

Smearred spectral functions: opportunities and challenges

Shoji Hashimoto (KEK, SOKENDAI)

@ Lattice 2026, U of Maryland

Spectral functions from LQCD?

Reconstructing the R-ratio from lattice correlators = an **ill-posed problem**

Smearing may be a compromise, or a built-in feature

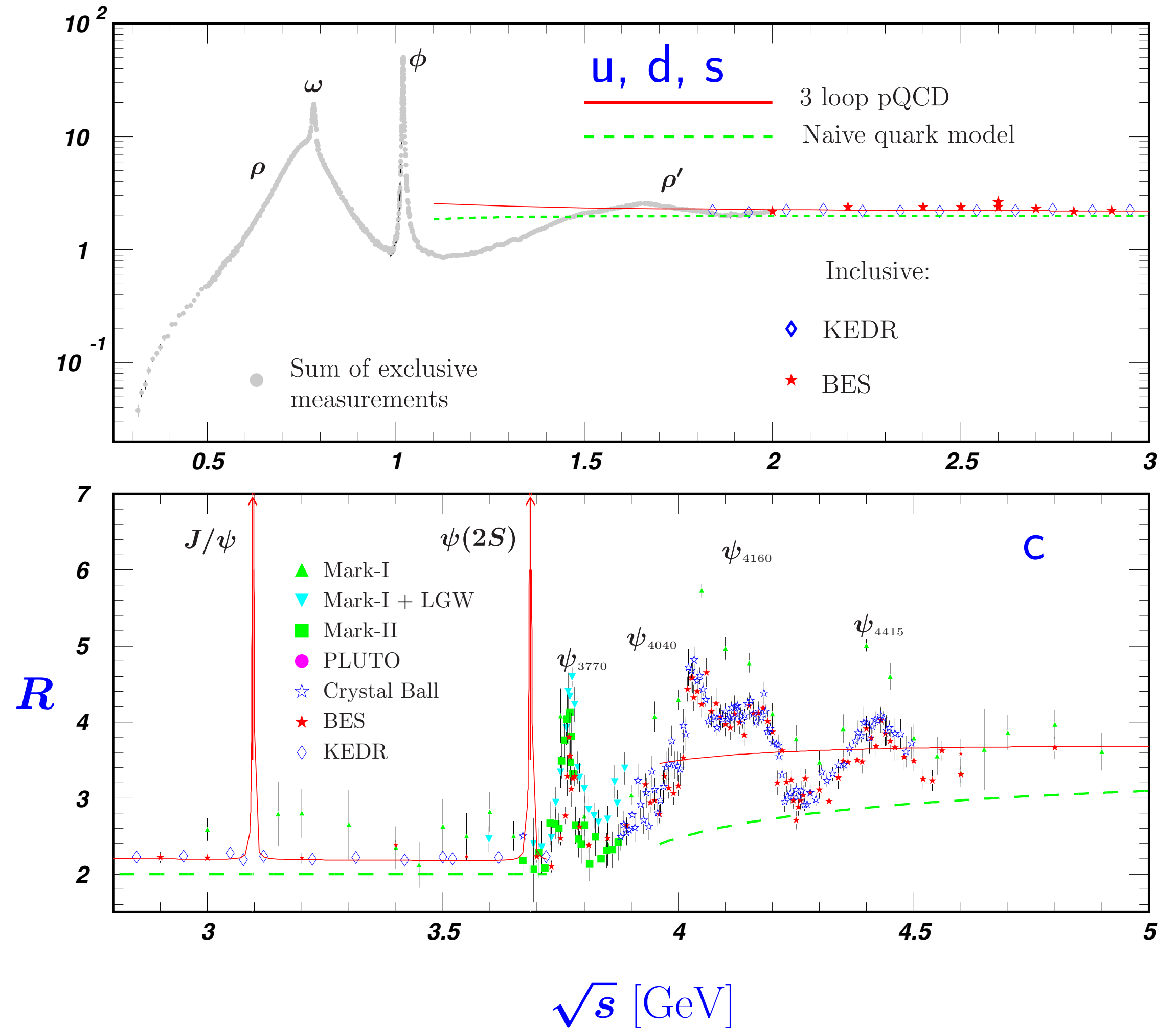
This talk:

Opportunities: possible applications

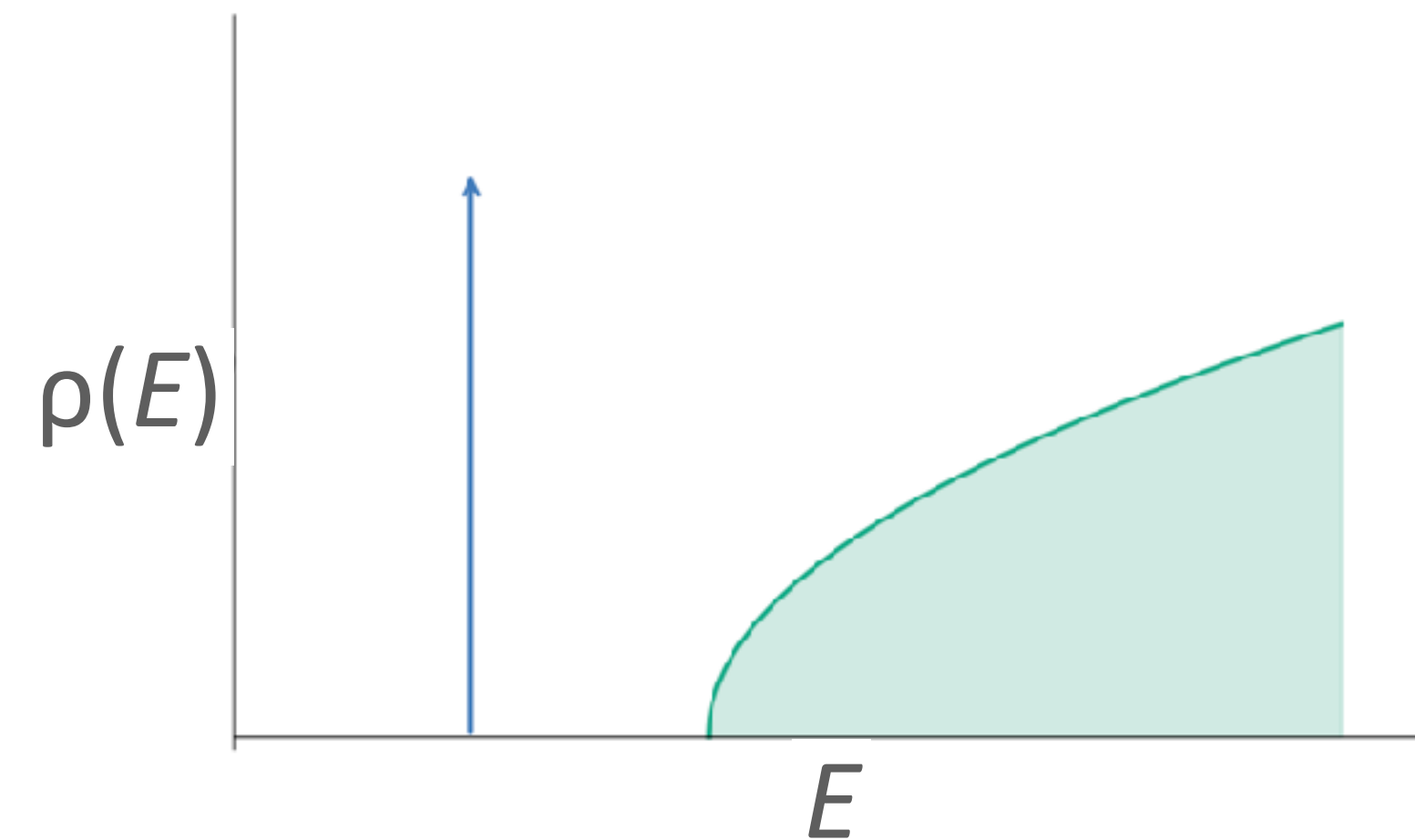
Challenges: available methods, limitations

But, not about finite T apps, sorry!

(see, Bala at Lattice 2025)



(smeared) spectral density



$$\rho_{AB}(E) = \sum_n \langle F | \mathcal{O}_A | n \rangle \langle n | \mathcal{O}_B | I \rangle \delta(E - E_n)$$

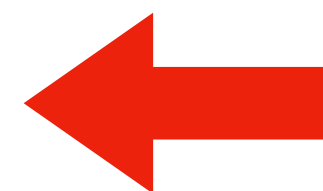
Smeared spectrum

$$S[\rho_{AB}] = \int_0^\infty dE K(E) \rho_{AB}(E)$$

with a chosen “kernel” $K(E)$

Lattice correlator

$$C_{AB}(t) = \langle I | \mathcal{O}_A(t) \mathcal{O}_B(0) | I \rangle = \int_0^\infty dE \rho_{AB}(E) e^{-Et}$$



= linear reconstruction problem $K(\hat{H}) = g_0 + g_1 e^{-\hat{H}} + g_2 e^{-2\hat{H}} + g_3 e^{-3\hat{H}} + \dots$

Related talks at this conference

Physics applications

136. [Towards the smeared R-ratio with Nf=2+1+1 highly improved staggered quarks](#)

 Matteo Saccardi (Colorado State University)

 2026/07/27 14:20
117. [The smeared R-ratio in isospin symmetric QCD from first-principles lattice simulations](#)

 Francesca Margari (University of Rome Tor Vergata)

 2026/07/31 16:50
313. [Isospin-breaking effects in inclusive hadronic tau data for the muon \(g-2\)](#)

 Mattia Bruno (Universita' di Milano-Bicocca)

 2026/07/31 16:30
172. [Spectral reconstruction of the inclusive tau decay-rate with \$C^*\$ boundary conditions](#)

 Erik Bäske (Humboldt-Universität zu Berlin)

 2026/07/27 15:00
109. [Inclusive \$B_s \rightarrow X_{\bar{s}c} \ell \bar{\nu}\$ decay from lattice QCD](#)

 Alessandro De Santis (Helmholtz-Institut Mainz, Johannes Gutenberg-Universität Mainz)

 2026/07/31 17:10
78. [Continuum and chiral extrapolations in the lattice calculation of inclusive \$D_s\$ -meson semileptonic decays](#)

 Ryan Kellermann (High Energy Accelerator Research Organization (KEK))

 2026/07/30 16:30
125. [A calculation of the time-like pion form factor via an infinite-volume approach](#)

 Mr Gabriele Morandi (Università degli Studi di Milano - Bicocca)

 2026/07/27 16:30
315. [Update on hadronic \$D\$ decays at the SU\(3\) flavour symmetric point](#)

 Max Hansen

 2026/07/28 15:00

Finite-volume effects / other uses

263. [Finite-volume corrections to smeared spectral densities](#)

 Francesca Argia Bresciani (University of Milano-Bicocca)

 2026/07/28 15:00
139. [Disentangling finite-volume and reconstruction errors on smeared spectral functions](#)

 Matteo Saccardi (Colorado State University)

Methods/Formulations

320. [Bootstrapping spectral functions](#)

 Ryan Abbott (Columbia University)

 2026/07/27 15:00
267. [Kernel transformations for smeared spectral functions](#)

 William Jay (Colorado State University)

 2026/07/27 15:20
250. [Nevanlinna-Pick interpolation from uncertain Euclidean correlation functions](#)

 Sarah Fields (Columbia University)

 2026/07/27 14:40
285. [Spectral density reconstruction from lattice QCD correlators: an approach via numerical anti-Laplace transform. Analytical characterisation](#)

 Jose Alejandro Vidal Maxia (University of Parma & INFN)

 2026/07/29 9:20
310. [Spectral density reconstruction from lattice QCD correlators: an approach via numerical anti-Laplace transform. Numerical treatment.](#)

 Francesco Di Renzo

 2026/07/29 9:40
303. [Direct and indirect inverse-Laplace strategies for spectral reconstruction](#)

 Demetrianos Gavriel

 2026/07/29 10:00
138. [Reconstructing spectral densities from integral transforms I](#)

 Matteo Saccardi (Colorado State University)

 2026/07/28 14:00
149. [Reconstructing spectral densities from integral transforms II](#)

 Diego Toniolo (University of Milan-Bicocca)

 2026/07/28 14:20
28. [Systematic uncertainties in spectral reconstruction from lattice correlators](#)

 Ryutaro Tsuji (KEK)

 2026/07/28 15:20
288. [Spectral reconstruction for reduced excited-state contamination in Euclidean correlators](#)

 Davide Laudicina (Ruhr Universität Bochum)

 2026/07/28 14:40
225. [Resolving vector-current spectral functions with singular value decomposition](#)

 Sakura Itatani (SOKENDAI / KEK)

 2026/07/29 9:20

Opportunities

R ratio

ETMC, PRL 130 (2023) 24, 241901, arXiv:2212.08467

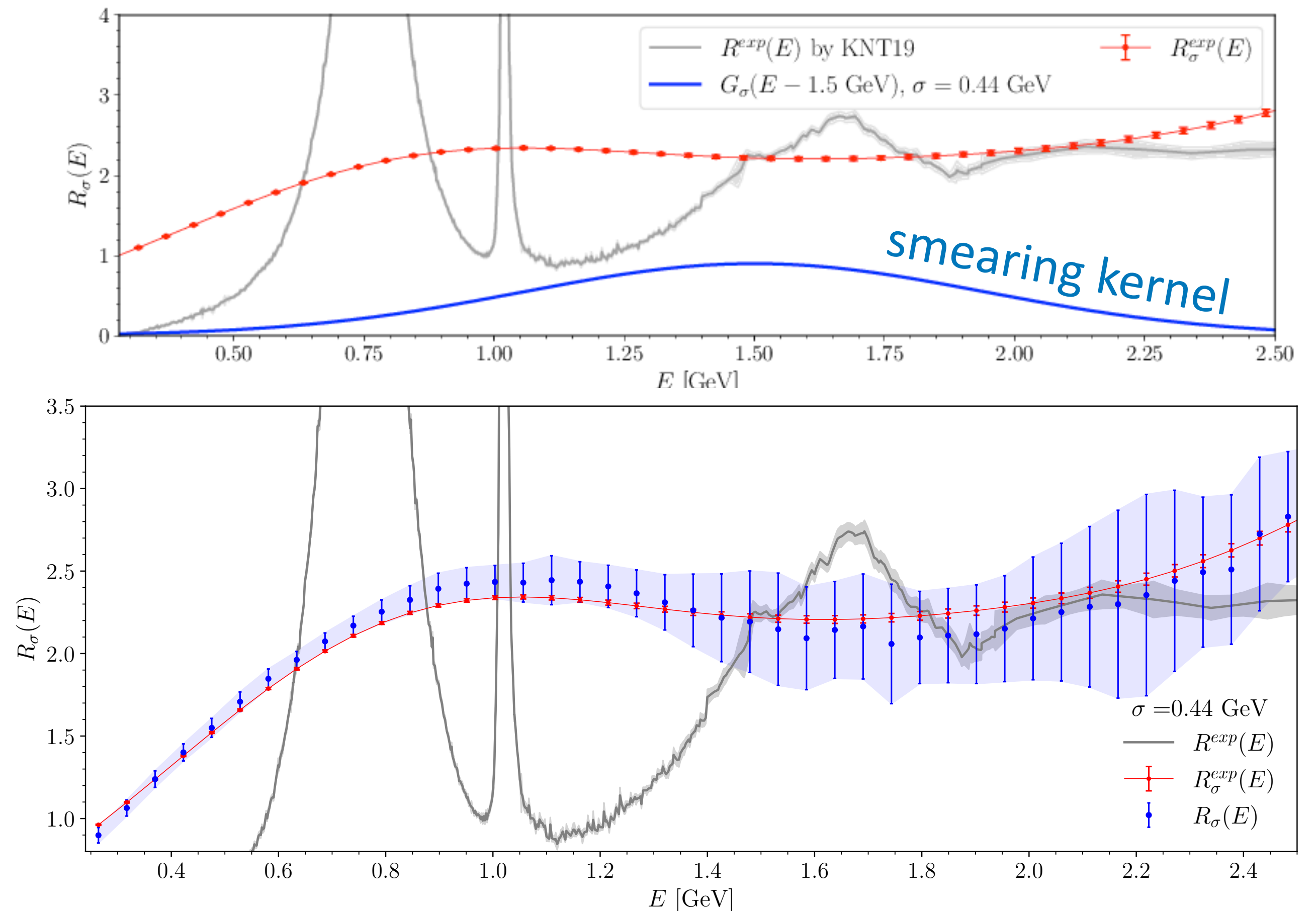
see also, talks by Saccardi and Margari

$$e^+e^- \rightarrow q\bar{q}$$

a workable example of a smeared spectrum; also serves as a test of the HLT (Hansen-Lupo-Tantalo) method

An alternative way of testing QCD

- Reasonable agreement
- But, smearing too broad to probe resonance structures



Inclusive τ decays

Evangelista et al. (ETMC), PRD108, 074513 (2023); see also, Baske's talk

A kind of smeared spectrum

$$\Gamma(\tau) \sim \int_0^{m_\tau} \frac{d\omega}{\omega^3} \left(1 - \frac{\omega^2}{m_\tau^2}\right) \left(1 + 2\frac{\omega^2}{m_\tau^2}\right) \cdot \omega^2 \rho_T(\omega)$$

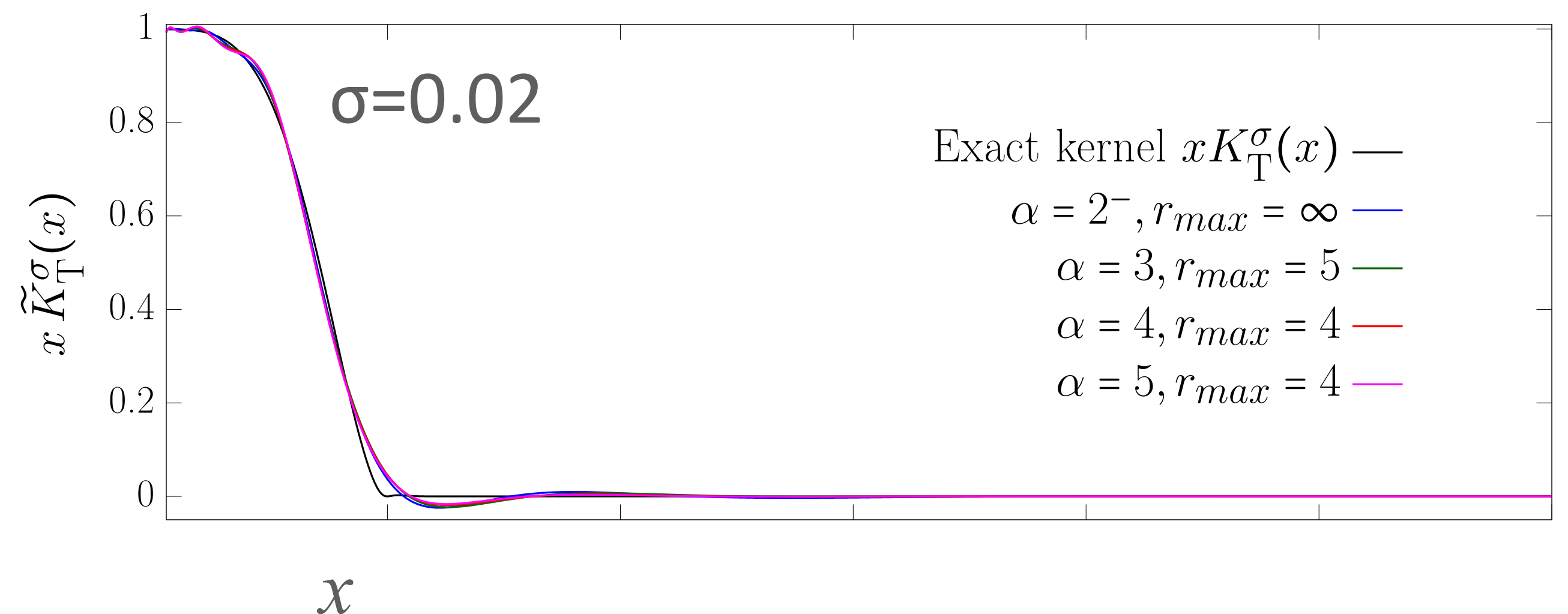
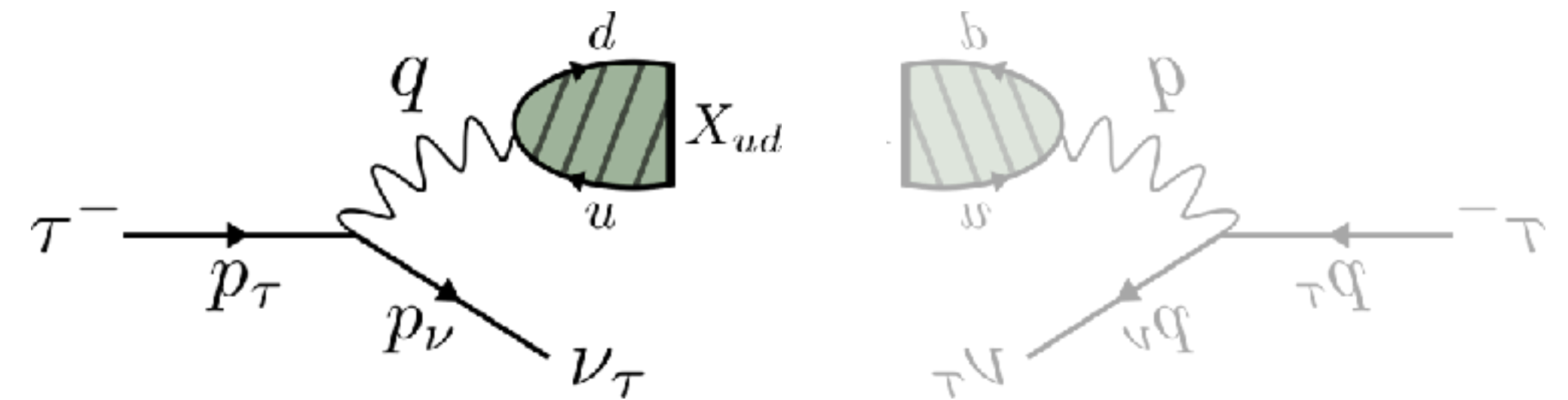
$\uparrow$
 $\times \theta(m_\tau - \omega)$

to implement the upper limit

Implemented using the HLT method with a sigmoid function

$$\theta_\sigma(x) \equiv \frac{1}{1 + e^{-x/\sigma}}$$

to better control the approximation;
the limit $\sigma \rightarrow 0$ is then taken.



