Disentangling finite-volume and reconstruction errors on smeared spectral functions

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Purpose of this work

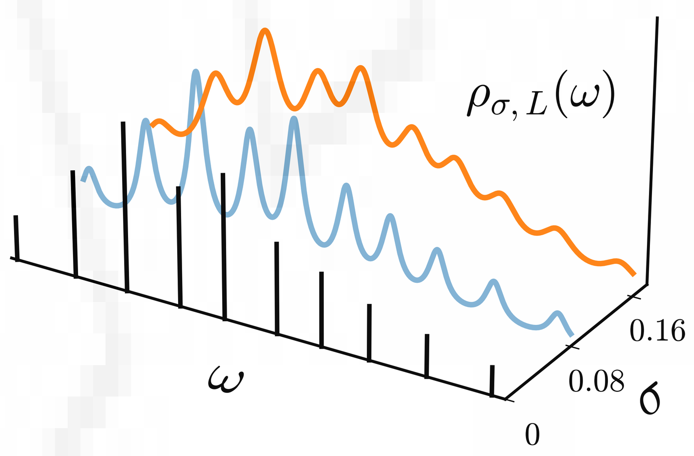
Lattice calculations in a finite spatial volume L^3 provide noisy Euclidean correlators,

$$C_L(t) = \int_0^\infty \mathrm{d}\omega\, \rho_L(\omega) e^{-\omega t}\,.$$

Since ρ_L is a weighted sum of delta functions, we target *smeared* spectral densities

$$\rho_{\sigma,L}(\omega) = \int_0^\infty \mathrm{d}\omega'\, \delta_\sigma(\omega,\omega')\rho_L(\omega')$$

$$\rho(\omega) = \lim_{\sigma\to 0}\lim_{L\to\infty} \rho_{\sigma,L}(\omega)\,.$$



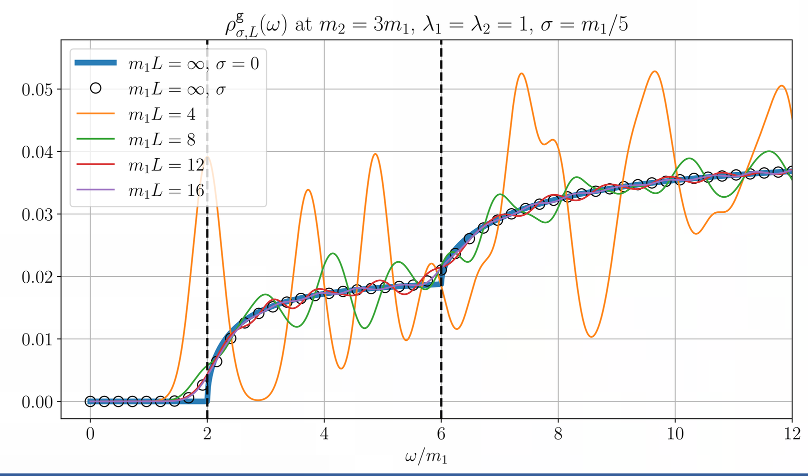
Here we present closure tests to separately study
- **finite-volume effects** when extrapolating ρ from exact $\rho_{\sigma,L}$
- **numerical reconstruction artifacts** in the reconstruction of $\rho_{\sigma,L}$ from C_L

Toy Models

$$\rho_L(\omega) = \lambda_1\Phi_{2,L}(\omega,m_1) + \lambda_2\Phi_{2,L}(\omega,m_2)$$

