Lattice2026 @ University of Maryland, College Park

Dealing with left-hand cuts in multi-hadron spectroscopy

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Aug. 1st, 2026

Apologize for the remote presentation.

For further discussion, please contact me at:

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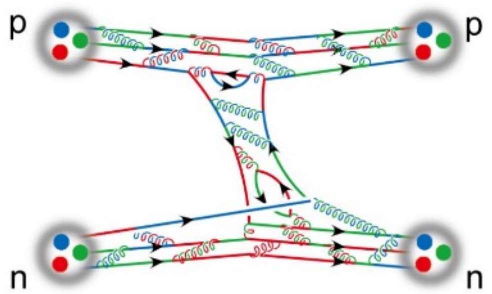
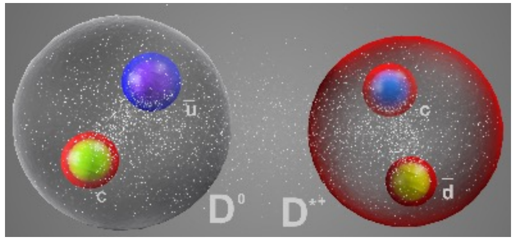
Left-hand cut problems

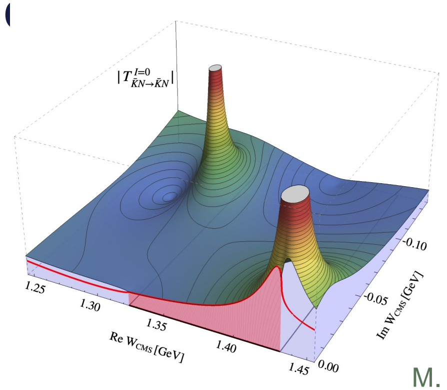
- To get the hadron-hadron interactions and resonances from the first prin



Nuclear force



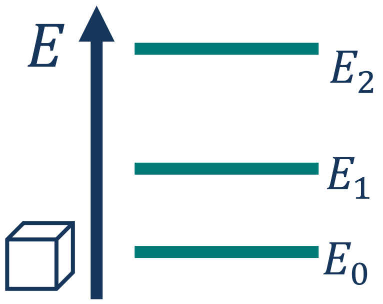
Hadronic molecule



M. Mai, '21

Resonance

- From finite volume energy levels to pole positions



Lüscher quantization condition (LQC)