Inclusive semileptonic decays $B \rightarrow X \ell \nu$

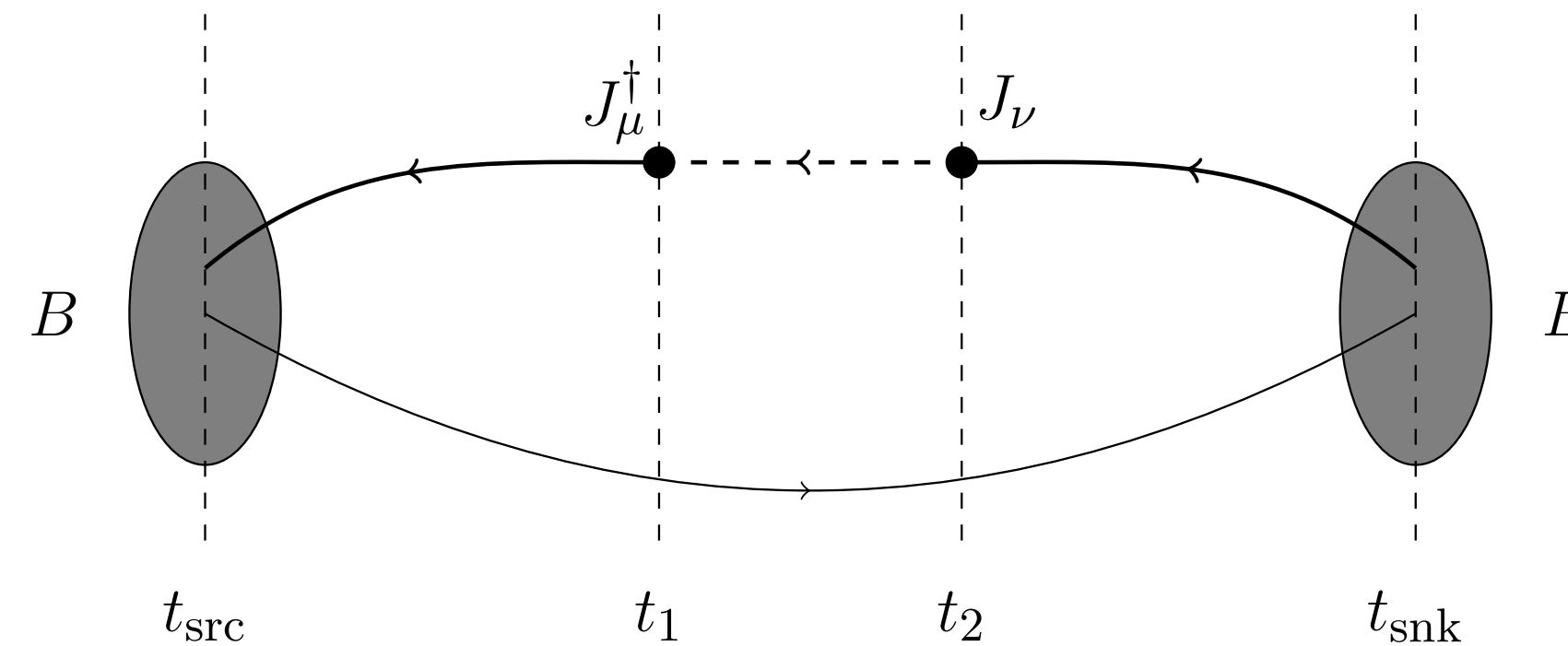
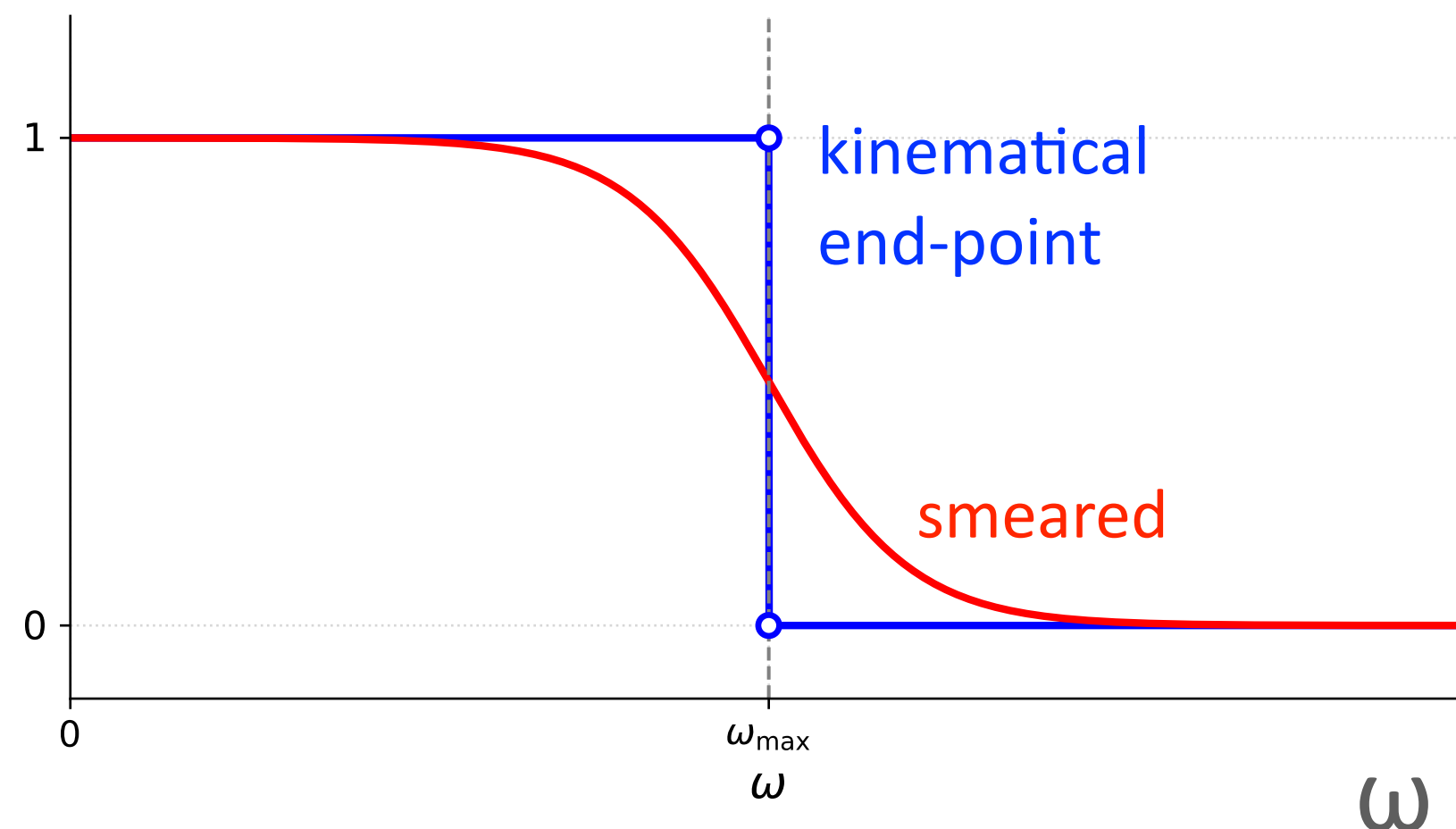
also, for νN scattering

Gambino, SH, PRL 125 (2020) 3, 032001, arXiv:2005.13730

Fukaya, SH, Kaneko, Ohki, PRD 102 (2020) 114506;
arXiv:2010.01253

$$\Gamma \propto \int_0^{q_{\max}^2} d\mathbf{q} \int_{\sqrt{m_D^2 + \mathbf{q}^2}}^{m_B - \sqrt{\mathbf{q}^2}} d\omega K(\omega; \mathbf{q}^2) \underbrace{\rho(\omega) = \langle B(\mathbf{0}) | \tilde{J}^\dagger(-\mathbf{q}) \delta(\omega - \hat{H}) \tilde{J}(\mathbf{q}) | B(\mathbf{0}) \rangle}_{\text{smearing kernel}}$$

The smearing kernel is essentially a step function



The ill-posed inverse problem returns for non-smooth kernel function; some smearing is needed and must be extrapolated away.

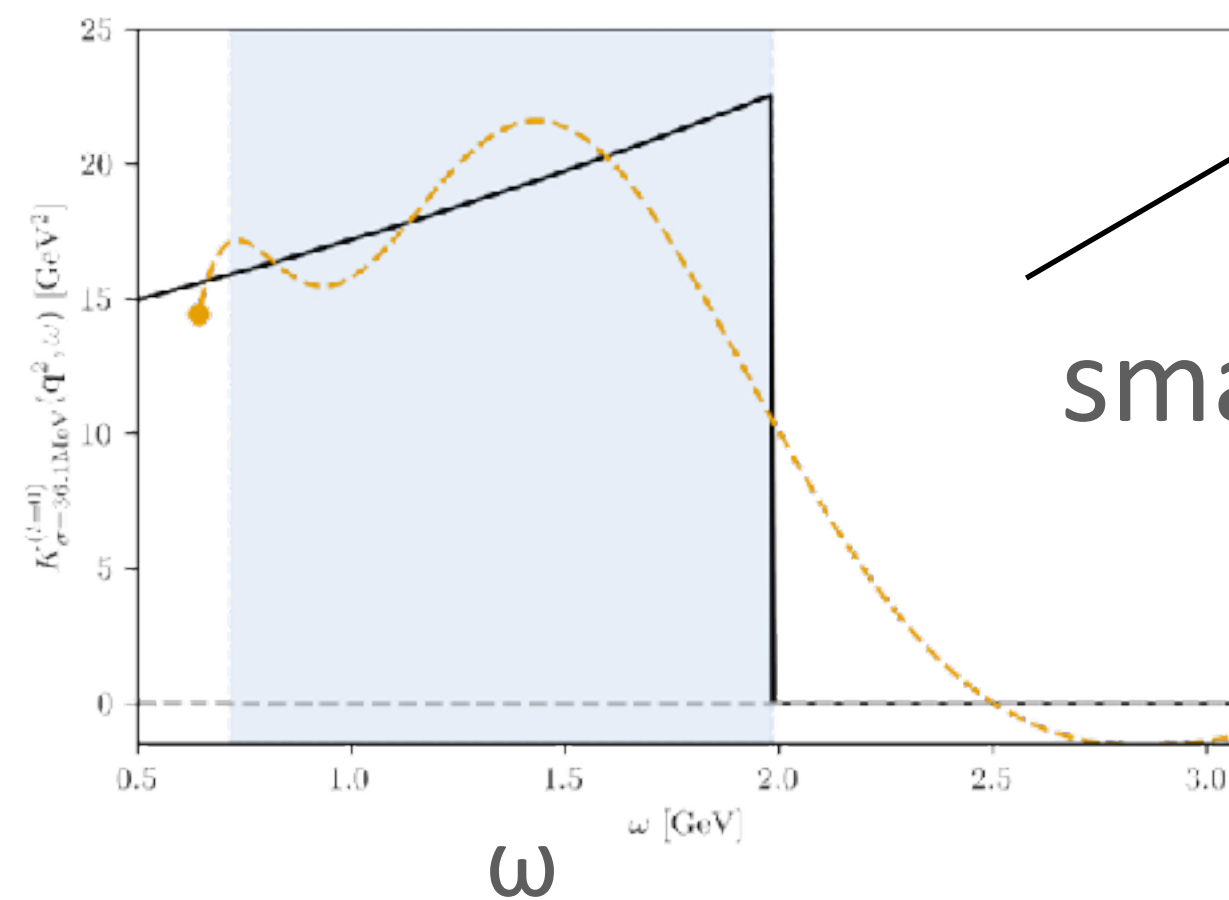
Kinematical endpoint

phase-space is squeezed

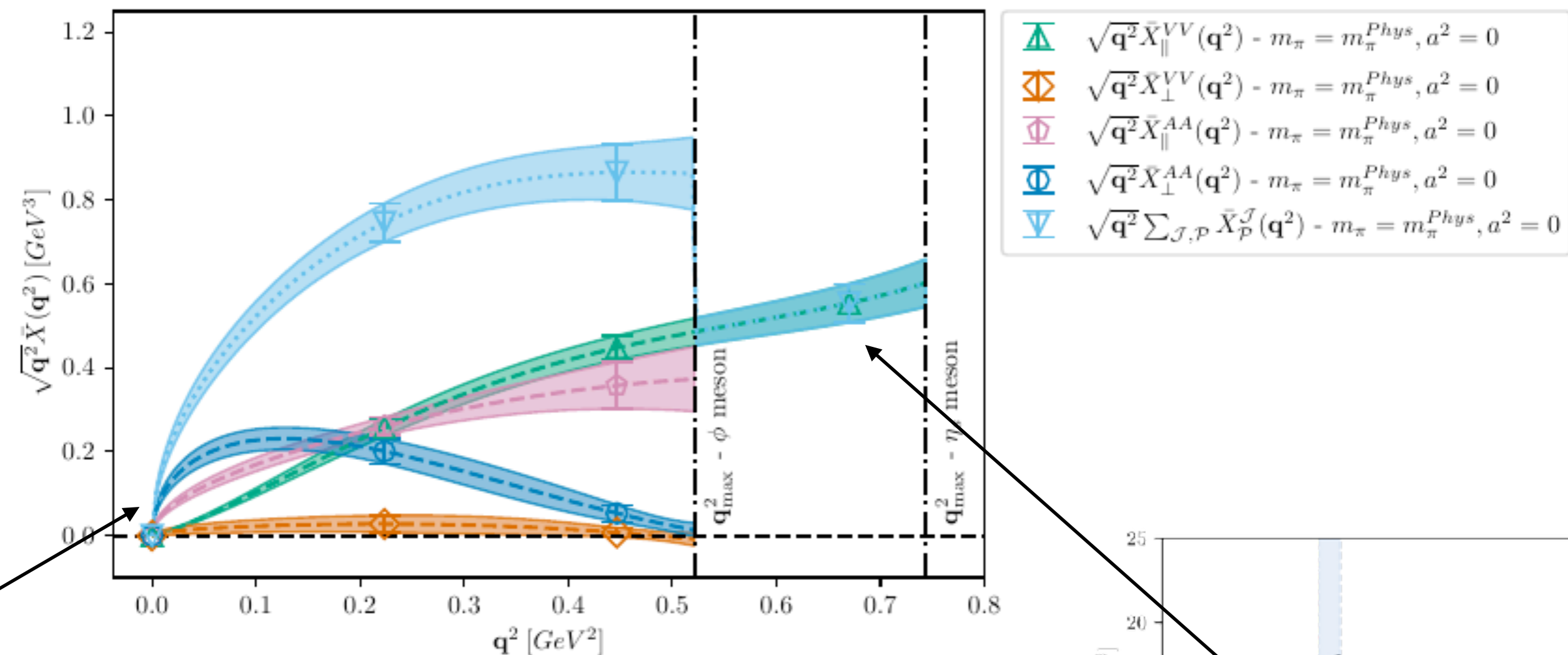
Inclusive semileptonic D_s decays

Kellermann et al., arXiv:2604.22201;
see also, Kellermann's talk

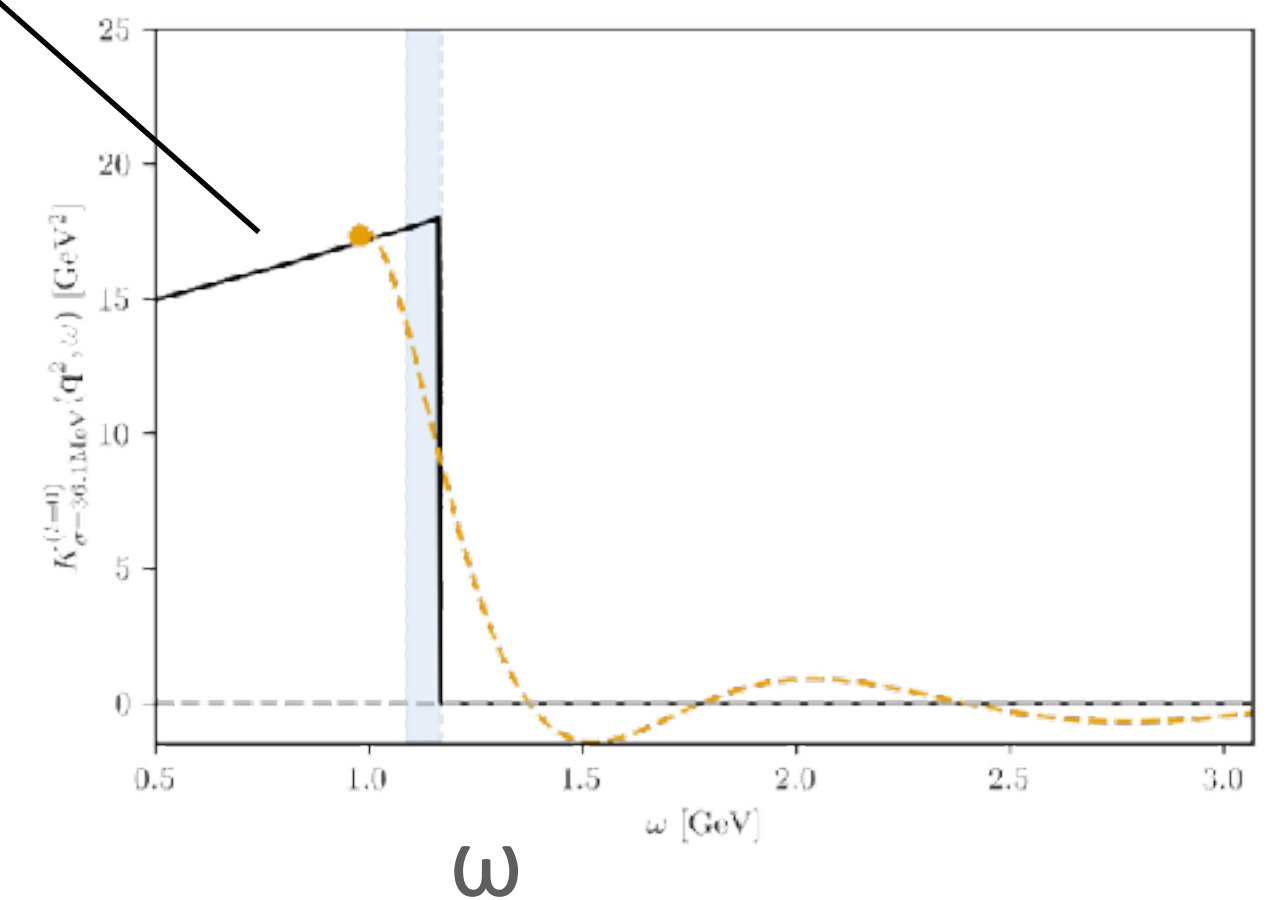
Kernel $K(\omega)$



small recoil



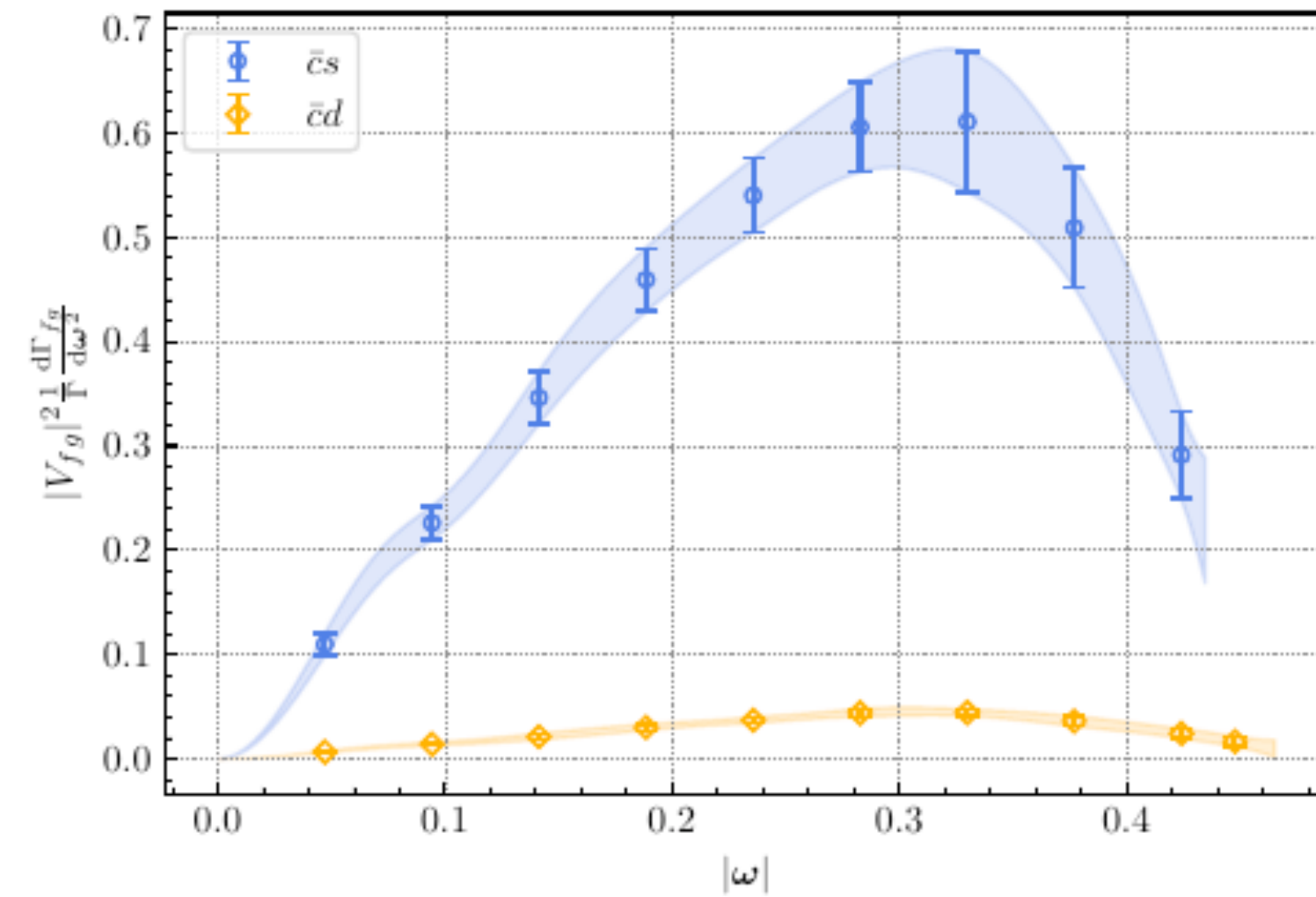
large recoil



Need careful study of truncation errors;
also a physics question depending on the physical spectrum