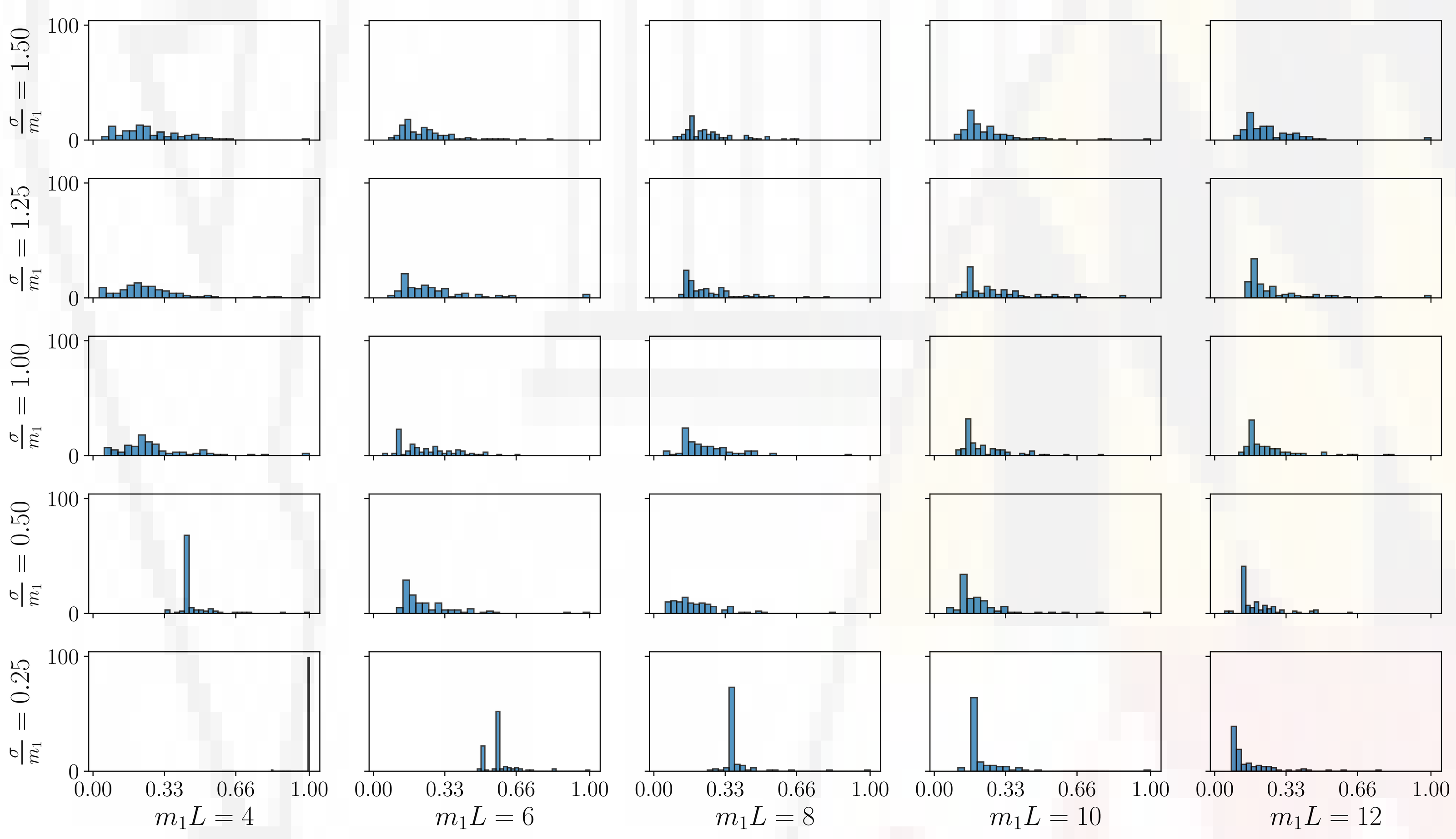
$$\Phi_{2,L}(\omega,m_i) = \frac{1}{L^3}\sum_{\mathbf{p}}\frac{1}{4\omega_i^2}(2\pi)\delta(\omega-2\omega_i)\,|F(2\omega_i;m_i,M_i,\Gamma_i)|^2\ ,\ \omega_i^2 = m_i^2+\mathbf{p}^2$$

- Toy-model spectral densities inspired by systems with **two open channels**, e.g., $\pi\pi$ and $K\overline{K}$
- Free parameters $\{\lambda_1,\lambda_2,m_1,m_2\}$
- Possibly resonant with F = Gounaris–Sakurai