Lüscher '90

Amplitude formula + analytic continuation

Poles

Finite volume lhc problems

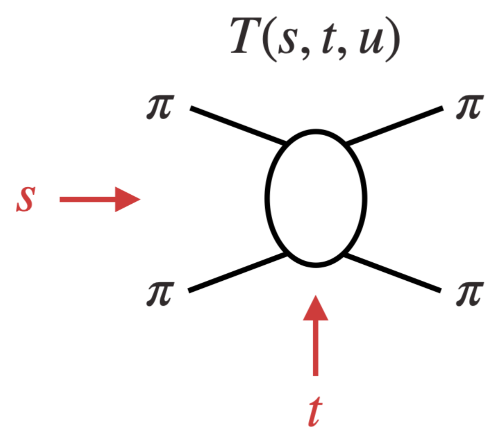
Infinite volume lhc problems

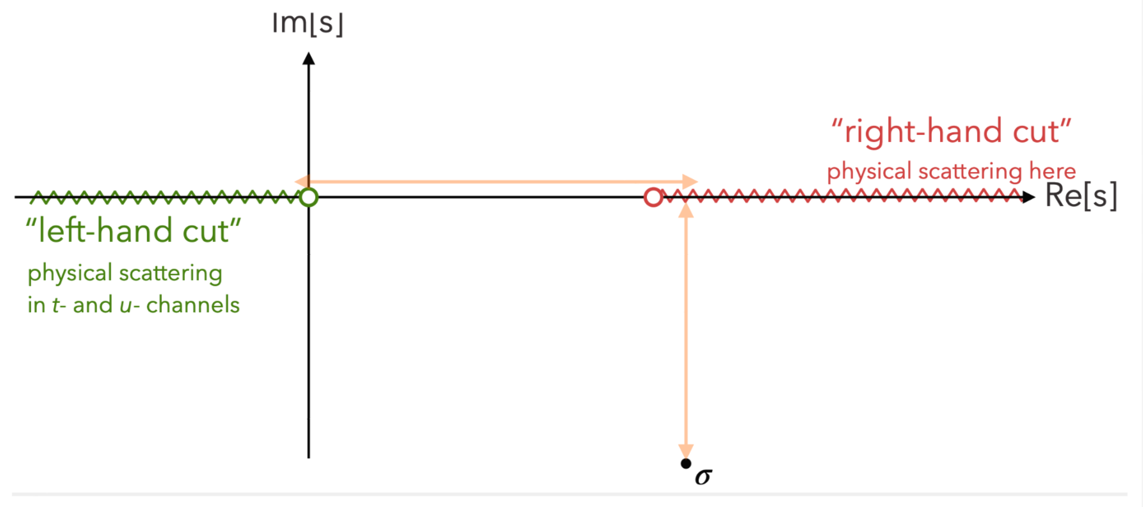
Left-hand cut (lhc) problems

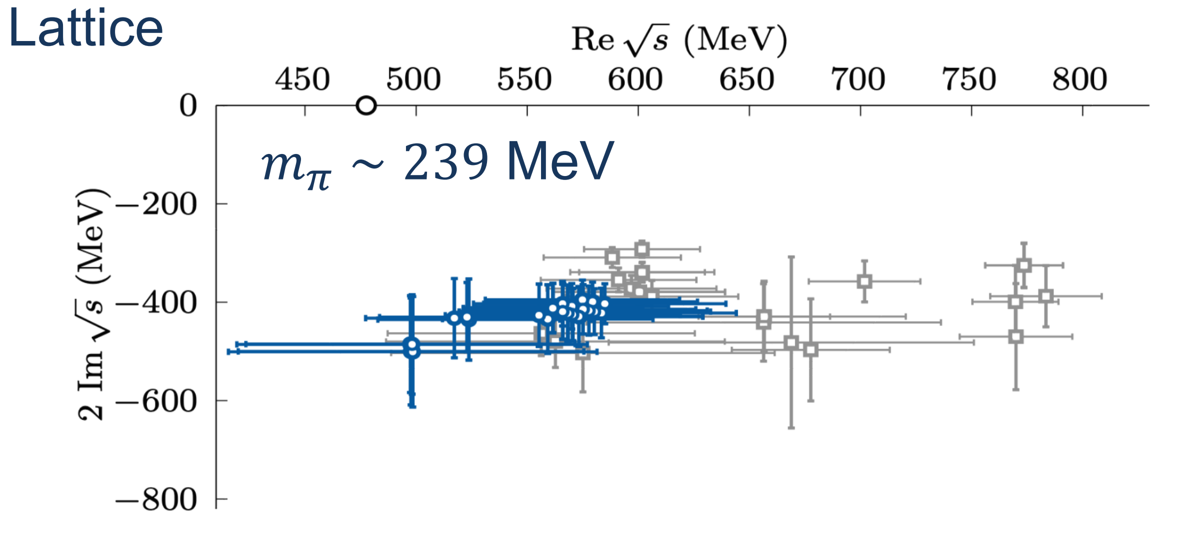
Left-hand cuts in $\pi\pi$ scattering

IFV lhc problem

- Large uncertainties of the $f_0(500)$ pole position
  - Both in Exp. and Lattice
  - One of the sources: **left-hand cuts**
  - Broad resonance: comparable distances to real axis and the lhc
  - Solutions: dispersion relations with **crossing symmetry**







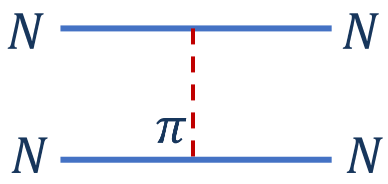
Blue: uncertainties reduced by dispersion relation

A. Rodas, J. J. Dudek, and R. G. Edwards, PRD109, 034513 (2024).

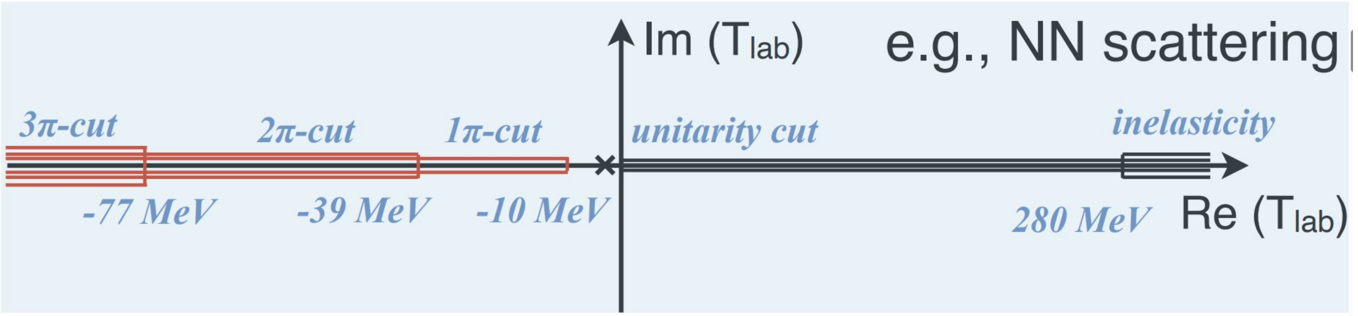
Similar analyses of Exp. data

L. Caprini, G. Colangelo, and H. Leutwyler, PRL96, 132001 (2006)
R. Garcia-Martin, R. Kaminski, et al., PRL107, 072001 (2011)
B. Moussallam, EPJC71, 1814 (2011)