Applications to inclusive $D_{(s)}$ decays

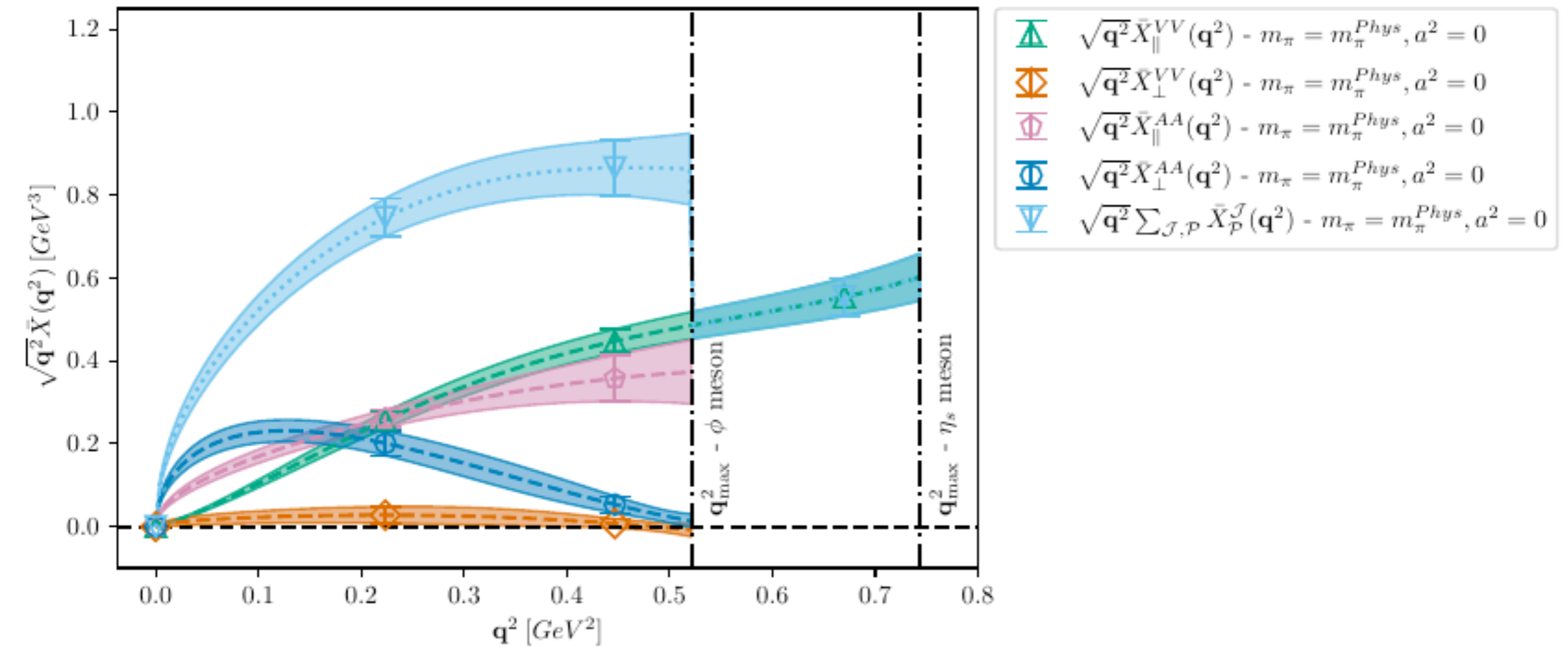
De Santis et al., PRL 135 (2025) 12, 121901,
arXiv:2504.06064; PRD 112 (2025) 5, 054503,
arXiv:2504.06063



$$|V_{cs}| = 0.951(35)$$

$$|V_{cs}|^{\text{PDG}} = 0.975(6)$$

Kellermann et al., arXiv:2604.22201



$$|V_{cs}| = 0.967(28)^{+0.024}_{-0.029}$$

with BESIII exp data

The $\sigma \rightarrow 0$ extrapolation is insignificant at the current level of accuracy.

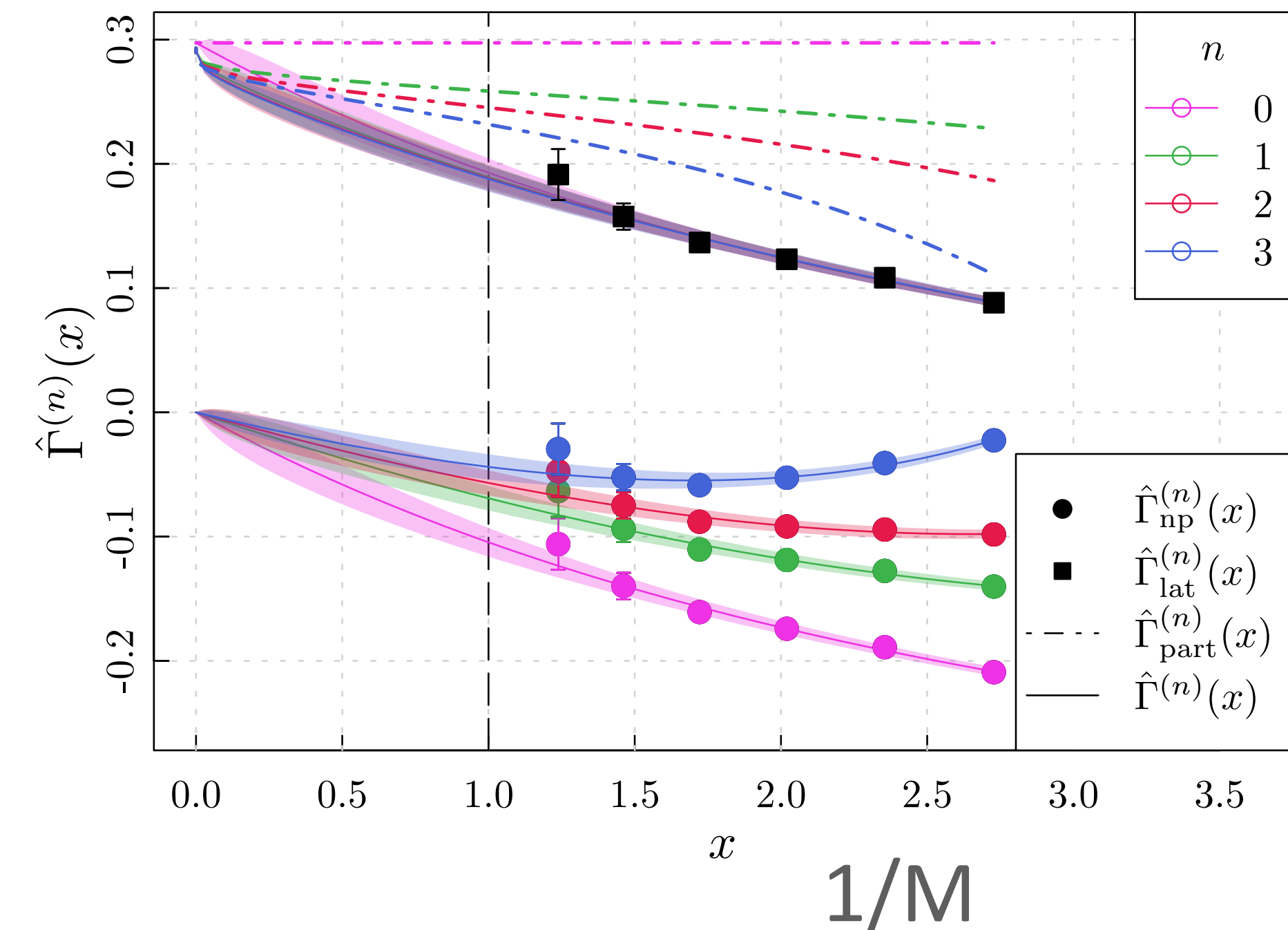
Extrapolation from charm to bottom

De Santis et al., arXiv:2607.01116;

see also De Santis' talk

- Calculation gets harder because of larger recoil momenta = larger noise and discretization effects.
- Asymptotic scaling toward $m_Q \rightarrow \infty$ can be used for extrapolation.

perturbative limit

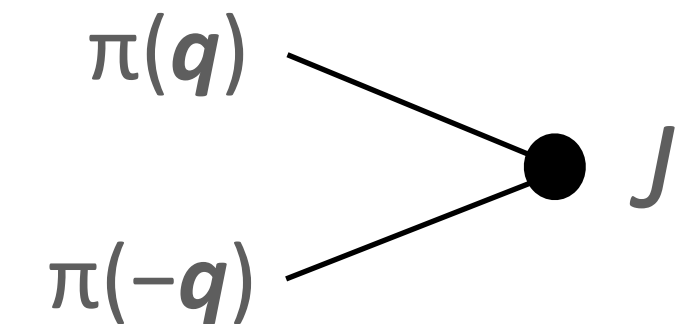


A step toward the resolution of the inclusive versus exclusive puzzle

Maiani-Testa problem

Maiani, Testa, “Final state interactions from euclidean correlation functions,” Phys. Lett. B245, 585 (1990).

Euclidean 3-pt function: $C(t_1, t_2; \mathbf{q}, -\mathbf{q}) = \langle \phi_{\mathbf{q}}(t_1) \phi_{-\mathbf{q}}(t_2) J(0) \rangle$



$$C(t_1, t_2) \rightarrow e^{-E(\mathbf{q})t_1} e^{-E(\mathbf{q})t_2} \langle \pi(\mathbf{q}) \pi(-\mathbf{q}) | J(0) | 0 \rangle$$

$$+ \sum_n e^{-E(\mathbf{q})(t_1 - t_2)} \langle \pi(\mathbf{q}) | \phi_{-\mathbf{q}}(0) | n \rangle e^{-E_n t_2} \langle n | J(0) | 0 \rangle$$

Intermediate states with zero total momentum start from 2 pions at rest.

Only the $\mathbf{q}=0$ state is directly accessible:

$$C(t_1, t_2) \rightarrow e^{-M_\pi t_1} e^{-M_\pi t_2} f(4M_\pi^2) \left(1 - a \sqrt{\frac{M_\pi}{4\pi t_2}} + \dots \right)$$

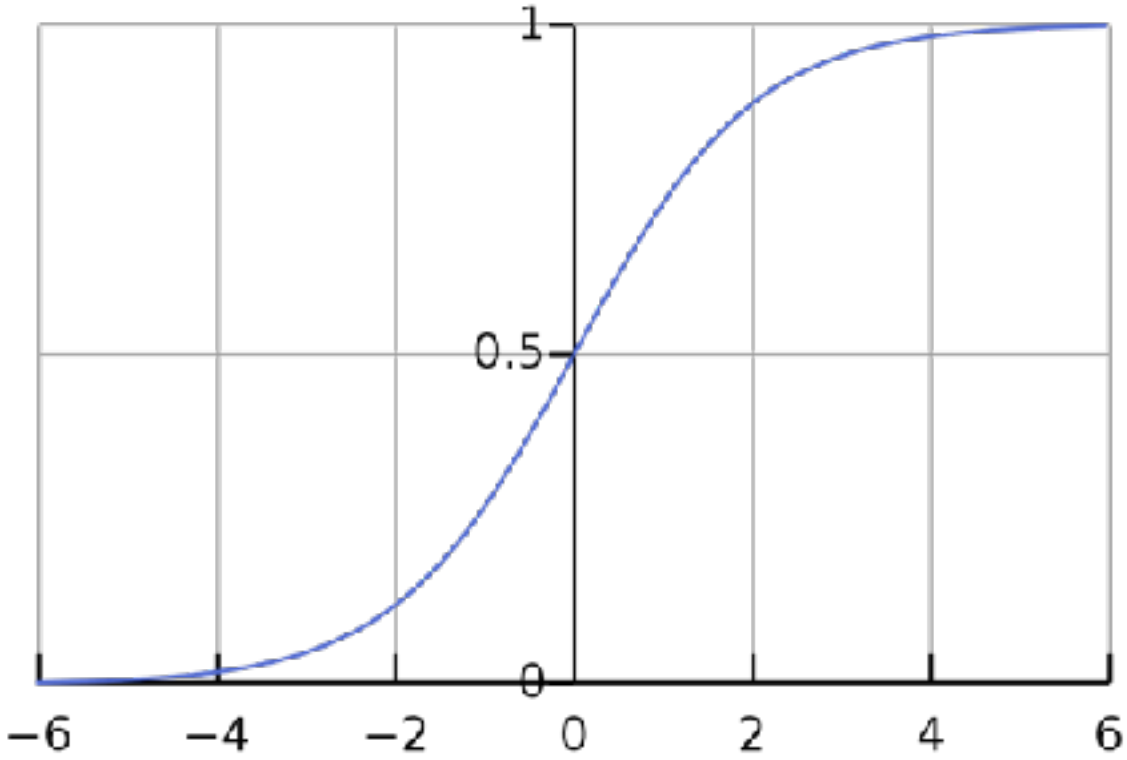
Time-like pion form factor

scattering length

see also R. Lehner's talk

High-pass filter

Bruno, Hansen, JHEP 06 (2021) 043, arXiv:2012.11488



energy filtering
= smeared spectrum

$$\langle \pi(\mathbf{q}) | \phi_{-\mathbf{q}}(t) \Theta(\hat{H} - \sqrt{s}, \Delta) J(0) | 0 \rangle$$

$$\rightarrow e^{-E(\mathbf{q})t} \left[\underbrace{\Theta(0, \Delta) \text{Re}[f(s)]}_{\text{known functions}} - \underbrace{2\mathcal{J}^{(0)}(t, s, \Delta) \text{Im}[f(s)]}_{\sqrt{s}} + \dots \right]_{s=(2E(\mathbf{q}))^2}$$

$f(s) = \langle \pi\pi; s | J(0) | 0 \rangle$ time-like form factor

or **low-pass filter**

to enhance the ground state: see Laudicina’s talk

or **energy-moments**

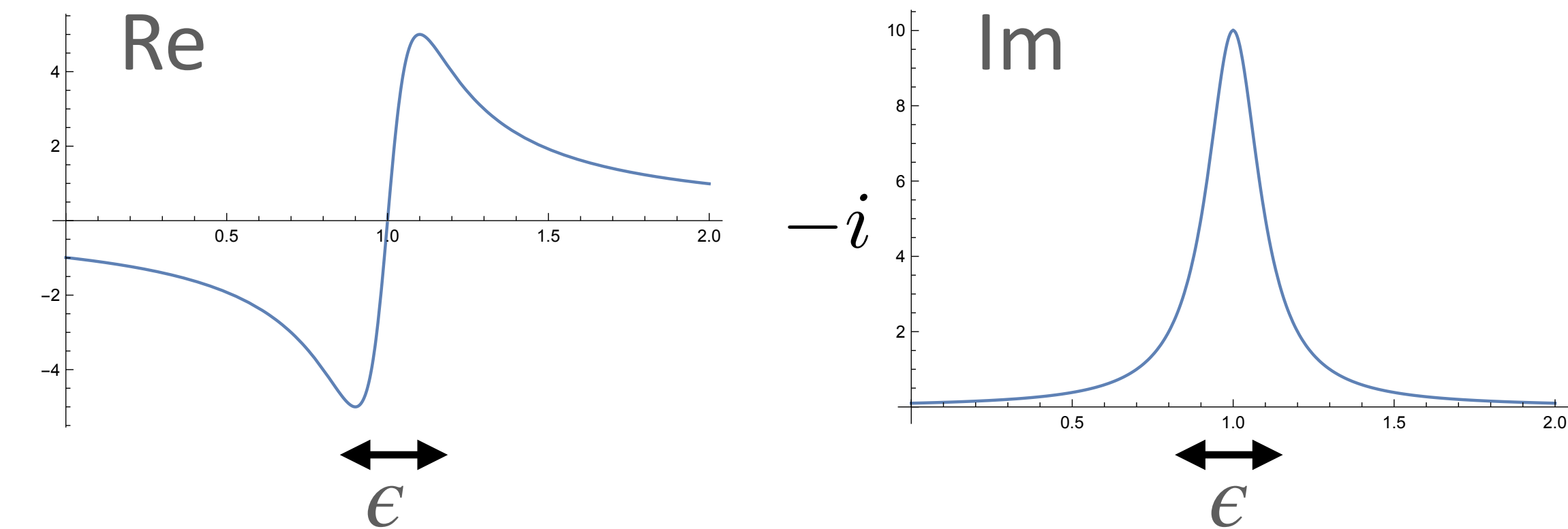
to focus on the state of interest: see Itatani’s talk

Scattering amplitudes

Bulava, Hansen, PRD 100 (2019) 3, 034521, arXiv:1903.11735
 Frezzotti, Tantalo et al., PRD 108 (2023) 7, 074510, arXiv:2306.07228

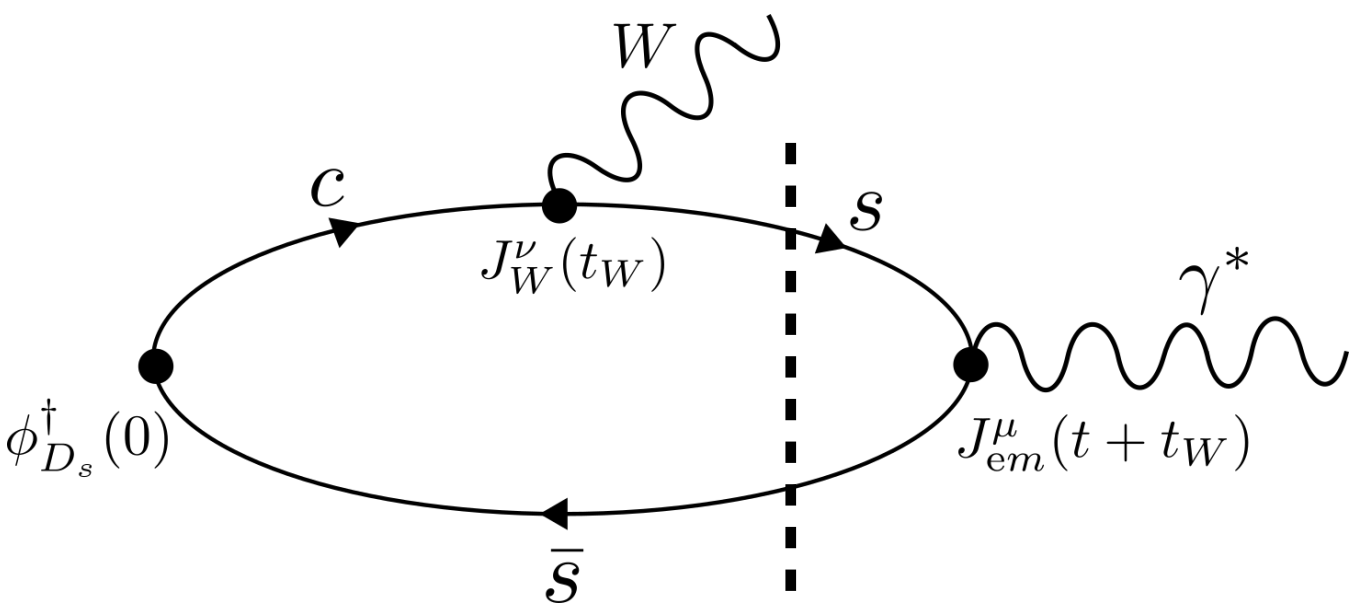
Amplitude can be constructed as

$$A(E) = \lim_{\epsilon \rightarrow 0} \int \frac{dE'}{2\pi} \frac{\rho(E')}{E' - E - i\epsilon}$$

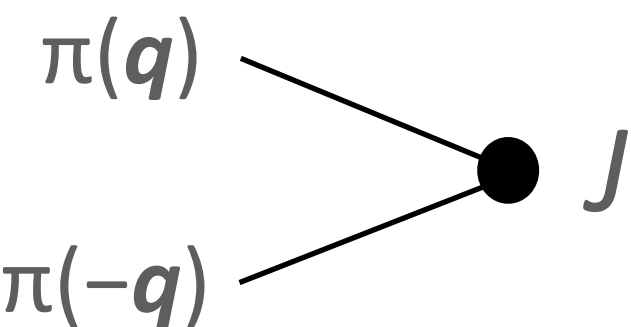


Pick up the info around a specified energy
 = smeared spectrum

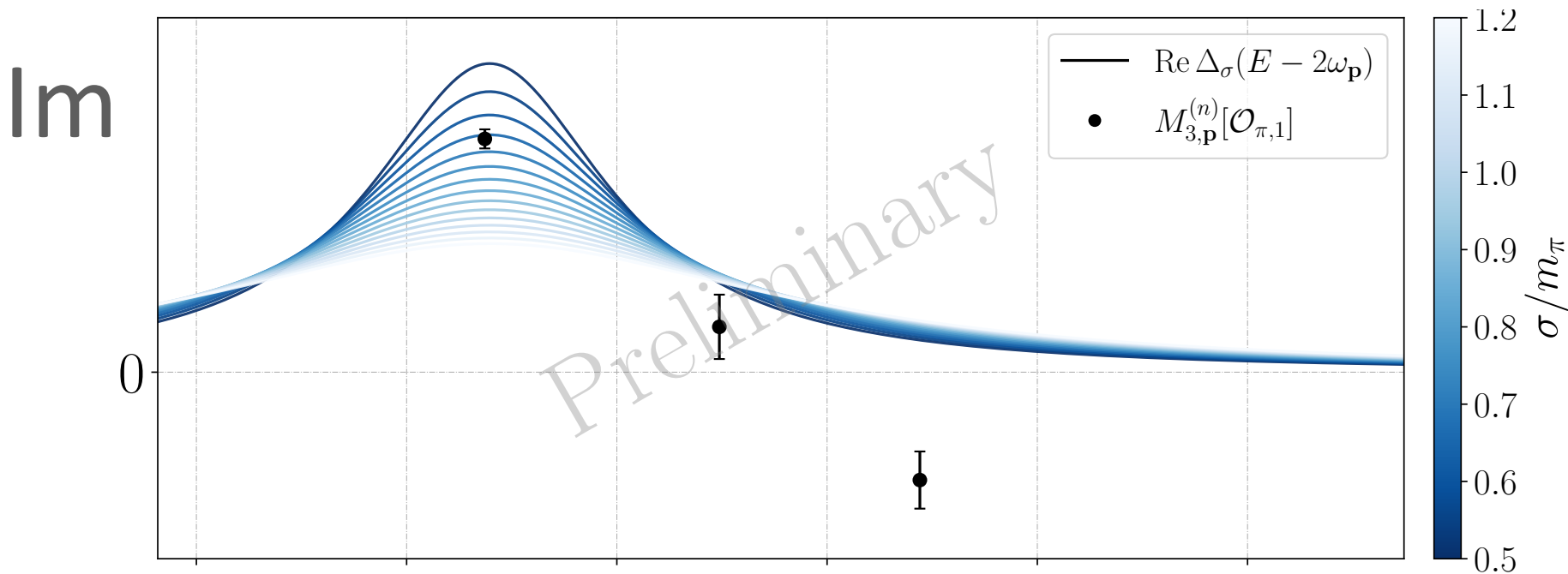
e.g. : $D_s \rightarrow \ell \nu \gamma^*$



e.g.: $\gamma^* \rightarrow \pi\pi$



see, Morandi’s talk



ϵ has to be sent to 0 ... a major challenge.