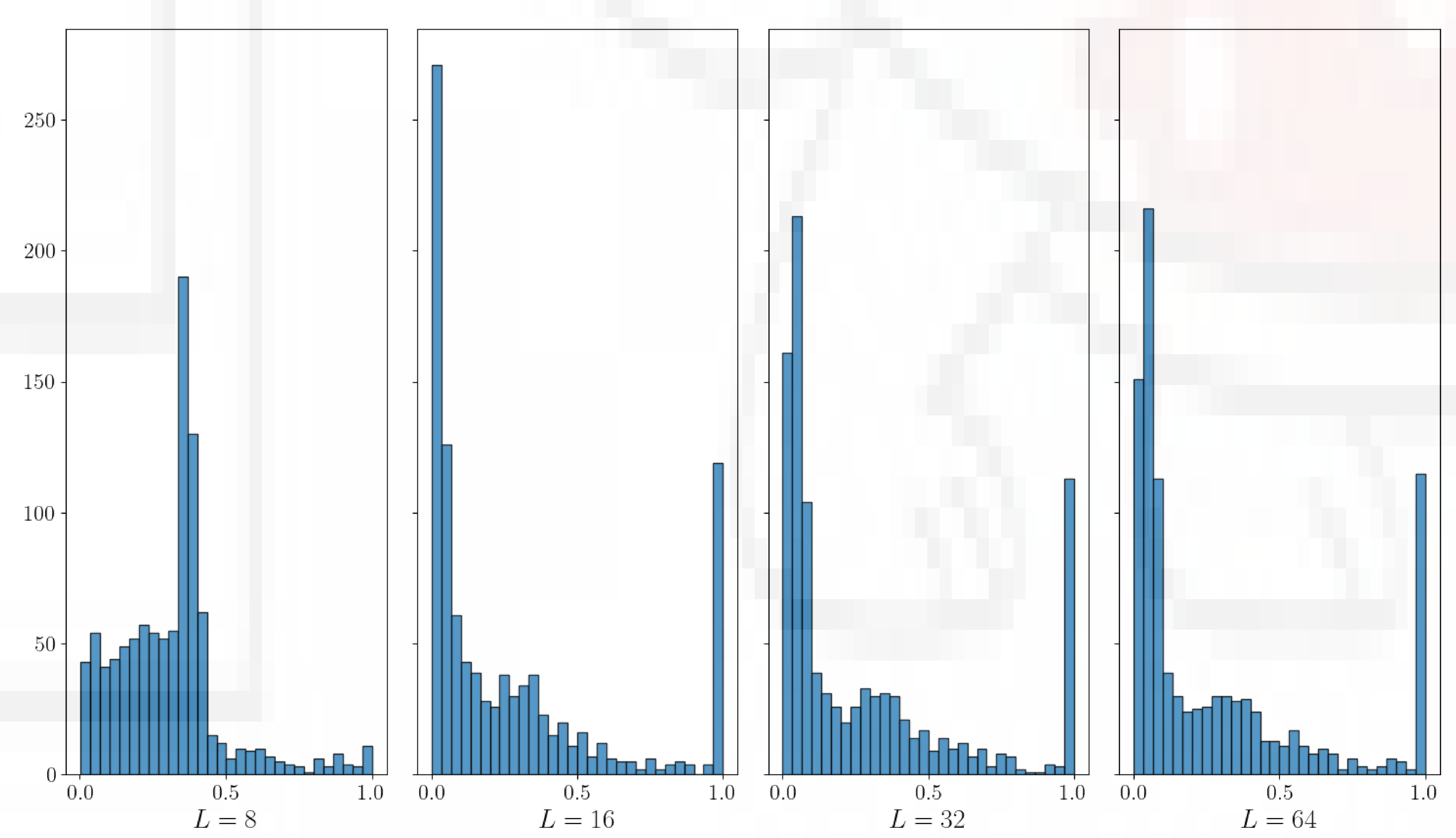
Finite-Volume and Smearing Interplay

Distribution of Systematic Error $\left\|\frac{\rho(\omega)-\rho_{\sigma,L}(\omega)}{\rho(\omega)}\right\|_{[2m_1+\sigma,10m_1]}$ vs (L,σ) across 100 theories



