QFT Thermodynamics from Entanglement Entropy

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Based on [2603.07635, 2607.06286]

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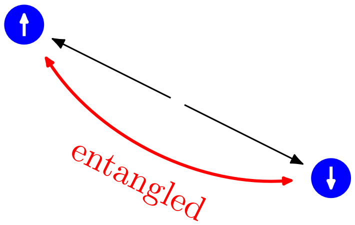
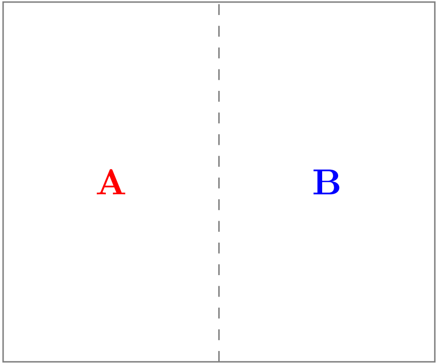
Entanglement

- Entangelement measures give a precise way to quantify quantum correlations
- For QFTs, applications in
  - Critical phenomena
  - RG flow
  - Confinement
  - Thermodynamics



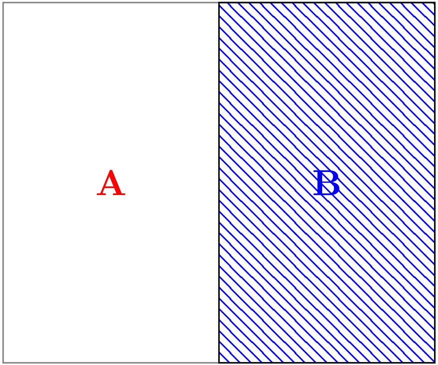
Entanglement Entropy

- An entanglement measure for a bipartite system



Entanglement Entropy

- An entanglement measure for a bipartite system
- Reduced density matrix
  - $\rho_A = \mathrm{tr}_B(\rho_{AB})$
- Entanglement entropy:
  - $S_{EE}(A) = -\mathrm{tr}(\rho_A \log(\rho_A))$



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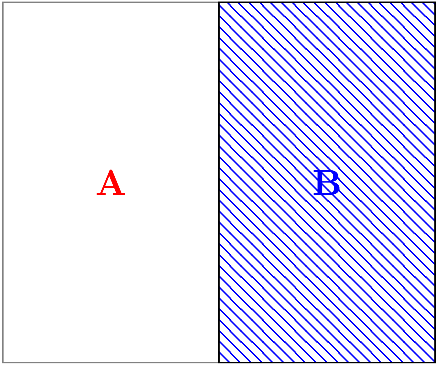
- At finite T contributions from thermal entropy



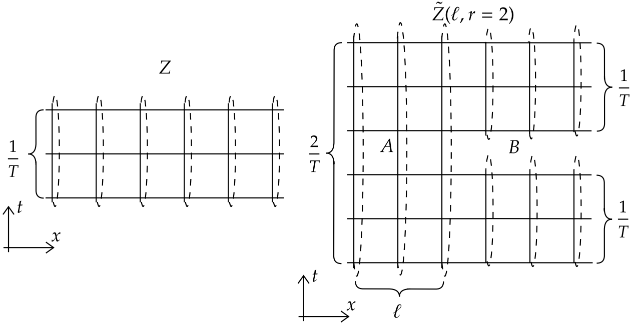
EE in QFTs and the replica trick

- The replica trick [Calabrese, Cardy; hep-th/0405152], $\mathrm{tr}(\rho_A^r) = \frac{\tilde{Z}(\ell, r)}{Z^r}$
- Entanlement entropy:

$$S_{\mathrm{EE}}(\ell) = -\lim_{r\to 1} \frac{\partial \log \mathrm{tr}(\rho_A^r)}{\partial r} = -\lim_{r\to 1}\left(\frac{\partial \log \tilde{Z}(\ell, r)}{\partial r} - \log Z\right).$$