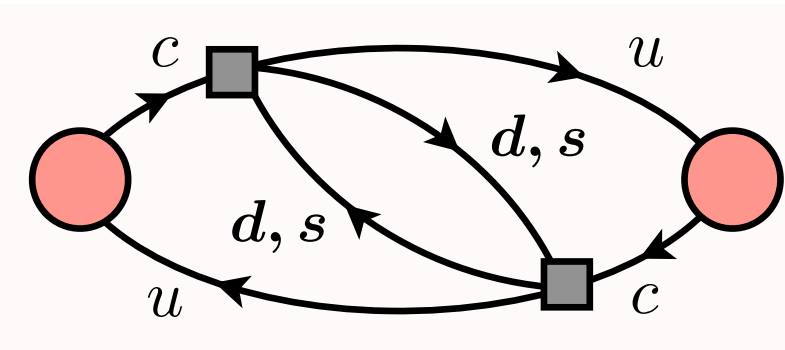
$D - \bar{D}$ mixing

Di Carlo, Erben, Hansen, JHEP 07 (2025) 229, arXiv:2504.16189

Theoretical predictions:

- ▶ Short-distance (box etc): too small by $\times 10^{-3}$
- ▶ Long-distance: too hard to calculate

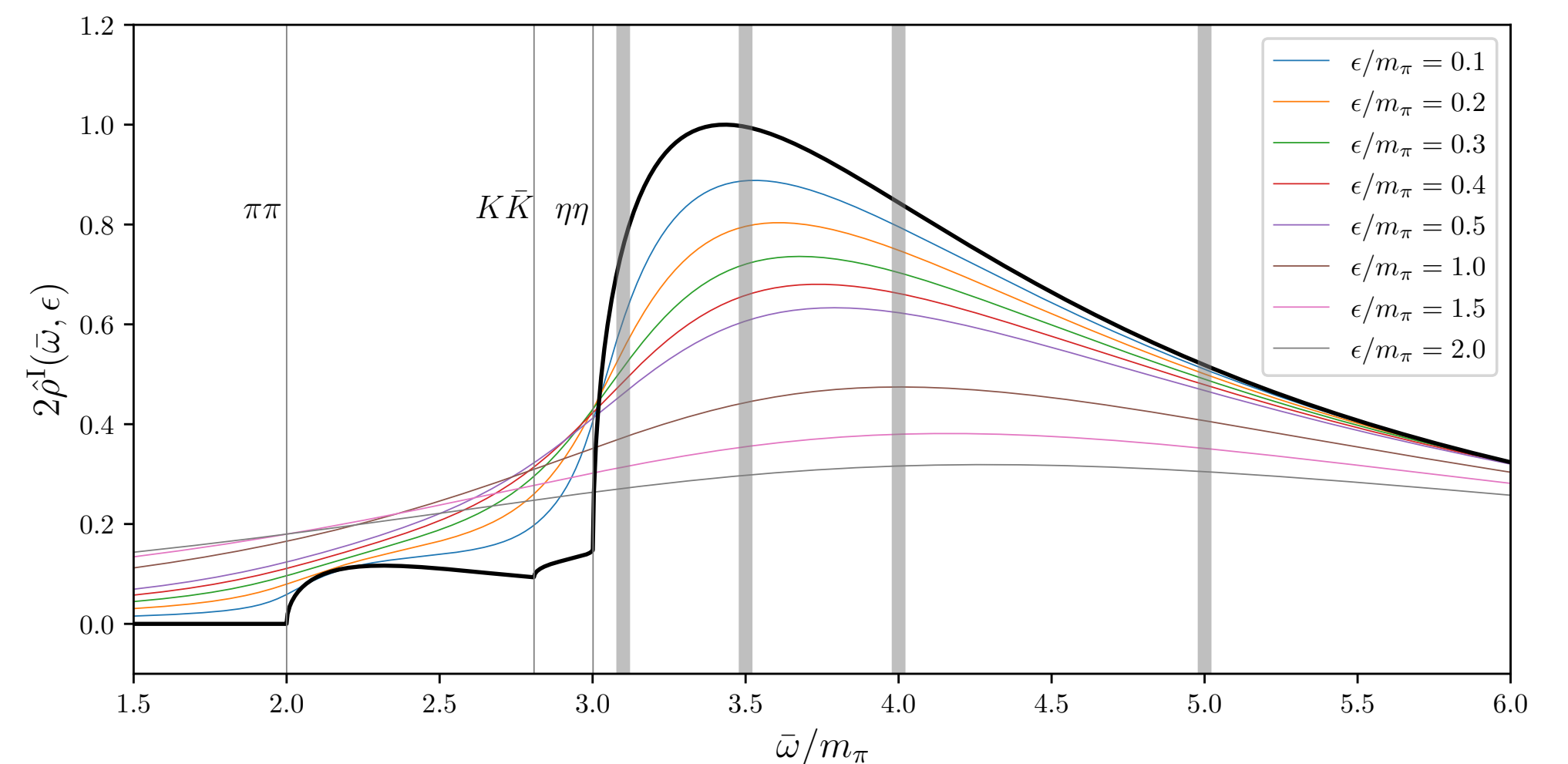
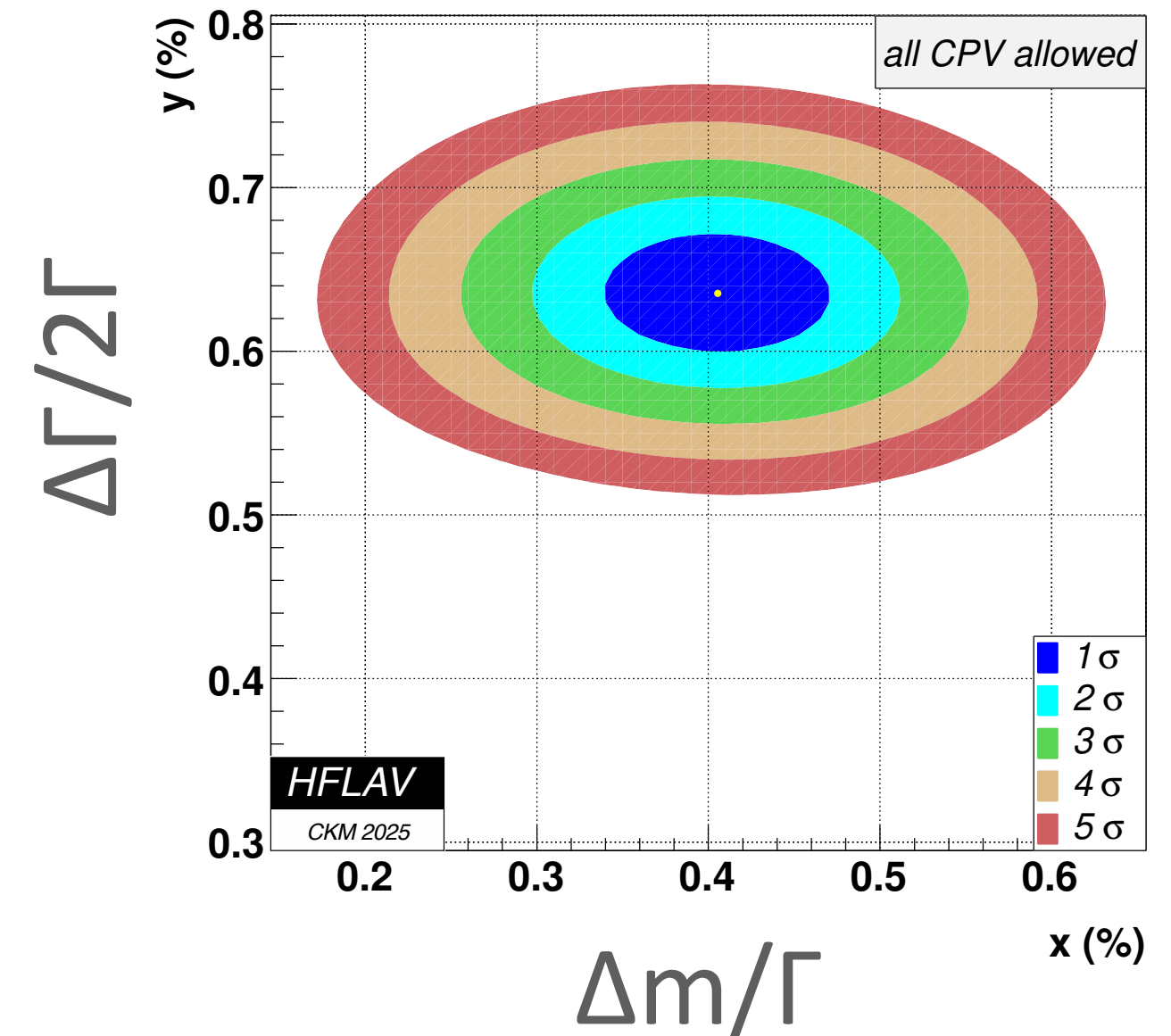
$$\Gamma_{12} \sim \sum_f A(D^0 \rightarrow f)^* A(\bar{D}^0 \rightarrow f)$$



Possible final states:

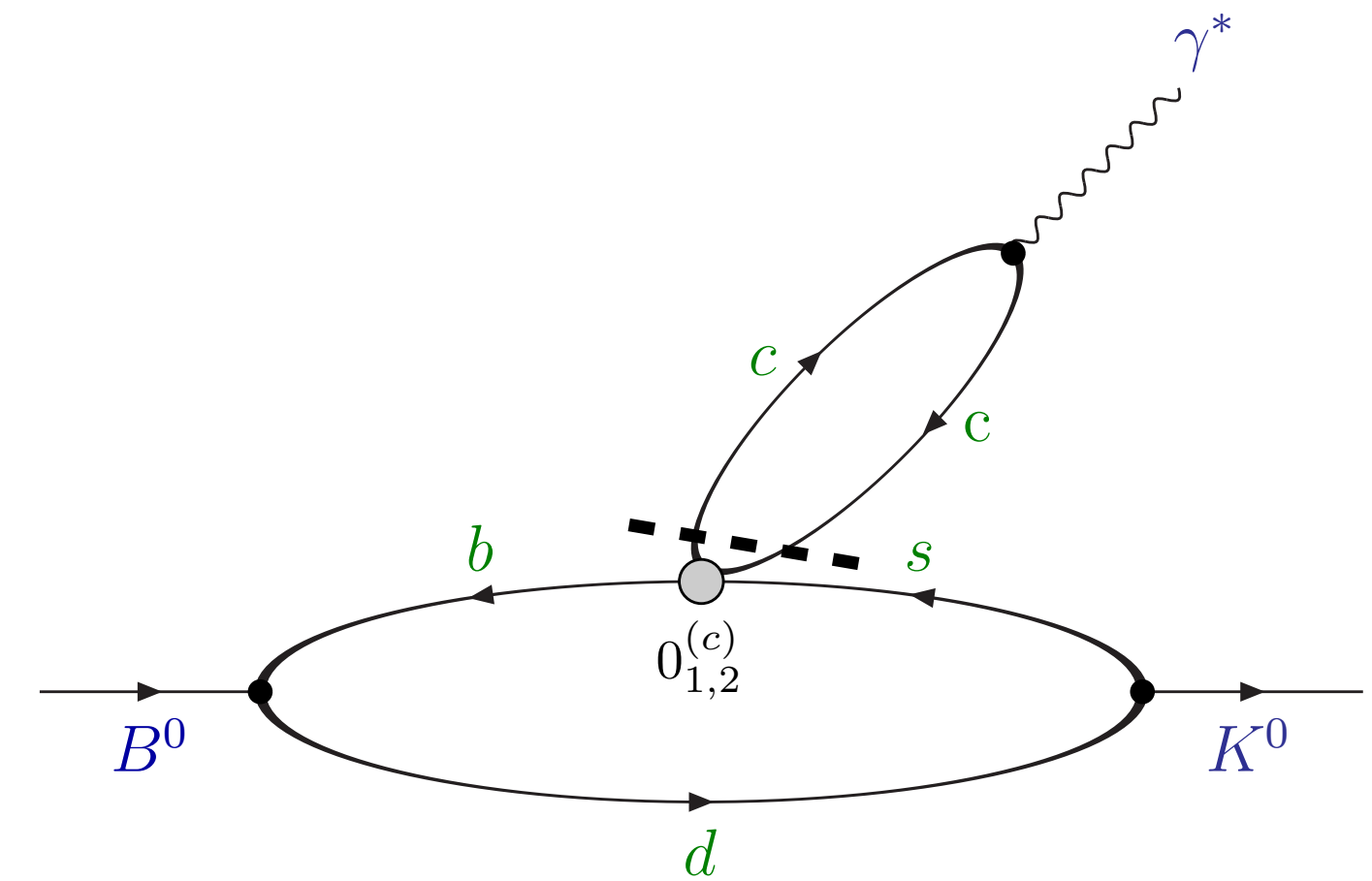
- ▶ $\pi\pi, \pi K, KK, \eta\eta, \pi K^*, \rho K, \dots$

A strategy proposed for the lattice calc.
Again the question of taking $\epsilon \rightarrow 0$

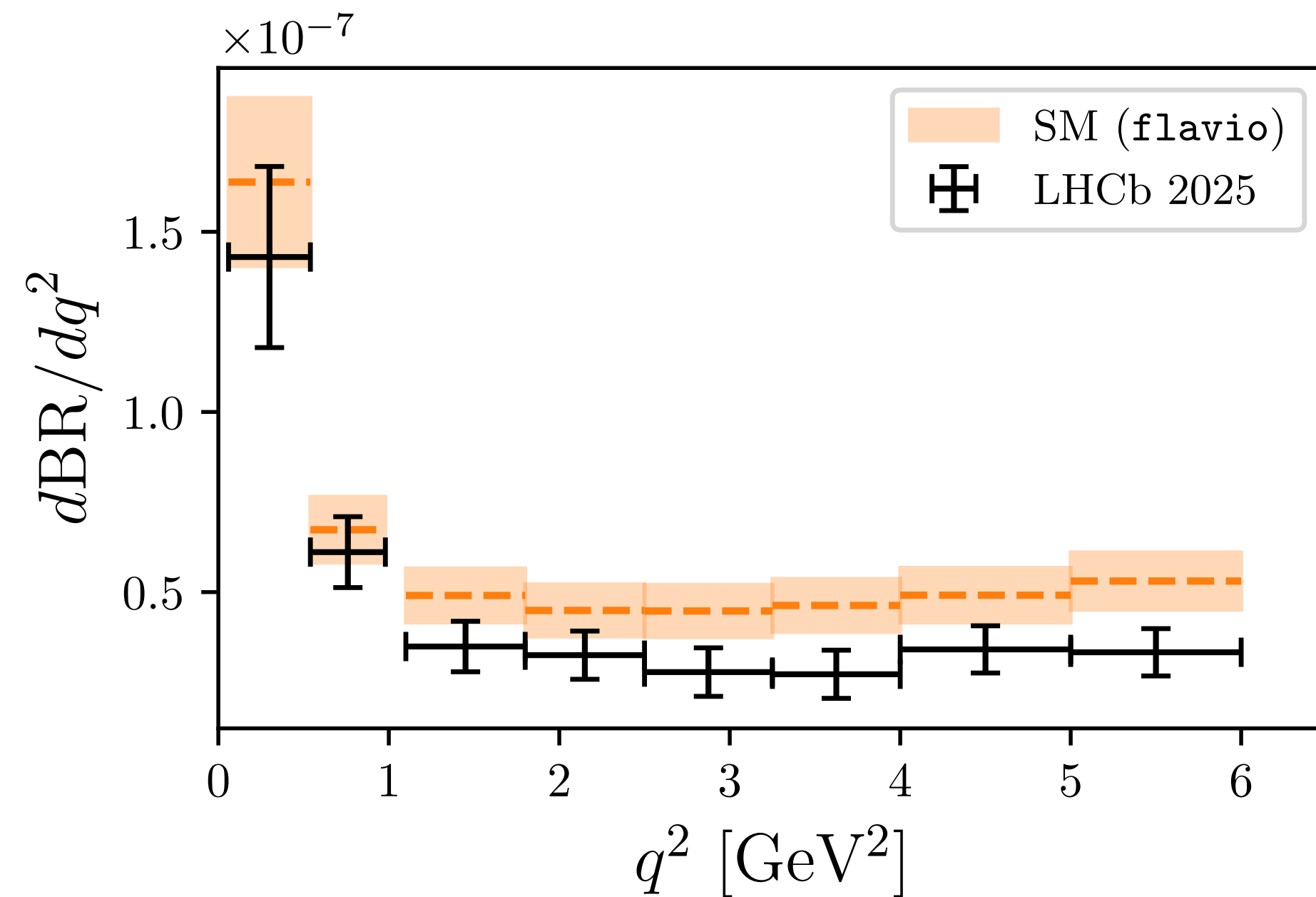


A big challenge: $B \rightarrow K^{(*)} \ell^+ \ell^-$

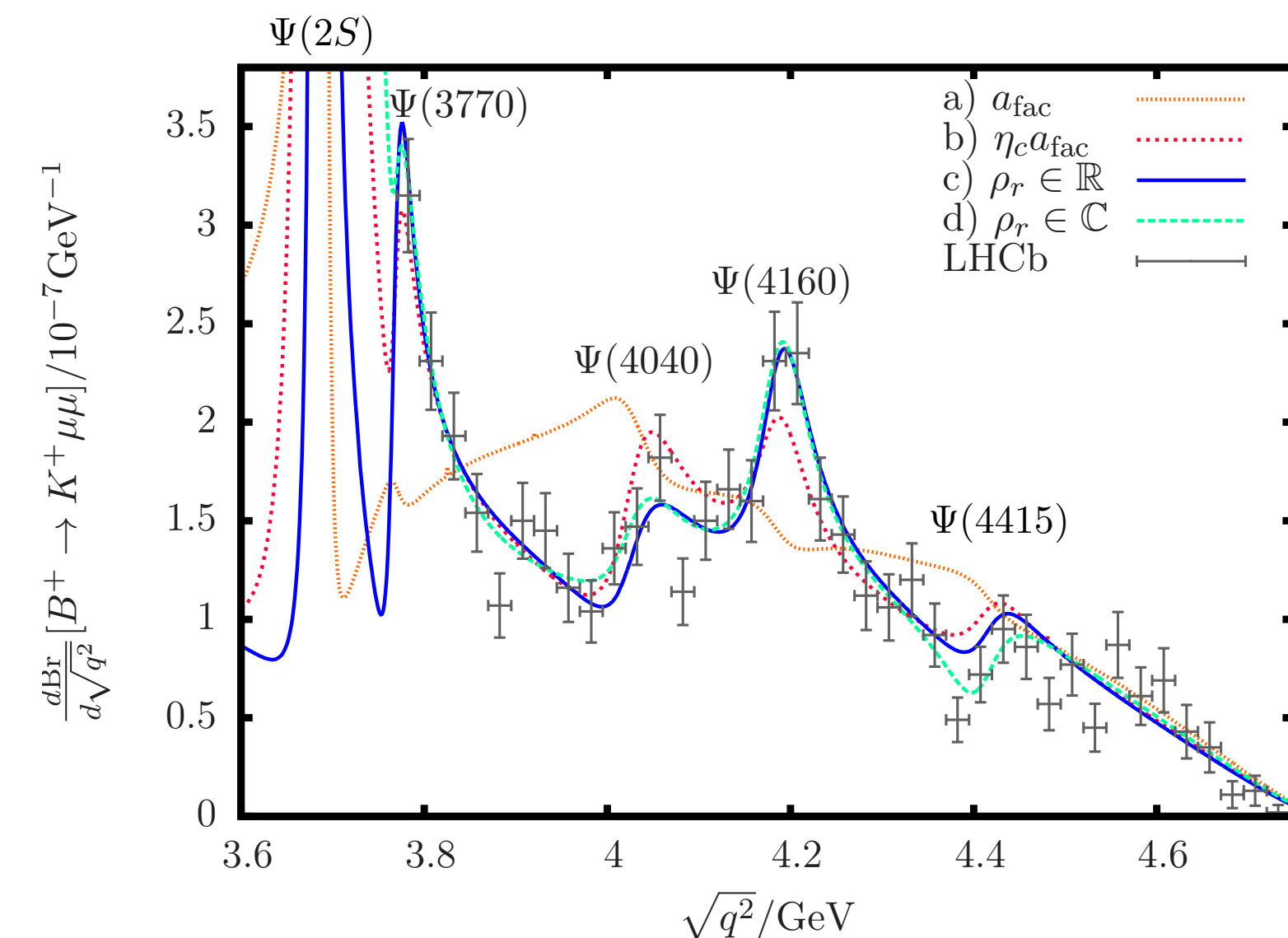
Potential new physics !