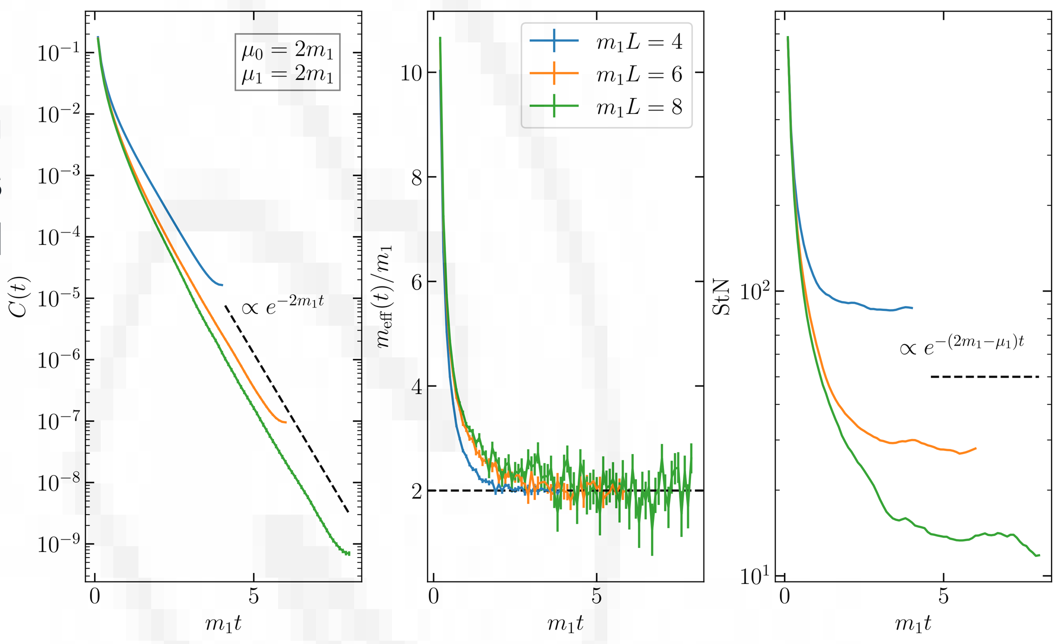
- At fixed L, lowering σ initially improves the target, then FVEs dominate
- At fixed σ, increasing L reduces FVEs until $\rho_{\sigma,L}\approx\rho_\sigma$
- This helps define $\{\sigma,L\}$ for extrapolations at $\sigma\to 0$ (here: $\sigma L\gtrsim 1$)

Distribution of Systematic Error $\left|\frac{\rho(\omega)-\rho^{g}_{\sigma,L}(\omega)}{\rho(\omega)}\right|$ vs (L,σ) across 1000 theories, at $\omega=3.00$



Noisy Correlation Functions

Noisy correlators are drawn from multivariate gaussians with central value $C_L(t)$ and model covariance controlling
- σ_0: noise size
- μ_0: correlation decay
- μ_1: variance decay (StN)