Left-hand cuts in NN scattering



Left-hand cut
on-shell pion



- lhc from the **one-pion exchange**

$$V(r)=\frac{e^{-mr}}{r},\quad V(\vec{p},\vec{p}')=\frac{1}{(\vec{p}'-\vec{p})^2+m^2}$$

Long-range interaction & Chiral dynamics

- Partial-wave decomposition, e.g. the S-wave

$$V_{l=0}(p,p')=-\frac{1}{2pp'}\log\left(\frac{(p-p')^2+m^2}{(p+p')^2+m^2}\right)$$

Multivalued function with cuts

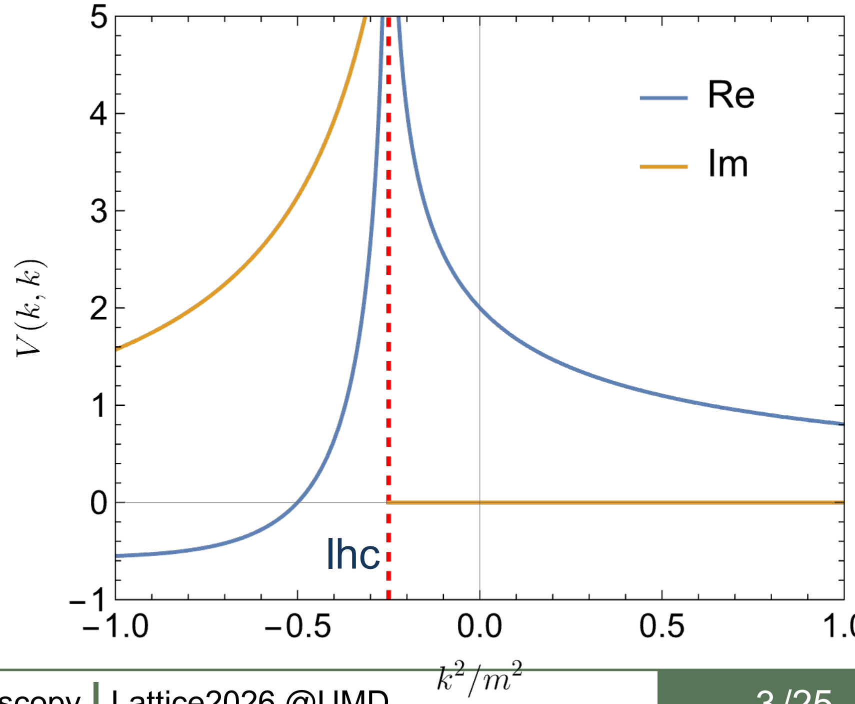
- On-shell $p=p'=k$, with $k^2=2\mu E$

$$V_{l=0}(k,k)=\int_{-1}^{1}dz\frac{1}{2k^2(1-z)+m^2},\qquad 2k^2(1-z)+m^2=0\Rightarrow z=\frac{m^2}{2k^2}+1,$$

on-shell pion

▶ Branch point: $k^2<-\frac{m^2}{4}$

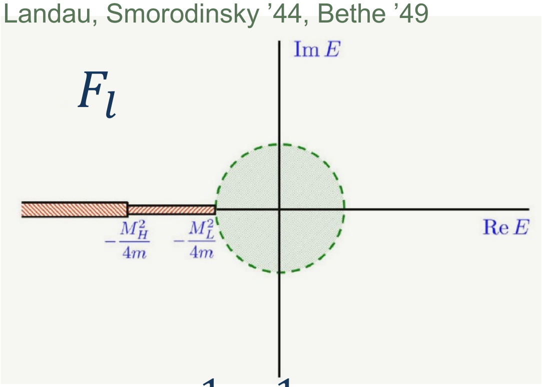
With imaginary part



Amplitude formalisms with lhcs

Two potential formalism $V(r) = V_L(r) + V_S(r)$

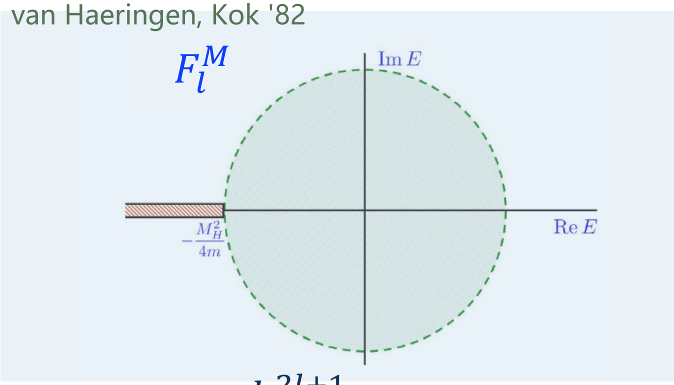
Landau, Smorodinsky '44, Bethe '49



$$F_l = -\frac{1}{a} + \frac{1}{2} r k^2 + \cdots$$

Effective range expansion (ERE)

van Haeringen, Kok '82



$$F_l^M \equiv M_l(q) + \frac{k^{2l+1}}{|f_l(k)|^2}[\cot(\delta_l - \sigma_l) - i]$$