Full calculation of the amplitude is extremely hard (charm-loop, rescattering etc) ; factorization does not explain the experimental data.



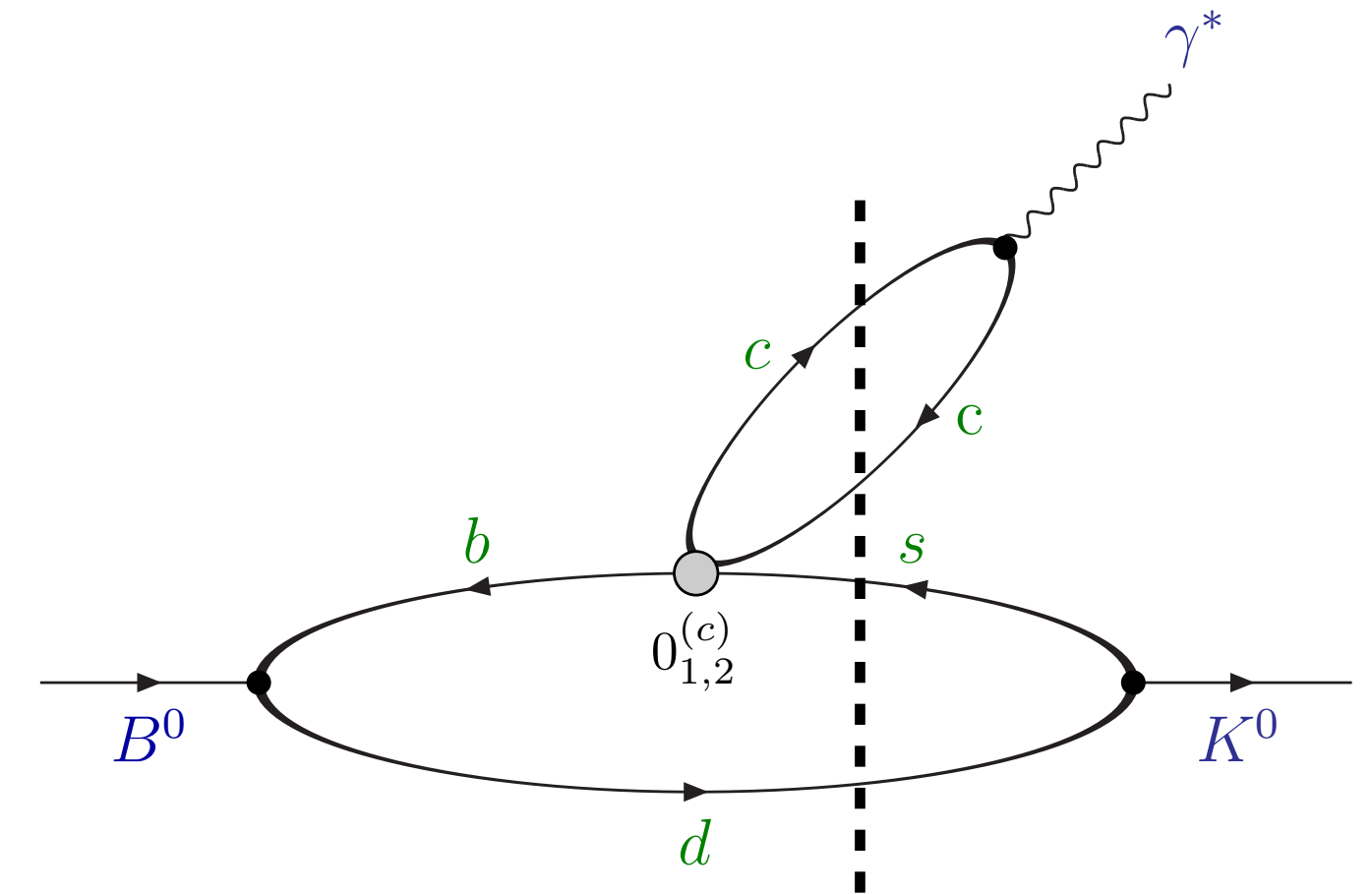
from Altmanshofer, Christensen, Stangle, 2603.27753



Lyon, Zwicky,
arXiv:1406.0566

A big challenge: $B \rightarrow K^{(*)} \ell^+ \ell^-$

Frezzotti, Tantalò et al, PRD 113 (2026) 3, 034509, arXiv:2508.03655



Following the same strategy

$$A(E) = \lim_{\epsilon \rightarrow 0} \int \frac{dE'}{2\pi} \frac{\rho(E')}{E' - E - i\epsilon}$$

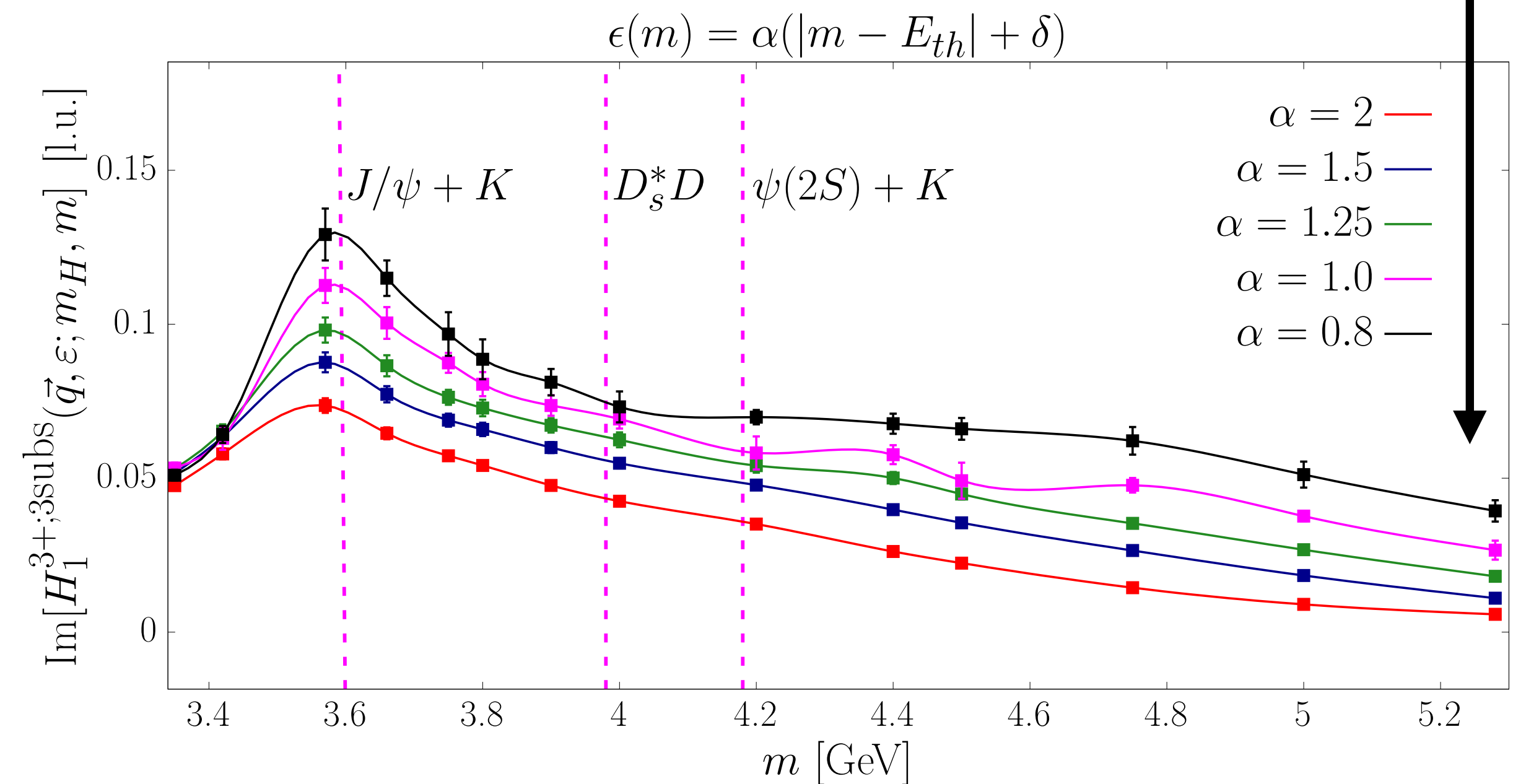
with $E = m_B$ and

$$\rho(E) = \langle K(-\mathbf{q}) | \tilde{J}_{\text{em}}(\mathbf{q}) \delta(\hat{H} - E) O_{1,2}(0) | B(\mathbf{0}) \rangle$$

A pilot study in the small recoil region;
the limit $\epsilon \rightarrow 0$ is yet to be taken.

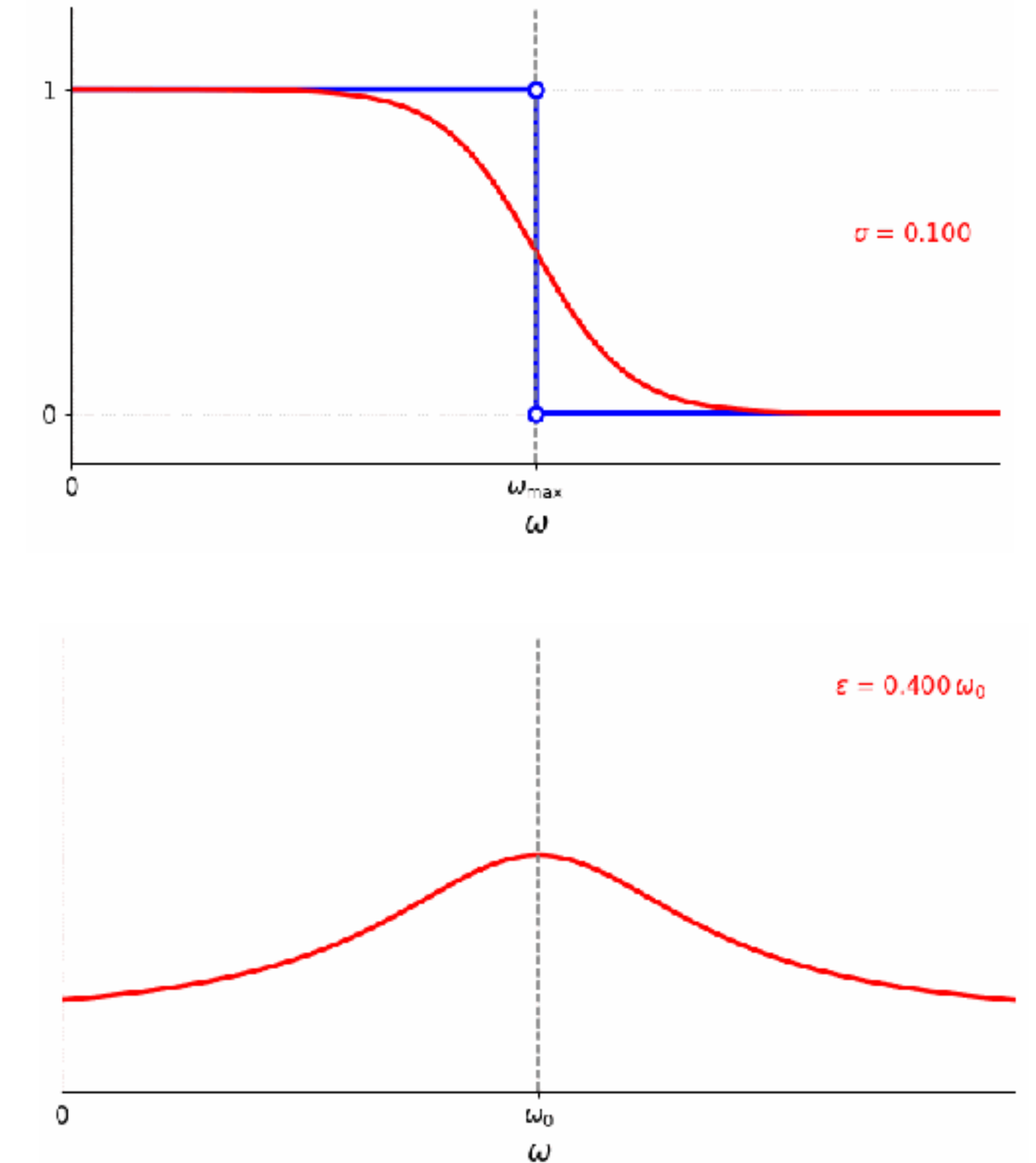
$|\mathbf{q}| = 0.25 \text{ GeV}$

$\epsilon \sim 1.5\text{-}3 \text{ GeV}$



Role of smearing

- **Smearing as given by the observable**
 - hadronic τ decays, inclusive decays... : the kernel is part of physics
- **Plus, smearing as a regulator**
 - Inclusive decays: modify Heaviside for approx;
 $\sigma \rightarrow 0$ is needed
 - Hadronic amplitudes: integ with $i\varepsilon$ -prescription;
 $\varepsilon \rightarrow 0$ is needed (δ function appears)

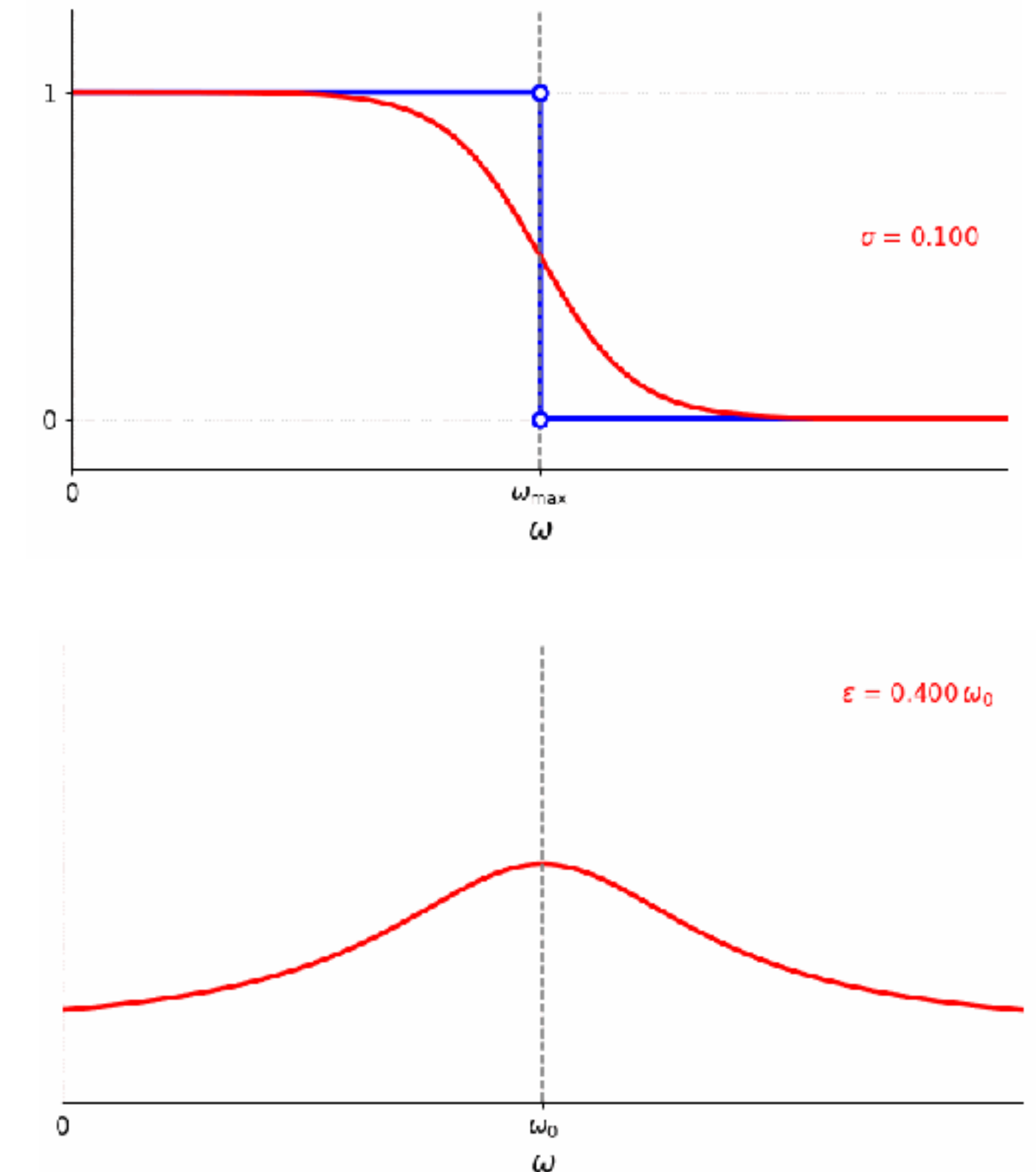


Crucial questions:

- **Physics:** How small σ (or ε) needed to take the limit reliably?
- **Technical:** How small σ (or ε) can be implemented in practice?

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Crucial questions:

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Challenges

Construct $\int_0^\infty dE \rho(E) K(E)$ from $\int_0^\infty dE \rho(E) e^{-Et}$

= linear reconstruction problem $K(\hat{H}) = g_0 + g_1 e^{-\hat{H}} + g_2 e^{-2\hat{H}} + g_3 e^{-3\hat{H}} + \dots$

A number of proposals

(likely, not a comprehensive list)

Maximum entropy: Asakawa, Hatsuda, Nakahara, hep-lat/0011040

Bayesian: Burnier, Rothkopf, PRL 111 (2013) 182003, arXiv:1307.6106; Del Debbio, Giani, Wilson, EPJC 82 (2022) 4, 330, arXiv:2111.05787

Machine Learning: Kades et al., PRD 102 (2020) 9, 096001, arXiv:1905.04305; Chen et al., arXiv:2110.13521; Wang, Shi, Zhou, PRD 106 (2022) L051502

Gaussian processes: Horal et al., PRD 105 (2022) 3, 036014, arXiv:2107.13464

Backus-Gilbert: Hansen, Meyer, Robaina, PRD 96 (2017) 9, 094513, arXiv:1704.08993

Sparse modeling: Otsuki, Ohzeki, Shinaoka, Yoshimi, PRE 95 (2017) 061302; Itou, Nagai, JHEP 07 (2020) 007, arXiv:2004.02426

Gaussian quadrature: Aliberti, Di Renzo, Dimopoulos, Gavriel, arXiv:2603.00764

More emphasis on smeared spectrum

HLT: Hansen, Lupo, Tantalo, PRD 99 (2019) 9, 094508, arXiv:1903.06476

Chebyshev: Bailas, SH, Ishikawa, PTEP 2020 (2020) 4, 043B07, arXiv:2001.11779

Conformal map (Nevanlinna-Pick): Bergamaschi, Jay, Oare, PRD 108 (2023) 7, 074516, arXiv:2305.16190; Abbott, Jay, Oare, arXiv:2508.01377; Fields, Christ, PRD 113 (2026) 5, 5, arXiv:2510.12136

Baysian $\rightleftharpoons$ HLT: Del Debbio, Lupo, Panero, Tantalo, EPJC 85 (2025) 2, 185, arXiv:2409.04413

ML: Buzzicotti, De Santis, Tantalo, EPJC 84 (2024) 1, 32, arXiv:2307.00808; De Santis, arXiv:2607.03311

An earlier review: W. Jay @ Lattice 2024, arXiv:2501.12259

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(likely, not a comprehensive list)

Maximum entropy: Asakawa, Hatsuda, Nakahara, hep-lat/0011040

Bayesian: Burnier, R
arXiv:2111.05787

Machine Learning: K
PRD 106 (2022) L051

Gaussian processes:

Backus-Gilbert: Han

Sparse modeling: O

Gaussian quadrature:

More emphasis on

HLT: Hansen, Lupo, T

Chebyshev: Bailas, S

Conformal map (Ne

arXiv:2508.01377; F

Baysian $\Leftrightarrow$ **HLT:** Del Debbio, Lupo, Panero, Tantalò, EPJC 85 (2025) 2, 185, arXiv:2409.04415

ML: Buzzicotti, De Santis, Tantalò, EPJC 84 (2024) 1, 32, arXiv:2307.00808; De Santis, arXiv:2607.03311

An earlier review: W. Jay @ Lattice 2024, arXiv:2501.12259

Details are different, but the broad consensus would be

1. Reconstruction of the local spectrum is indeed an ill-posed problem = no breakthrough (so far)
2. Smeared spectrum can be reconstructed with controlled errors (details matter)

Not a magic...