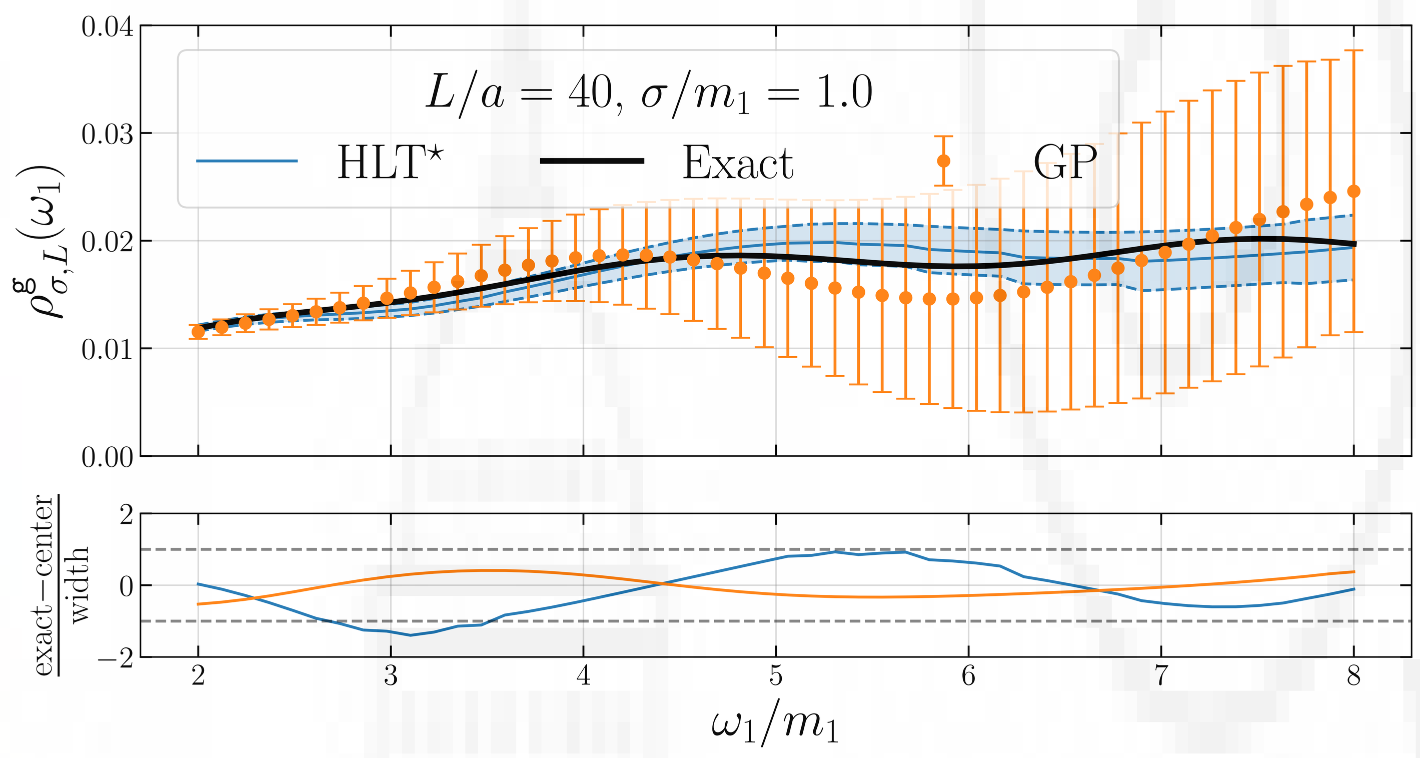


Reconstructions: HLT vs Gaussian Processes

- **HLT** [Hansen, Lupo and Tantalo 2019] is a linear estimator from a “bias-variance tradeoff”
- **GPs** [Del Debbio et al. 2025] promote and generalize HLT to a Bayesian framework, requiring hyperparameter marginalization, e.g. through