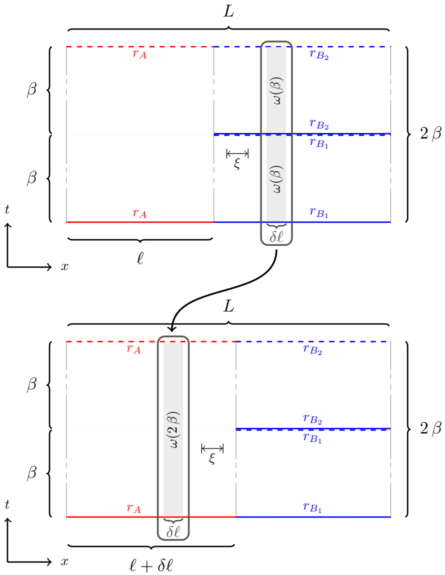
- Due to UV-divergence, measured erivative w.r.t ℓ instead

$$\frac{\partial S_{\mathrm{EE}}}{\partial \ell} = \lim_{r\to 1} \frac{\partial^2 \log \tilde{Z}(\ell, r)}{\partial \ell\, \partial r}$$



Entanglement thermodynamics

- Free energy density $\omega(\beta,\mu) = \frac{1}{V}\log Z(\beta,\mu,V)$
- Thermal entropy density: $s = \beta\,\frac{\partial\omega}{\partial\beta} - \omega$



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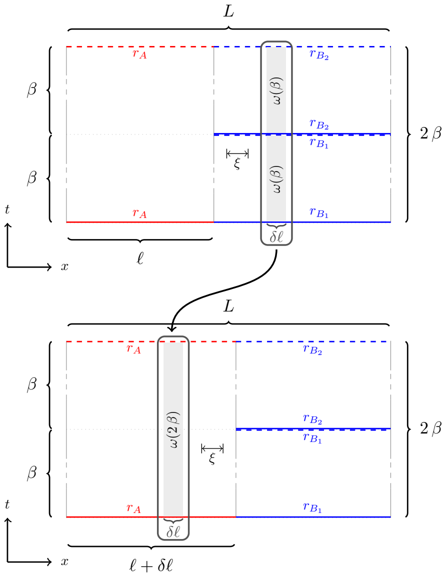
- ∂_ℓ can be expressed as

$$\frac{\partial \log\tilde{Z}(\ell+\delta\ell,\beta,V,\mu,r) - \log\tilde{Z}(\ell,\beta,V,\mu,r)}{\delta\ell}$$



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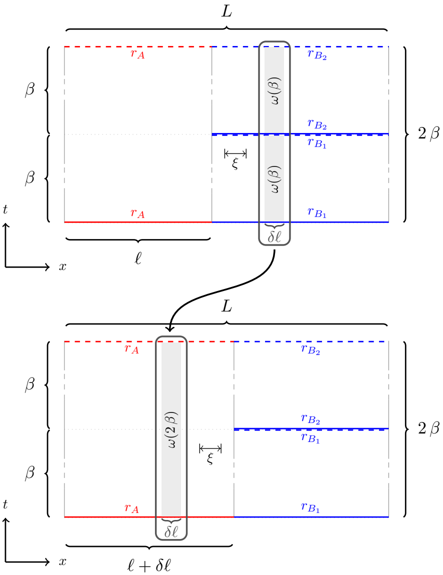
- at $\xi \ll \ell \ll L$

$$\frac{1}{V_\perp} \frac{\partial \log \tilde{Z}(\ell, r)}{\partial \ell} = \omega(r\,\beta, \mu) - r\,\omega(\beta, \mu)$$



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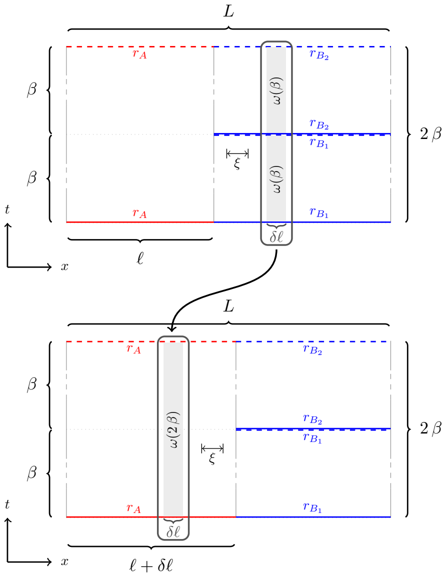
- So,

$$\frac{1}{V_\perp}\frac{\partial S_{\mathrm{EE}}(\ell)}{\partial\ell} = \beta\,\frac{\partial\omega}{\partial\beta} - \omega = s$$



Entanglement entropy on the lattice

- $\lim_{r\to 1}\partial_r\partial_\ell$ in $\partial_\ell S_{\mathrm{EE}}$ have to be discretized.

$$\left.\frac{\partial S_{EE}(\ell)}{\partial \ell'}\right|_{\ell'=\ell+1/2} \approx -\log \tilde{Z}(\ell+1,2) + \log \tilde{Z}(\ell,2)$$