2) 4, 330,

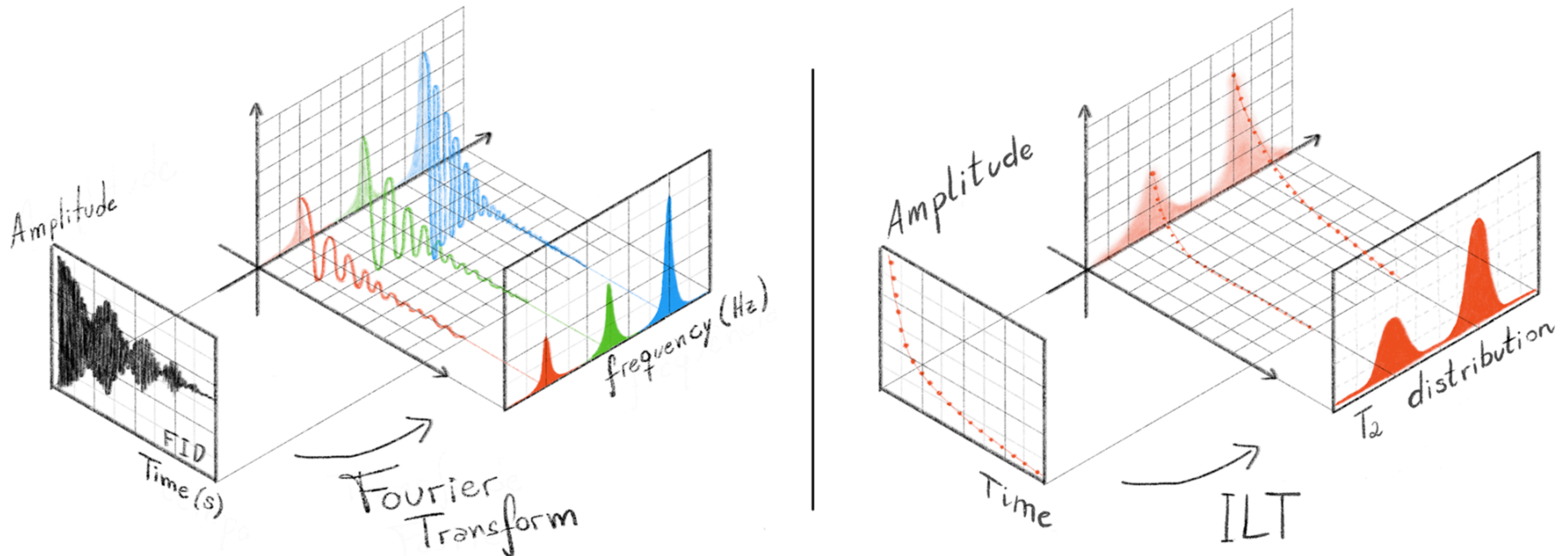
Vang, Shi, Zhou,

arXiv:2004.02426

ott, Jay, Oare,

The problem is not only for us: condensed matter, geophysics, ..., NMR!

Moraes, T. B., Von Atzingen, G. V., Mazzero, L. P., Mendes, W. S., Zacharias, M. B., and Cardinali, M. C. B.: An open-access WebApp for inverse Laplace transform analysis of time-domain nuclear magnetic resonance signals, Magn. Reson., 7, 39–51, <https://doi.org/10.5194/mr-7-39-2026>, 2026.



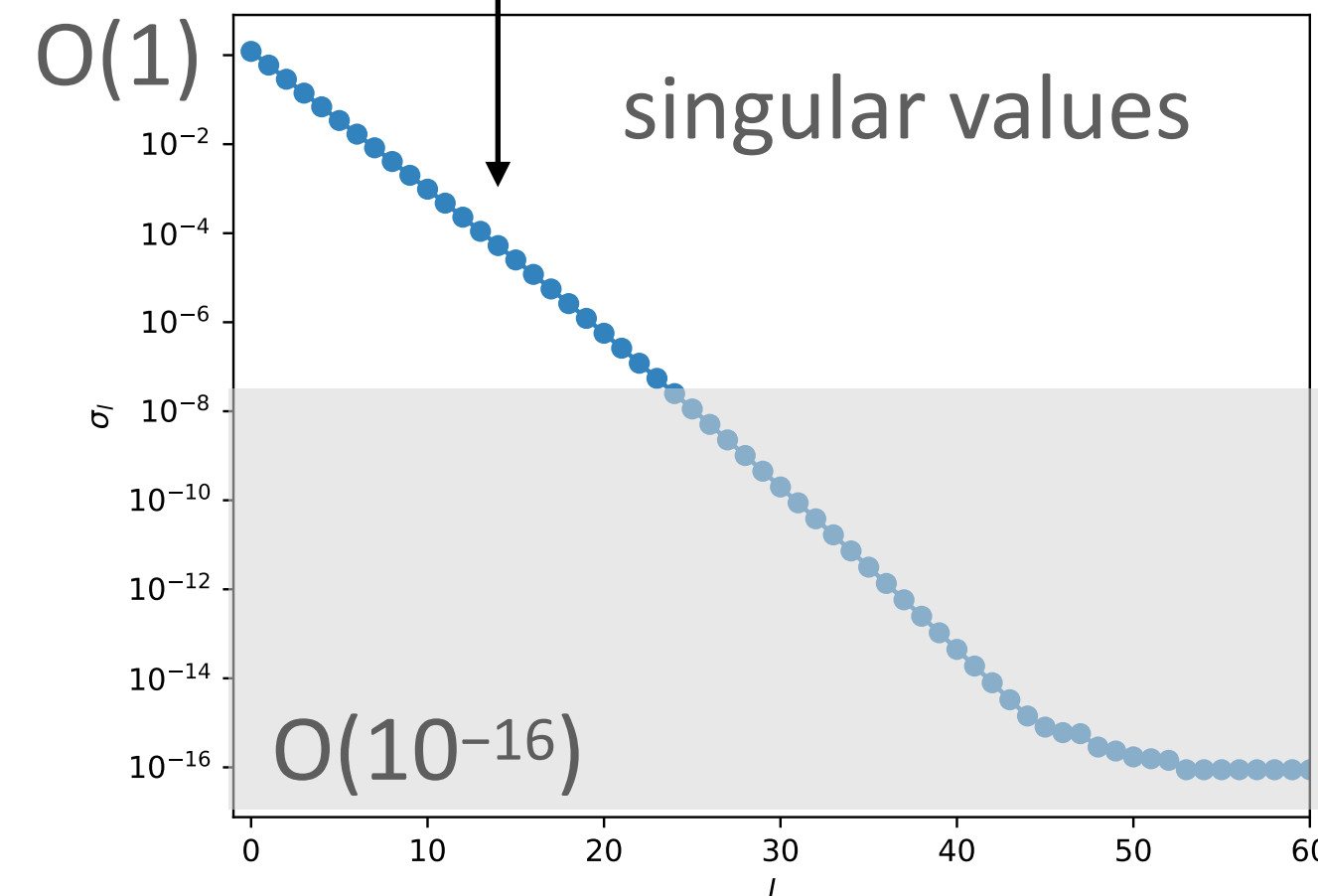
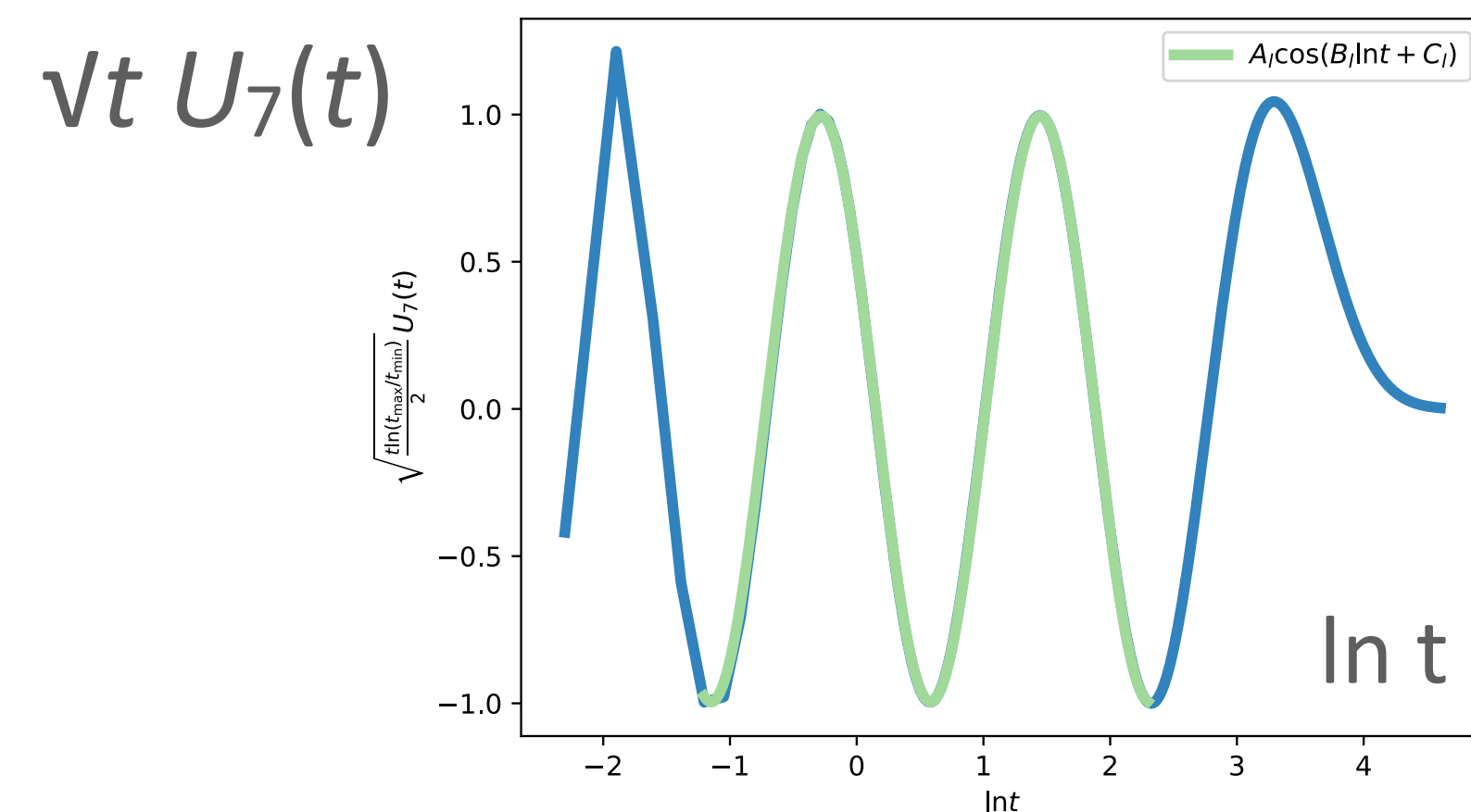
Why so difficult? — a simple argument

Tsuji, SH, arXiv:2605.15674; Shi, Wang, Zhou, CPC 282 (2023) 108547; see also Tsuji's talk

Singular Value Decomposition (SVD)

$$e^{-\omega t} = \sum_l U_l(t) \sigma_l V_l(\omega) \quad \text{for discrete sets of } t_i \text{ and } \omega_j$$

orthonormal basis



Modes of large l 's are too small to reconstruct.

Fine structure of
 $\ln \omega \sim 1/l$
is already lost in $e^{-\omega t}$

Reconstruction in the SVD basis

Tsuji, SH, arXiv:2605.15674; see also Lupo, Tantalò, arXiv:2605.14652; Rothkoph, arXiv:2002.09865 (SVD basis)

$$\int d\omega \rho(\omega) K(\omega) = \sum_l \rho_l K_l$$

kernel function projected in the SVD basis

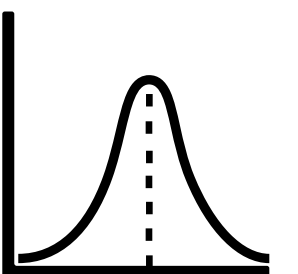
$$\rho_l = \frac{1}{\sigma_l} \sum_{t_i} U_l(t_i) C(t_i)$$

enhanced for large l
together with its noise

lattice correlator projected
in the SVD basis

$$K_l = \sum_{\omega_j} K(\omega_j) V_l(\omega_j)$$

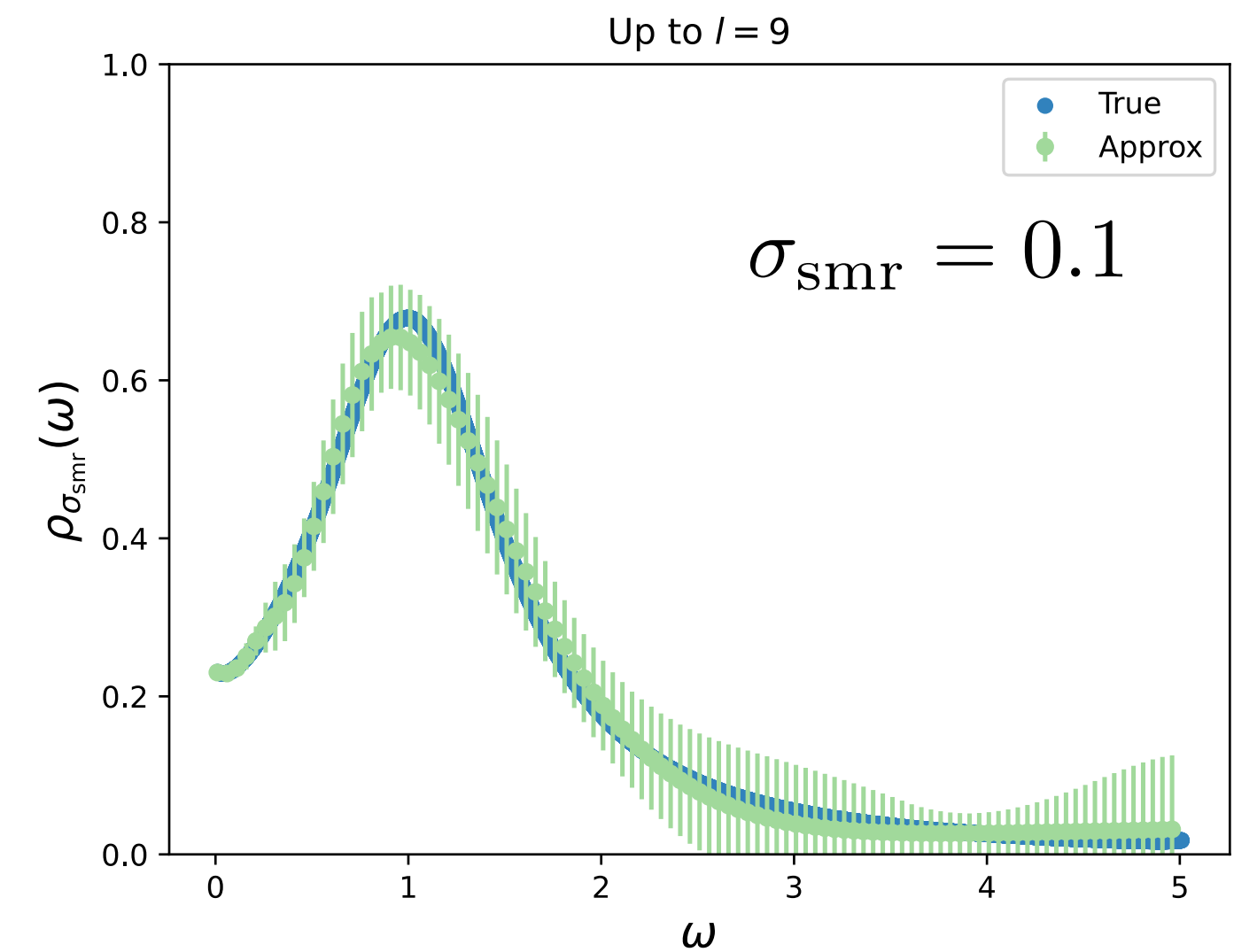
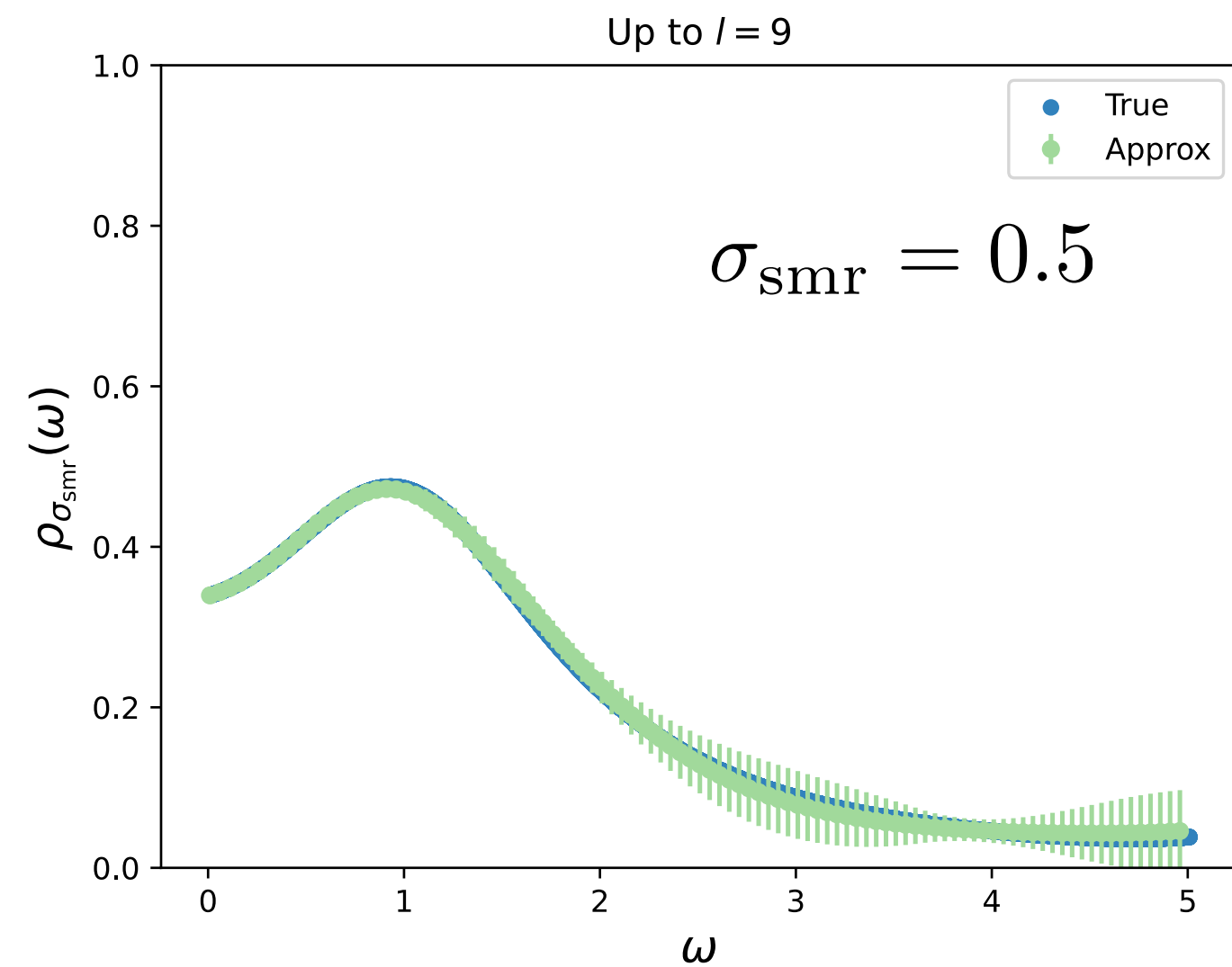
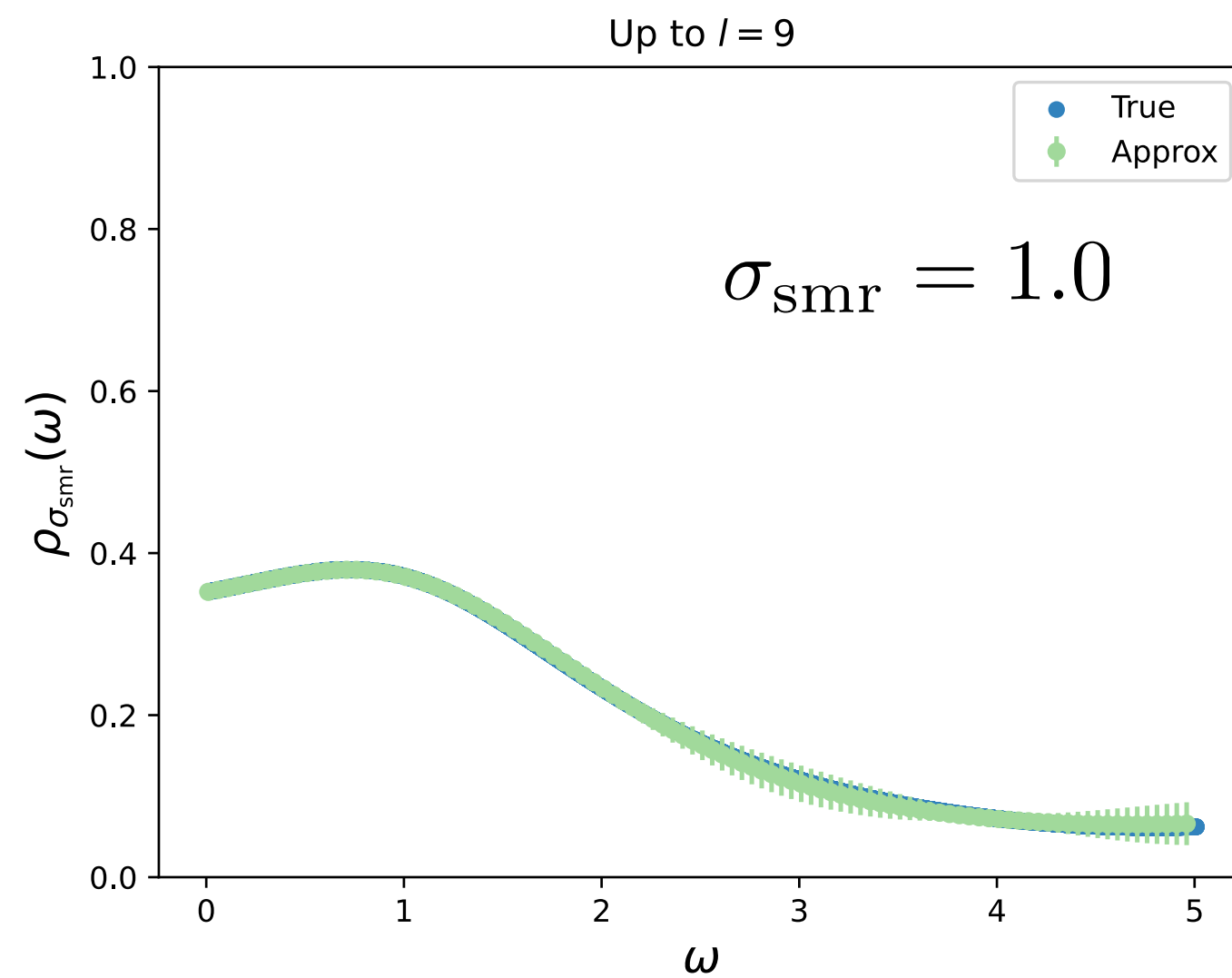
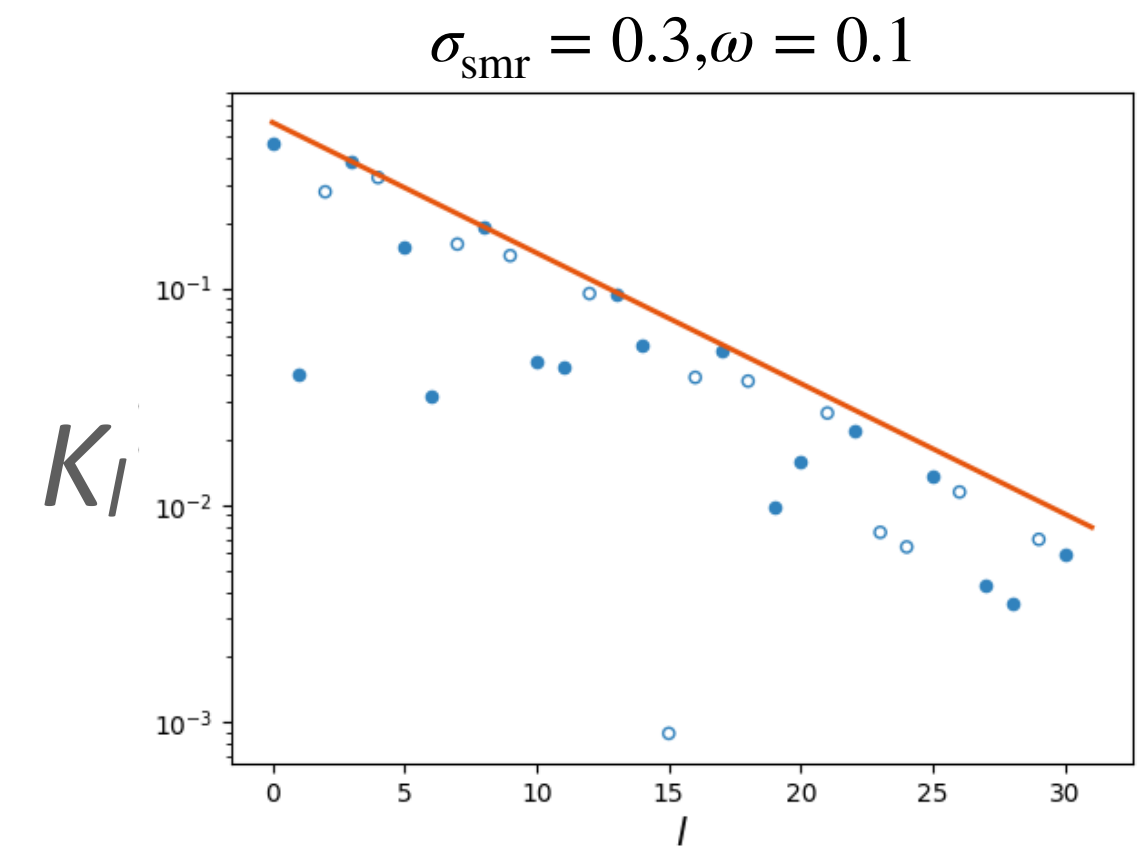
suppressed for large l , as $e^{-\sigma_{\text{smr}} l}$,
depending on the smearing width σ_{smr}