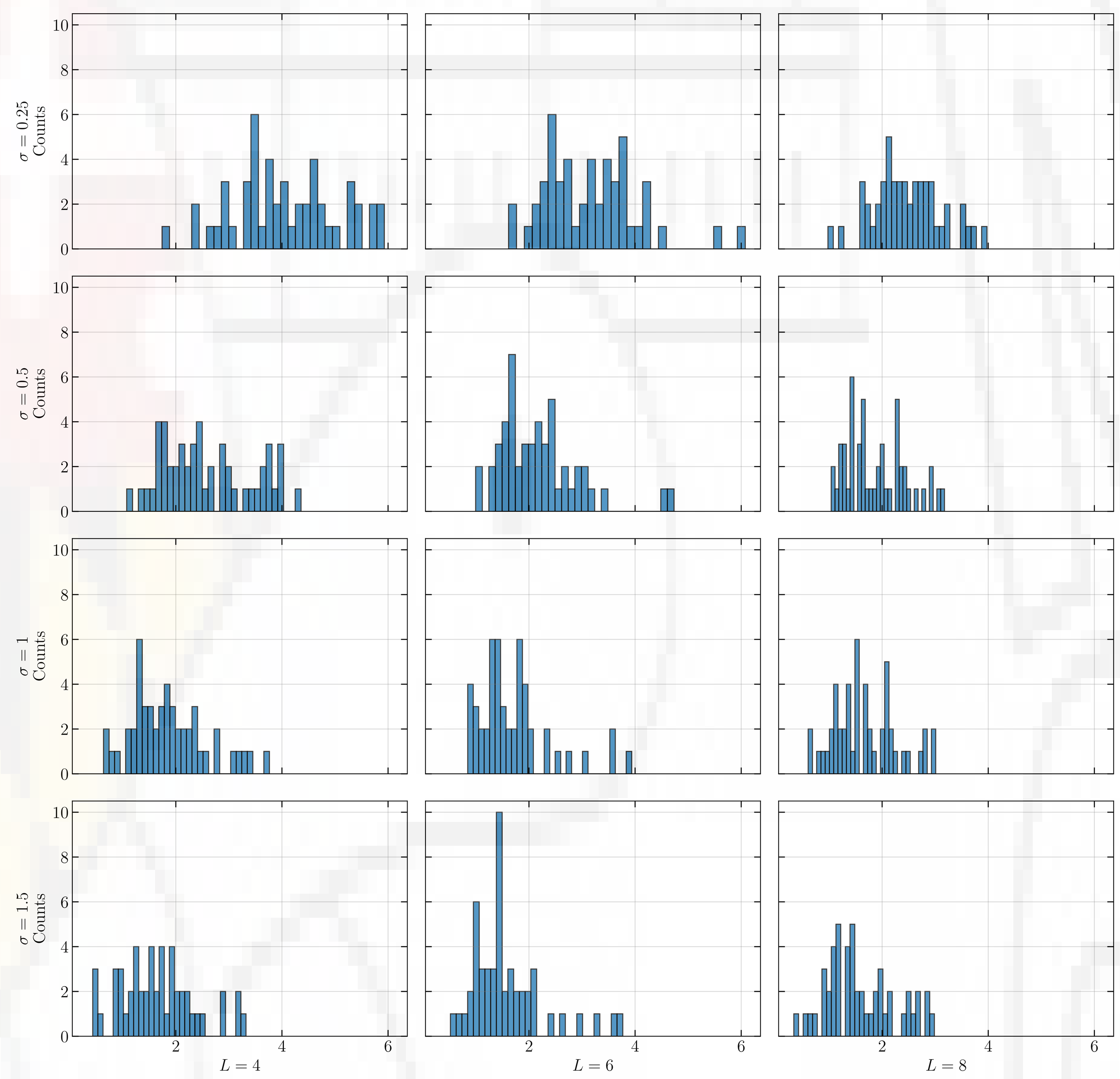
  ⇒ affine-invariant sampling [Goodman and Weare 2010]

  ⇒ nested sampling [Skilling 2004, 2006]



Closure Tests on Noisy Data

Distribution of Systematic Error $\max_\omega\left|\frac{\rho^{\mathrm{exact}}_{\sigma,L}(\omega)-\rho^{\mathrm{HLT}}_{\sigma,L}(\omega)}{\delta\rho^{\mathrm{stat}}_{\sigma,L}(\omega)}\right|$ vs (L,σ) across 50 theories



Takeaways and Future Work

- Finite-volume and smearing effects are studied in the continuum
- Closure tests from phenomenologically motivated models help diagnose systematic errors when reconstructing $\rho(\omega)$ from noisy $C_L(t)$
  - HLT successfully targets $\rho_{\sigma,L}$ in the noisy two-channel setup
  - GPs are generally more conservative and statistically robust
  - Other methods could be considered to assess systematic errors

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