- Corresponds to approximating S_{EE} as the 2nd Rényi entropy $H_2(\ell)$

$$\begin{aligned} H_r(\ell,\beta,V,\mu) &= \frac{1}{1-r}\log \mathrm{tr}(\rho_A^r) \\ &= \frac{1}{1-r}\Big(\log \tilde{Z}(\ell,\beta,V,\mu,r) - r\,Z(\beta,V,\mu)\Big) \end{aligned}$$

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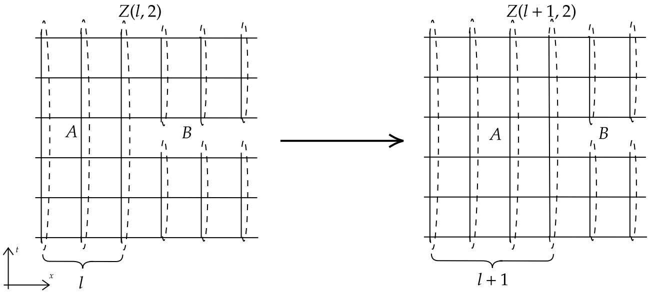

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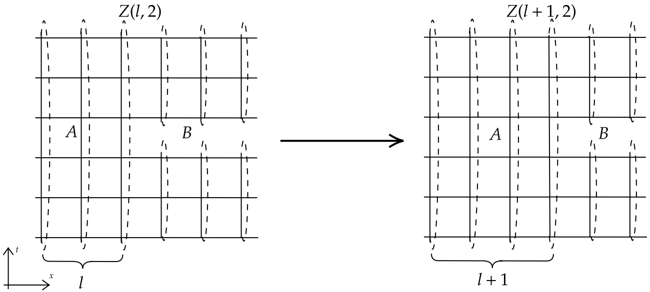
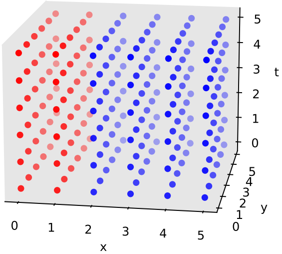
- The change from ℓ to $\ell+1$ is highly non-local $\to$ bad overlap



Boundary deformation method

- Deform the entangling surface and interpolate between ℓ and $\ell+1$ [Jokela et al.; 2304.08949]
- histograms h_i for each boundary configuration

$$\left.\frac{\partial S_{\mathrm{EE}}(\ell')}{\partial \ell'}\right|_{\ell'=\ell+1/2} \approx \log(h_N) - \log(h_0)$$



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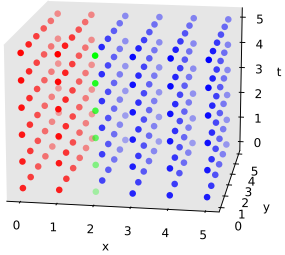

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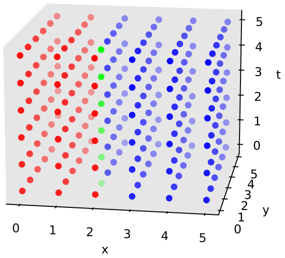

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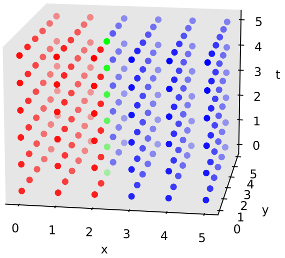

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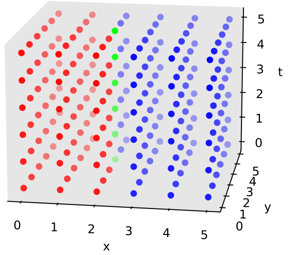

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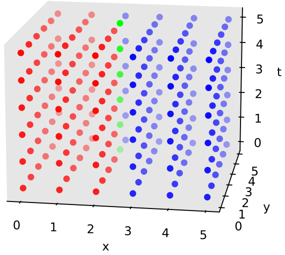

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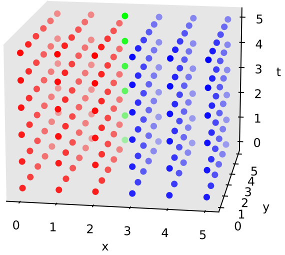