Need sufficiently large σ_{smr} for convergent expansion

Breit-Wigner reconstructed with $l < 10$

$$K(\omega) = \frac{2\sigma_{\text{smr}}}{(\omega - \omega_0)^2 + \sigma_{\text{smr}}^2}$$



Truncation errors are estimated assuming the exponential scaling of ρ_l and K_l

The smaller width, the harder to reconstruct.
= the common feature of all the methods.

Mellin transform

Bruno, Giusti, Saccardi, PRD 111 (2025) 9, 094515, arXiv:2407.04141; Giusti, Saccardi, Toniolo, arXiv:2606.28167

See also, talks by Saccardi and Toniolo

Limit of $\omega, t \in [0, \infty]$

$$e^{-\omega t} = \int_0^\infty ds |\lambda_s| \left(u_s^+(\omega) u_s^+(t) - u_s^-(\omega) u_s^-(t) \right)$$

“singular values”

basis functions (Fourier-like)

$$\sqrt{\frac{\pi}{\cosh(\pi s)}}$$

$$u_s^\pm(x) = \frac{\frac{\cos}{\sin}(s \ln(x) - \theta(s)/2)}{\sqrt{\pi x}}$$

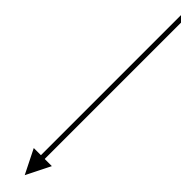
Exponentially suppressed for higher frequency modes

Provides a natural basis to represent smearing functions

- Formulae for limited and discretized range of t are also worked out.

Practical limitation

$$\delta[e^{\text{mt}} C(t)] \sim \Delta$$



Singular values (= correlator precision Δ) control the frequency resolution $\Delta\omega/\omega$

“singular values”

$$\sqrt{\frac{\pi}{\cosh(\pi s)}}$$

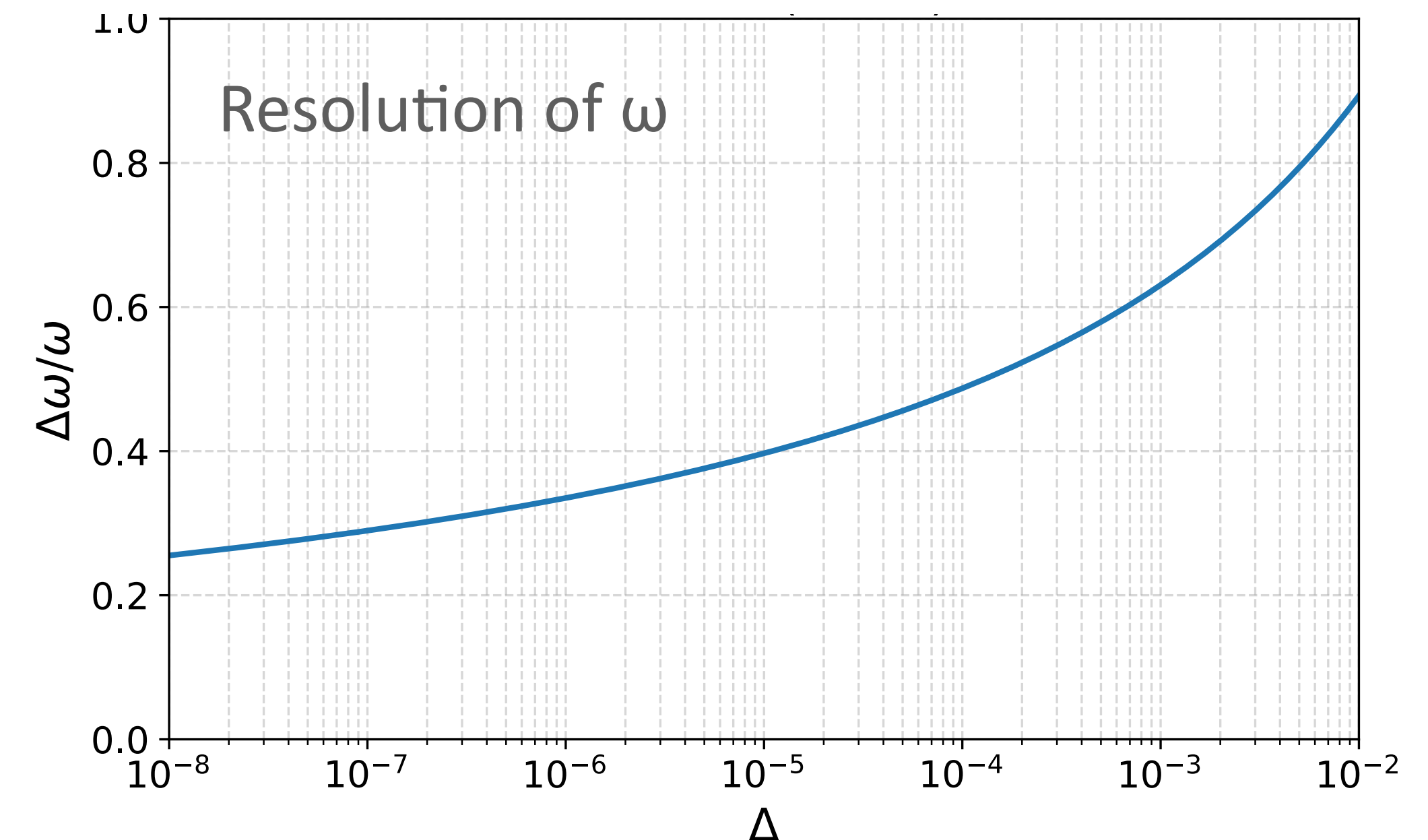
basis functions (Fourier-like)

$$u_s^\pm(\omega) = \frac{\cos \left(s \ln(\omega) - \theta(s)/2 \right)}{\sqrt{\pi \omega}}$$

The resolution of ω improves only logarithmically for more precise Δ .

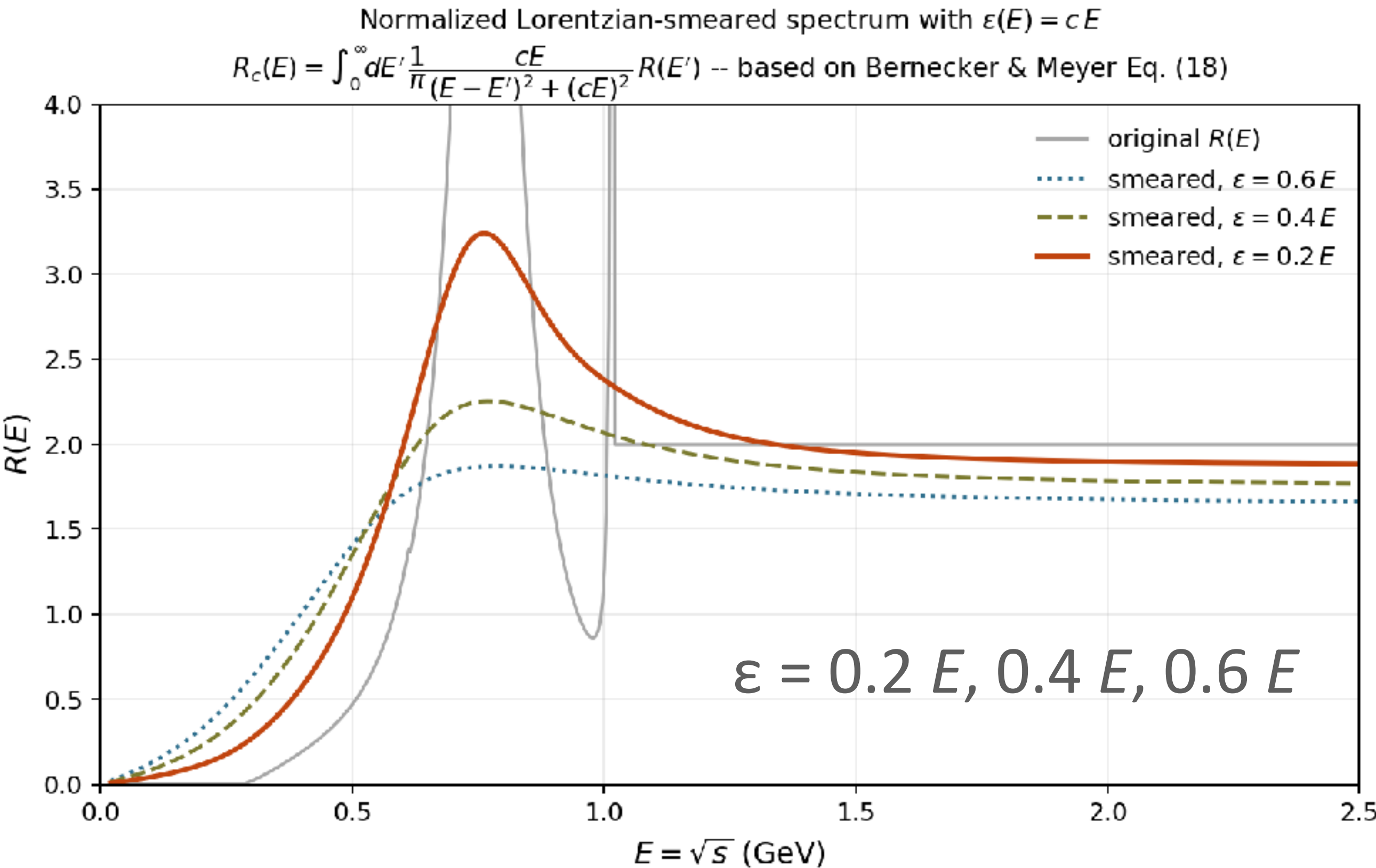
$$\frac{\Delta\omega}{\omega} \sim \frac{\pi^2}{2 \ln(\sqrt{2\pi}/\Delta)}$$

> 20-40% with currently available technology



Smeared R ratio with
(expected)

$$K(E, E') = \frac{1}{\pi} \frac{\epsilon}{(E - E')^2 + \epsilon^2}$$



sample spectrum from Bernecker-Meyer

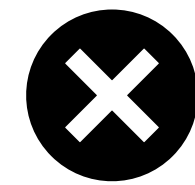
Many lenses, same aperture

linear reconstruction problem $K(\hat{H}) = g_0 + g_1 e^{-\hat{H}} + g_2 e^{-2\hat{H}} + g_3 e^{-3\hat{H}} + \dots$

Choices from

Basis (in terms of $z=e^{-\omega}$) :

- Simple powers z^n
- Chebyshev polynomials
- SVD or Mellin basis
- ...



Criterion (to determine g_n):

- Straight inversion (in a truncated space)
- Approximation theory
- Minimize the loss (with statistical noise)
- MEM, gaussian process, ML, ...

But, changing the lenses does not create information.

Note: Nevanlinna-Pick interpolation is non-linear; different discussion may apply.
Bergamaschi, Jay, Oare, PRD 108, 074516 (2023); Fields, Christ, PRD 113, 054502 (2026).
See also, Fields' talk

The focus should be on the method to estimate the truncation errors.

Approximation versus Bound

Lawrence, arXiv:2408.11766; Abbot, Jay, Oare, arXiv:2508.01377; Abbot, Fields, Jay, Oare, Saccardi, arXiv:2605.20509; Mutzel, Tilloy, arXiv:2606.09791; see also, talks by Abbot and Jay

Approximation:

$$\int_0^\infty dE \rho(E) K(E) \simeq \sum_t g_t C(t) \quad \leftarrow \quad K(E) \simeq \sum_t g_t e^{-Et} \quad \leftarrow \quad \int_0^\infty dE \rho(E) e^{-Et}$$

→ What is the truncation error?

Various strategies to estimate for HLT, Chebyshev, etc.

Bound:

Assuming $\rho(\omega) \geq 0$, maximize $\sum_t g_t C(t)$ while satisfying

$$K(E) \geq \sum_t g_t e^{-Et} \quad \text{for } \forall E$$

= convex optimization problem
(like the conformal bootstrap)
(extended also to the case with noise)

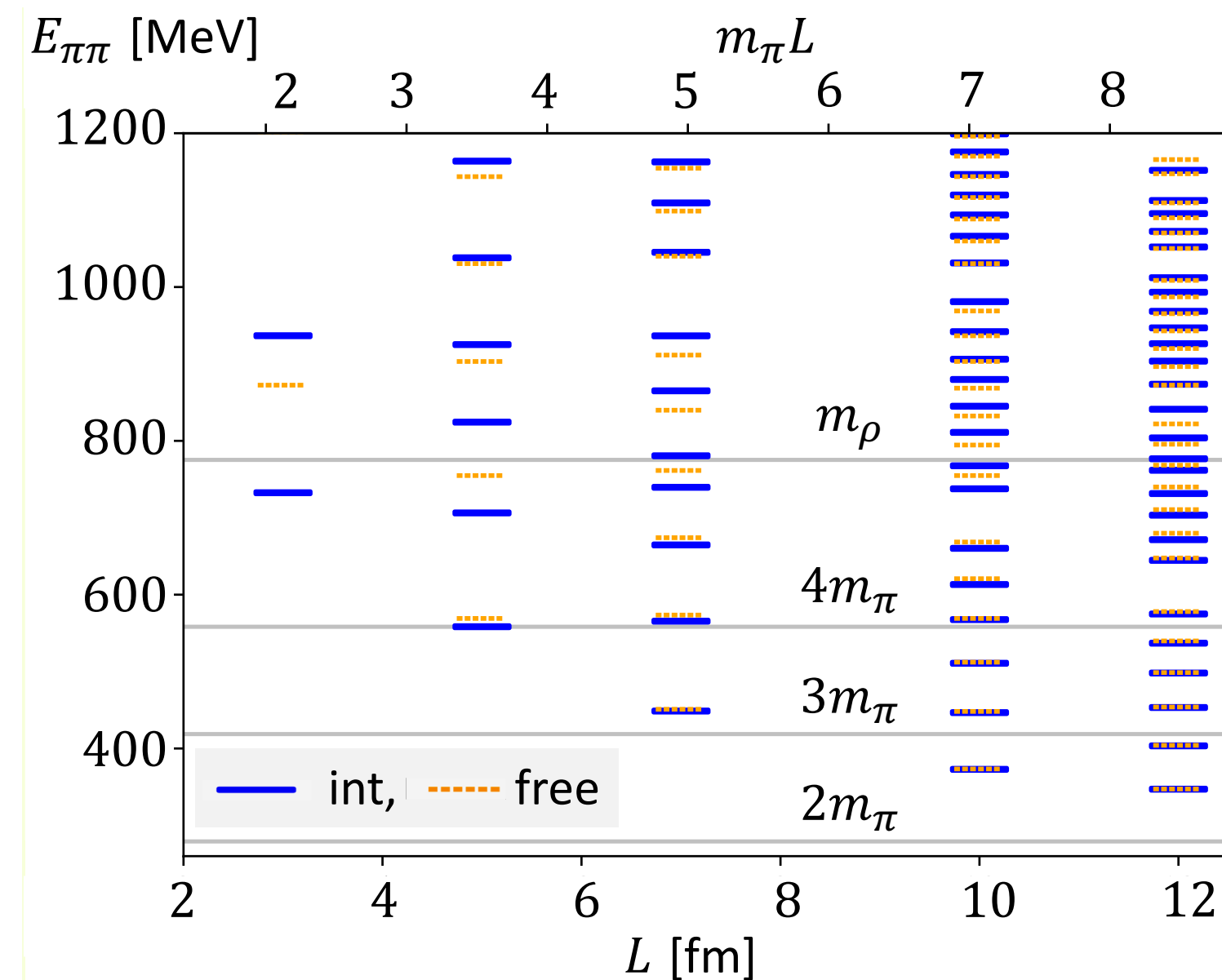
→ the lower bound (similarly the upper bound)

Potentially tighter than the error estimate. Explicit benchmark test would be desired.

