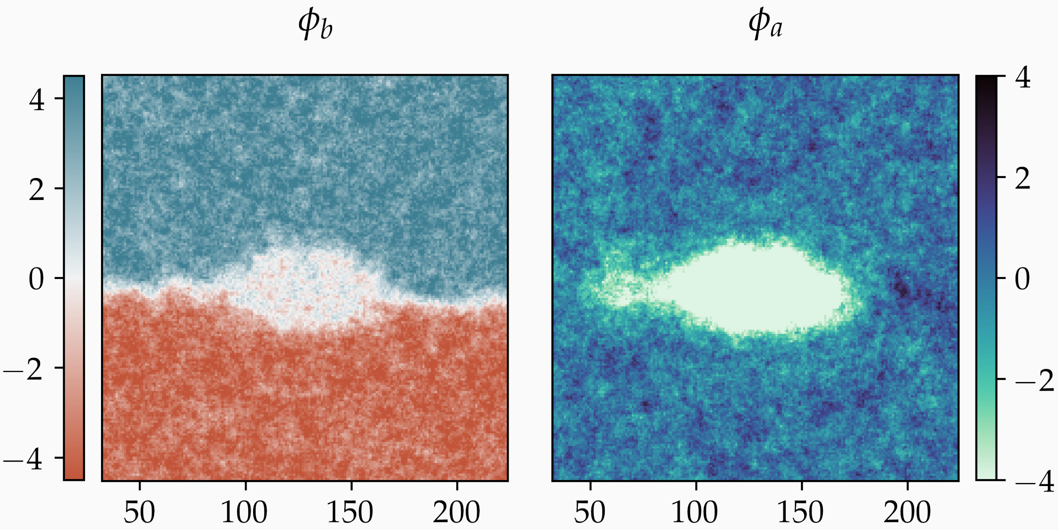
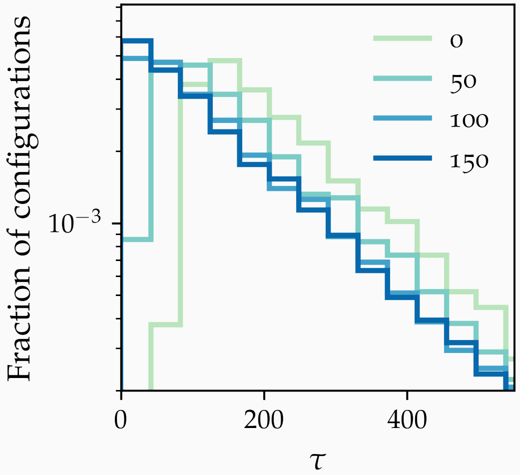


Analysing the nucleation times

- To control dependence on the initial conditions, construct from the full set of nucleation times $\{t_i\}$ the sets

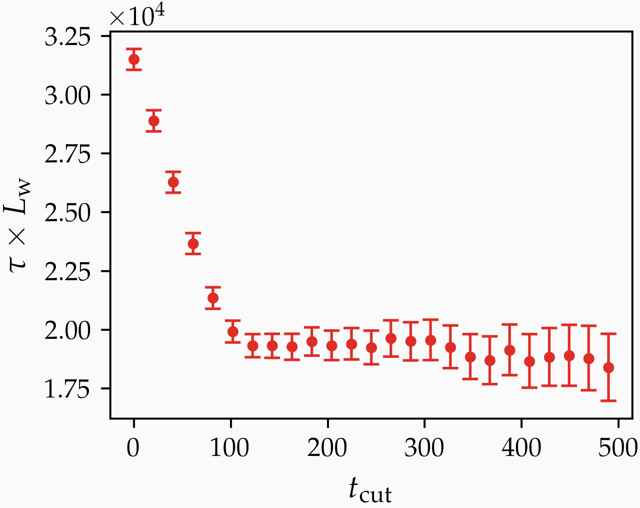
$$\{\, t_i - t_{\text{cut}} \mid t_i > t_{\text{cut}} \,\}$$

- Compute the mean nucleation time τ for larger and larger values of t_{cut}
- Extract the result from behaviour at large t_{cut}

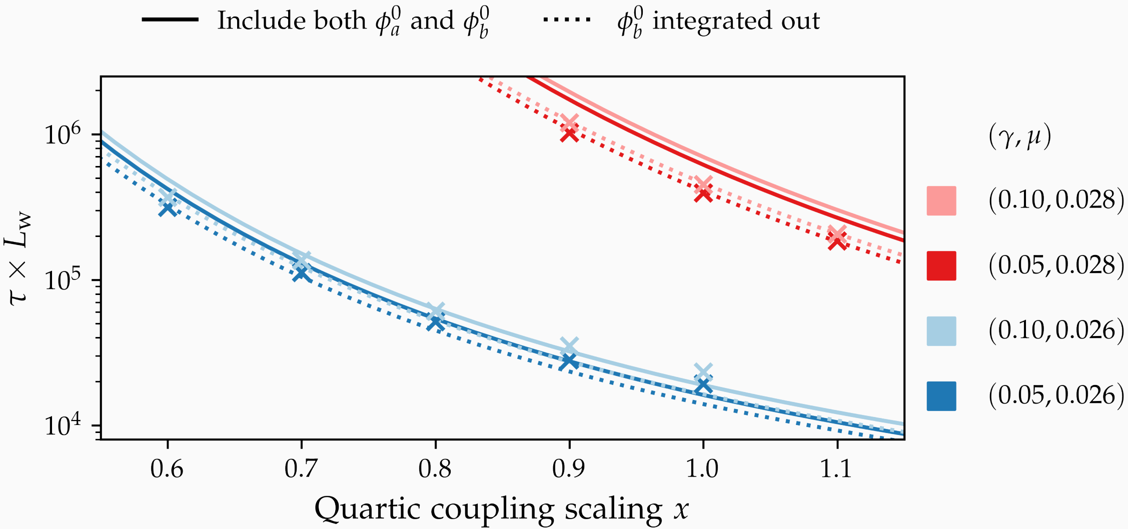


Analysing the nucleation times

- τ stabilises as $t_{\text{cut}} \to \infty$
- Works well if the thermalisation time is not much longer than the average nucleation time τ
- Estimate statistical errors by resampling the initial set of nucleation times $\{t_i\}$



Results



Include both ϕ_a^0 and ϕ_b^0
ϕ_b^0 integrated out
10^6
10^5
10^4
$\tau \times L_{\mathrm{W}}$
0.6
0.7
0.8
0.9
1.0
1.1
Quartic coupling scaling x
(γ, μ)
$(0.10, 0.028)$
$(0.05, 0.028)$
$(0.10, 0.026)$
$(0.05, 0.026)$