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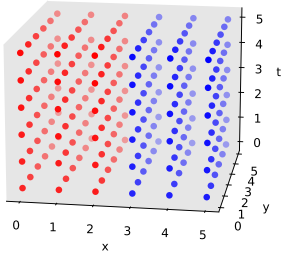


$O(N)$ models

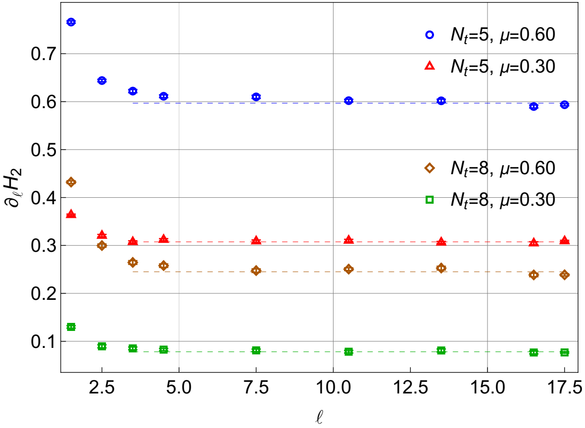
- A QFT with a N-component scalar field
- Interesting properties like asymptotic freedom and spontaneous symmetry breaking
- Relatively low computational cost
- Can be simulated directly at $\mu \neq 0$ with the worm algorithm [Rindlisbacher et al.;1602.09017,1512.05684]

Simulation parameters

- O(4)
- $\kappa = 1.2$ and $j_3 = 0.2$, symmetry broken phase and finite correlation length
- $d = 3$, lattices of size $2N_t \times N_x \times N_s$, $N_x = 36$, $N_s = 12$
- multiple μ and N_t values

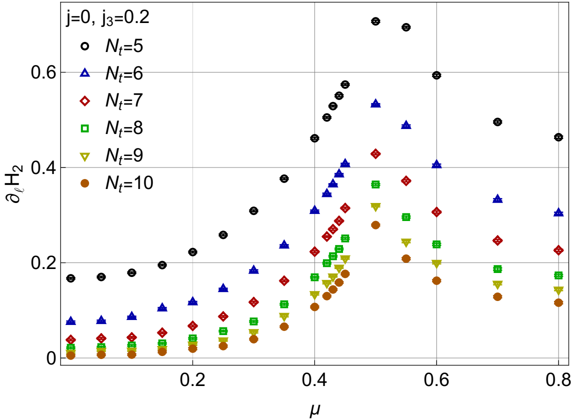
Results

- $\partial_\ell H_2$ plateaus quickly as a function of ℓ:



Results

- $\partial_\ell H_2$ as a function of μ at $\ell = 17.5$
- Finite density phase transition clearly visible with $\mu_c \approx 0.5$



Results

- Maxwell relation

$$\left.\frac{\partial s}{\partial \mu}\right|_T = \left.\frac{\partial n}{\partial T}\right|_\mu$$

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- Maxwell relation, replace $s \to 1/V_\perp \partial_\ell S_{\mathrm{EE}}$

$$\frac{1}{V_\perp}\frac{\partial^2 S_{\mathrm{EE}}}{\partial\mu\partial\ell} \overset{\xi \ll \ell}{=} \left.\frac{\partial n}{\partial T}\right|_\mu$$

Results

- Approximate with H_2

$$\frac{1}{V_\perp}\frac{\partial^2 H_2}{\partial\mu\,\partial\ell} \overset{\xi\ll\ell}{=} -2\,N_t\left(n(2\,N_t,\mu) - n(N_t,\mu)\right)$$

- Both sides can be computed on the lattice

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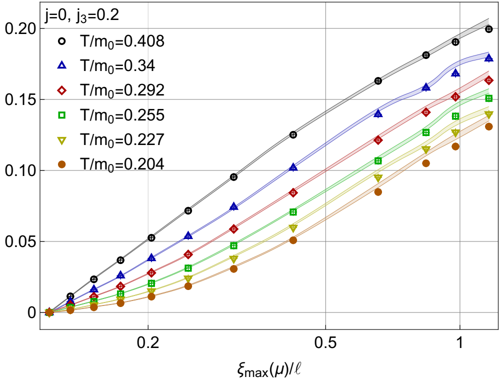
- Both sides can be computed on the lattice
- Relation seems to hold up to $\xi_{\max}(\mu)/\ell \approx 0.5 \rightarrow \mu \leq 0.4$



Summary & Outlook

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- Other entanglement quantities (entanglement asymmetry, symmetry resolved entanglement, ...)
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Thank You!!!