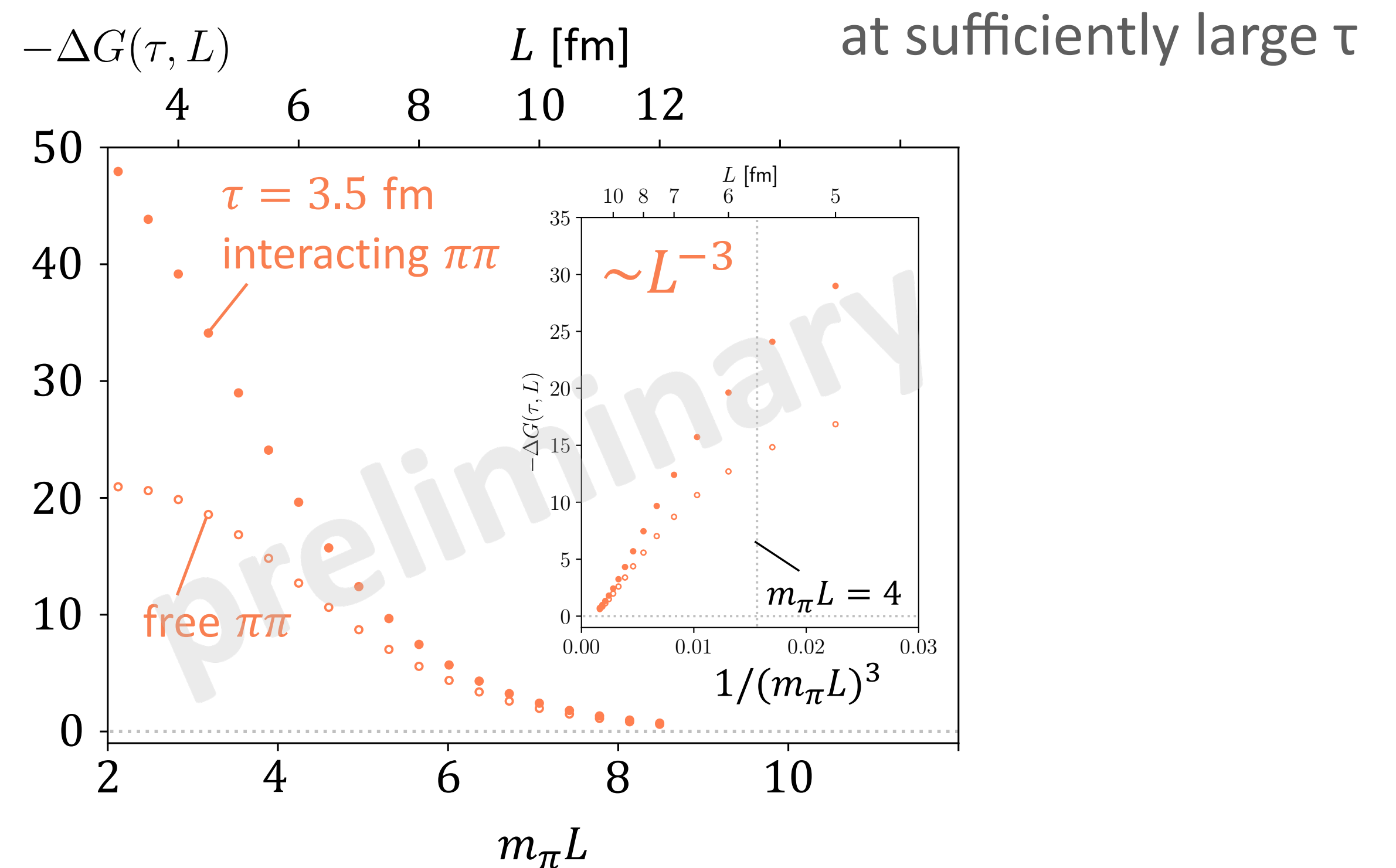
Finite volume effects

(Smeared) spectrum often represents two particle states: worry about FVE?

individual energy levels $\sim 1/L^2$



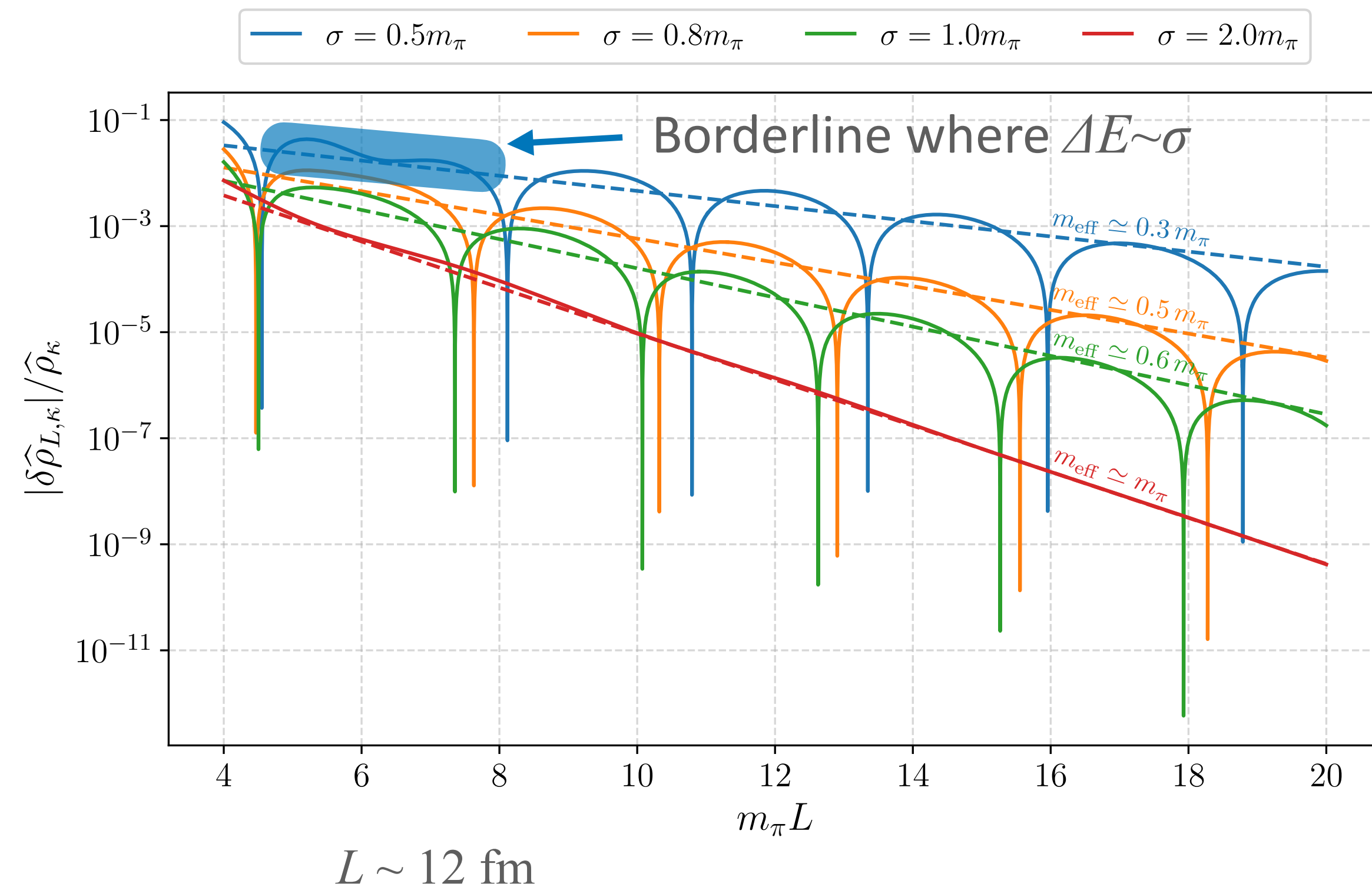
vector-current correlator $\sim 1/L^3$



Finite volume effects

Bresciani, Bruno, Hansen, arXiv:2606.14349; = Analysis as in a_μ^{HVP} by Hansen, Patella
 see also, Bresciani's talk

Exponentially suppressed for sufficiently large σ = covering more than 1 states



$$\sim \exp(-m_{\text{eff}} L)$$

with

$$m_{\text{eff}} = \frac{1}{2} \text{Im} \sqrt{(\omega^* + i\sigma)^2 - 4m^2}$$

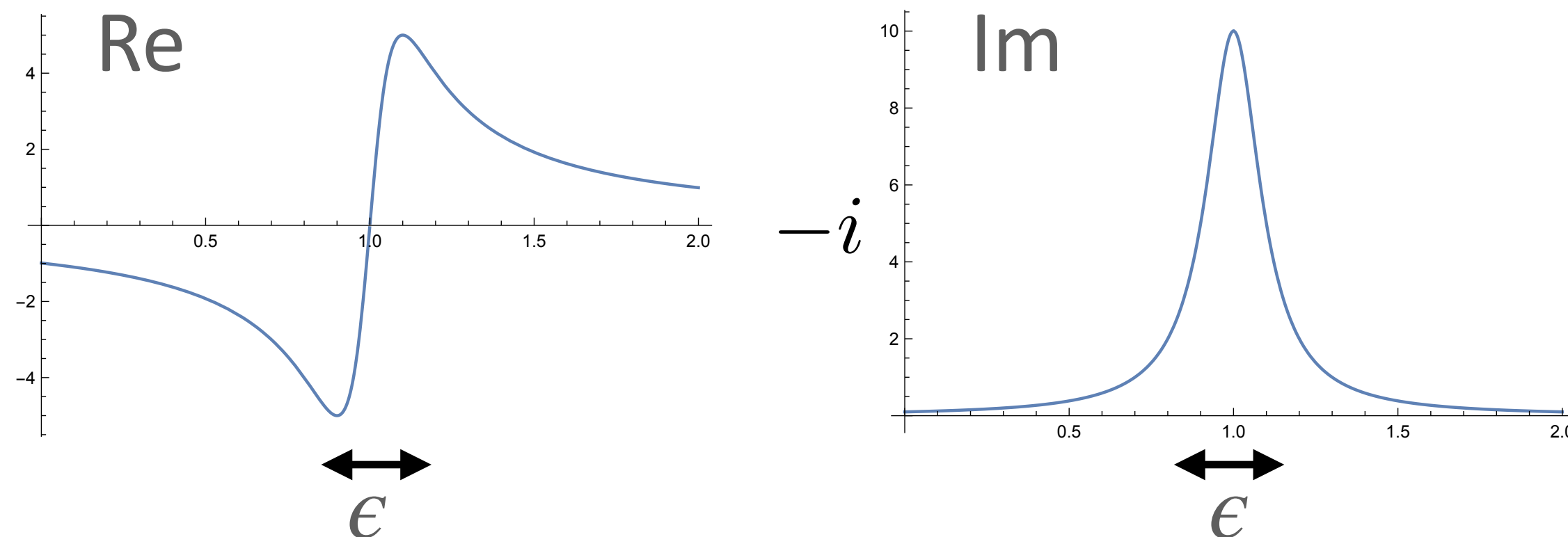
Concluding remarks

Resolution matters

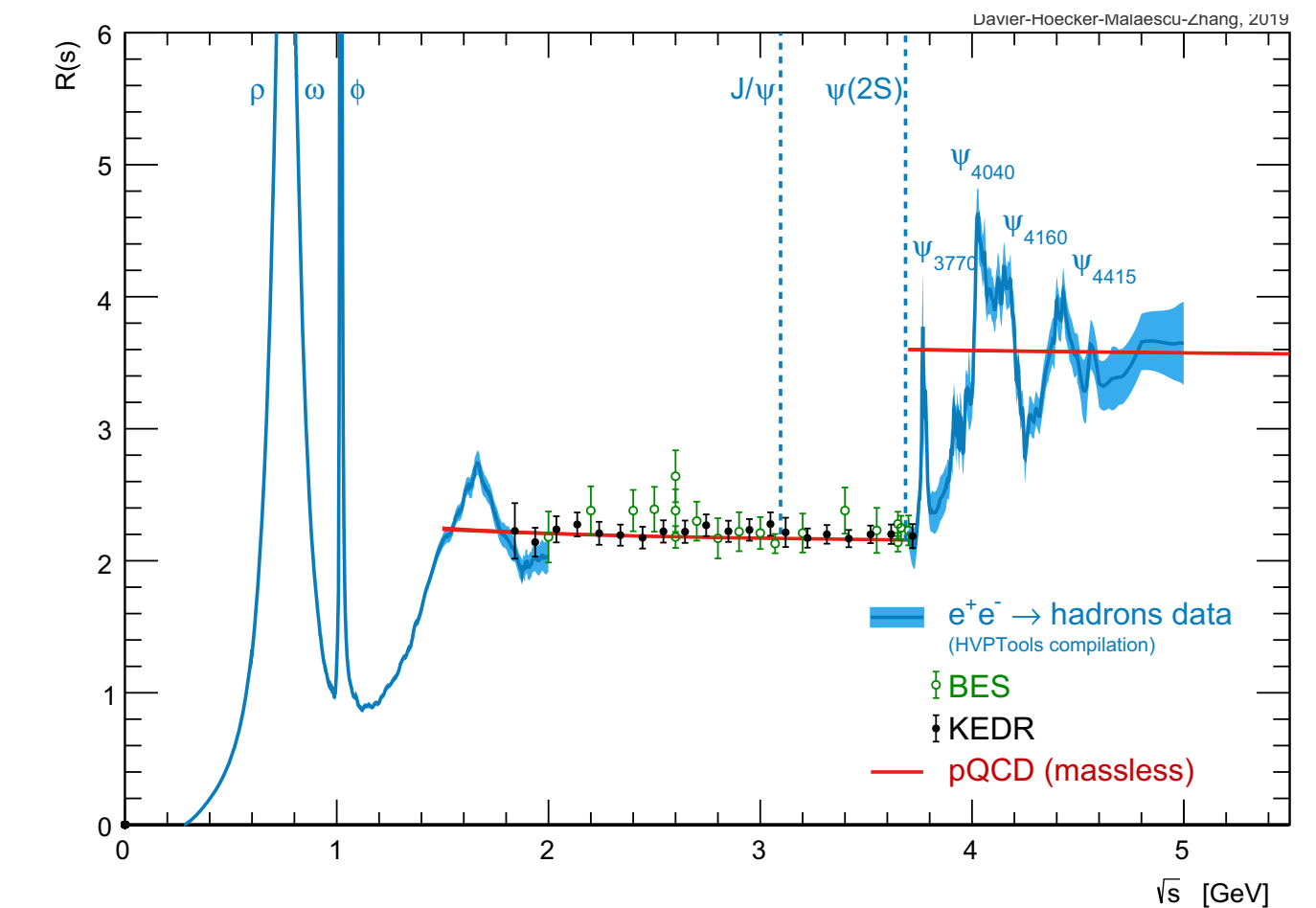
With the currently available correlator precision, a realistic resolution appears to be $\varepsilon > (0.2\text{--}0.4)E$. The limit $\varepsilon \rightarrow 0$ involves some assumptions.

= Challenges for the computation of amplitudes

$$A(E) = \lim_{\varepsilon \rightarrow 0} \int \frac{dE'}{2\pi} \frac{\rho(E')}{E' - E - i\varepsilon}$$



ex) R-ratio



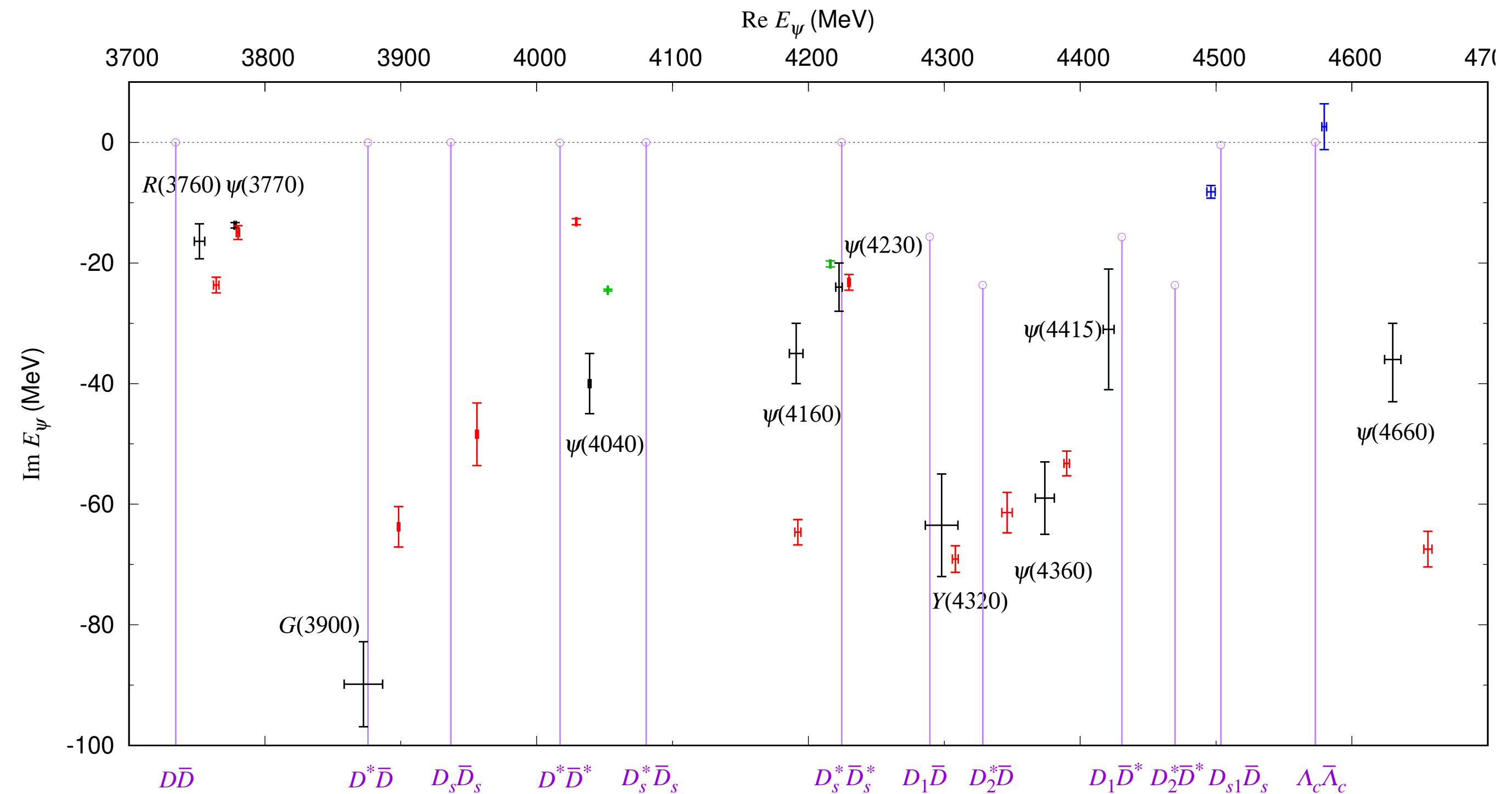
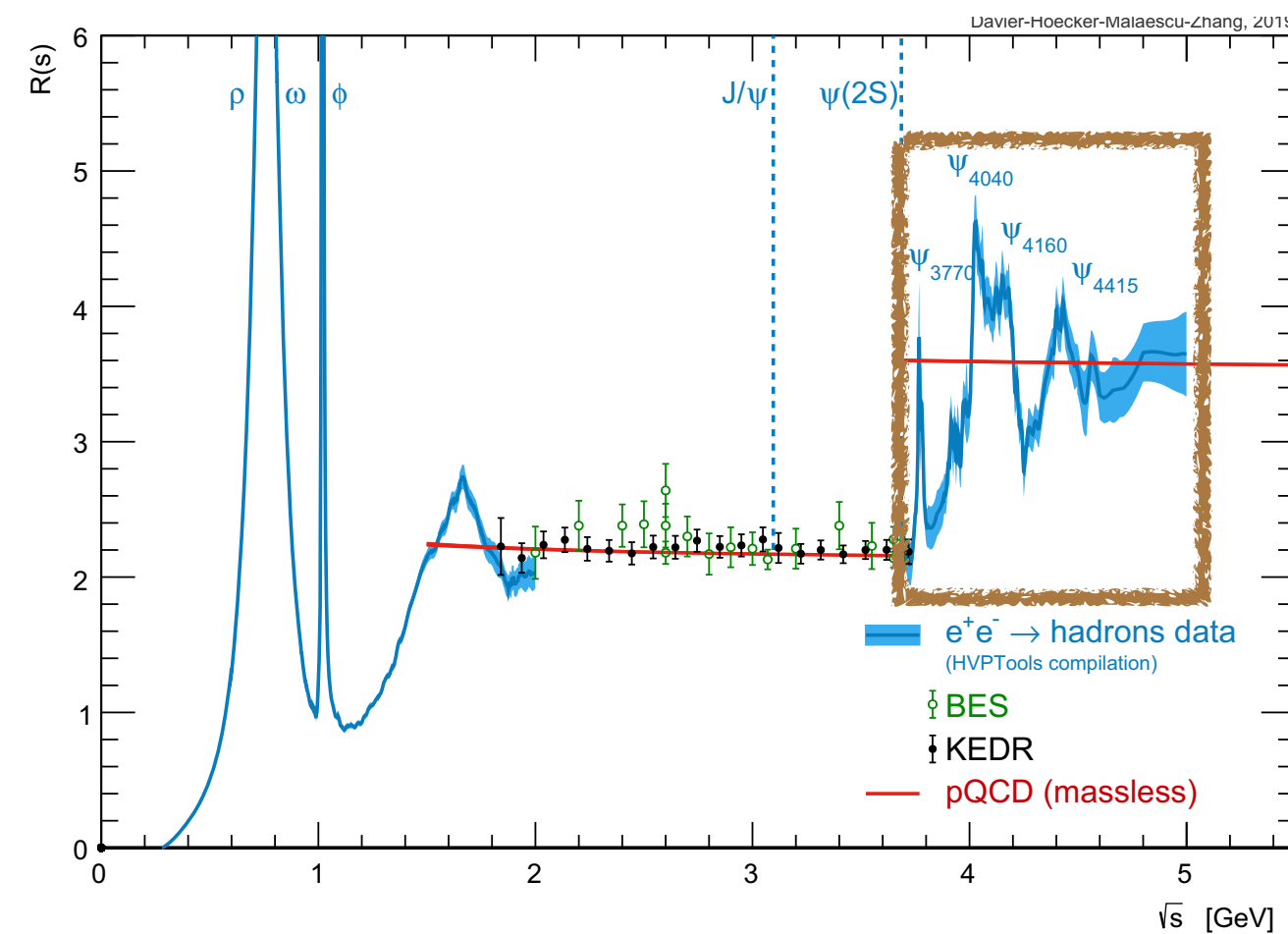
Complicated structure is seen experimentally. Need some understanding of multi-body states.

Physics can be wild

depending on the kinematical setup

Nakamura et al., arXiv:2312.17658

Coupled-channel analysis of charmonium exp data.



Rich structure of $\rho(\omega)$ over 100 MeV to be expected.
= Need $\varepsilon < 100$ MeV, or less (process dependent).

Final words

- Opportunities:
 - Smeared spectrum opens a broad area of new applications of LQCD. Access to hadronic amplitudes, in particular.
- Challenges:
 - No universal solution to the ill-posed problem, other than introducing smearing.
 - No universal solution to the S/N problem, then resolution is limited.
 - Hadronic amplitudes may need rather small ϵ ; care about underlying physics

*Don't ask if the spectral functions can be fully reconstructed;
Ask if phenomenologically useful observables can be constructed.*

e.g. Jüttner, arXiv:2603.15487, for rare $D \rightarrow \pi l^+ l^-$.