Modified Effective range expansion (MERE)

Amplitude formalisms with lhcs

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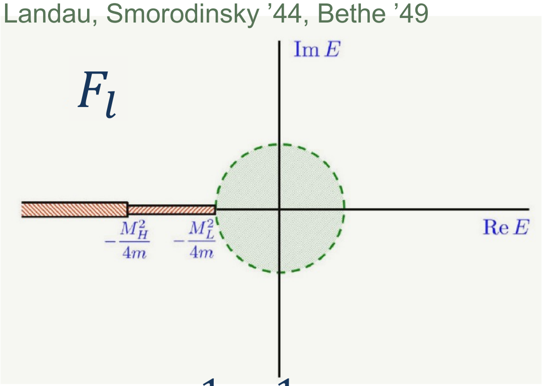
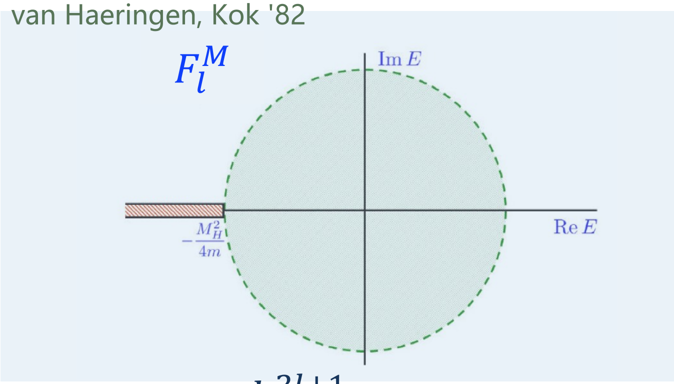


van Haeringen, Kok '82



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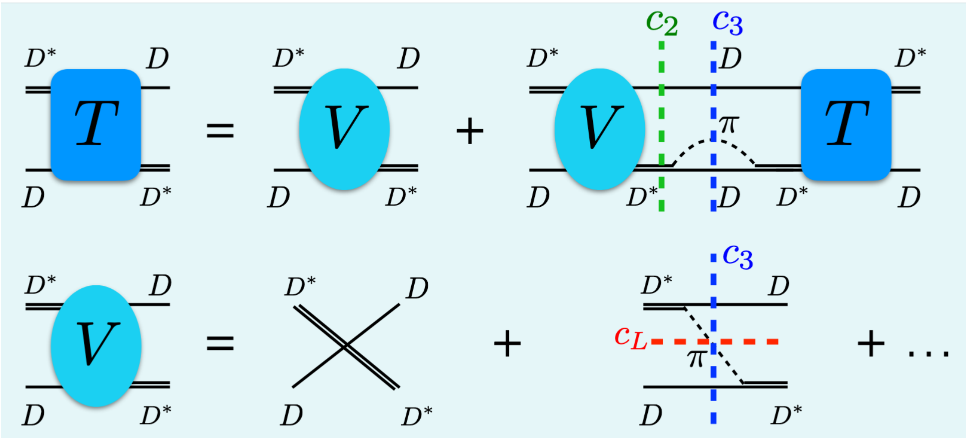
Effective range expansion (ERE)

Modified Effective range expansion (MERE)

Lippmann-Schwinger equation



E.g. M.-L. Du, et al., PRL**131**, 131903 (2023)

Amplitude formalisms with lhcs

Two potential formalism $V(r) = V_L(r) + V_S(r)$

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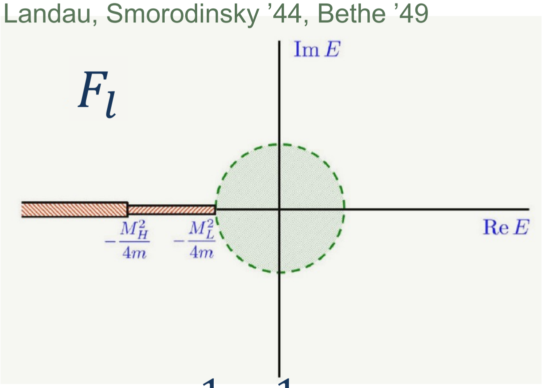
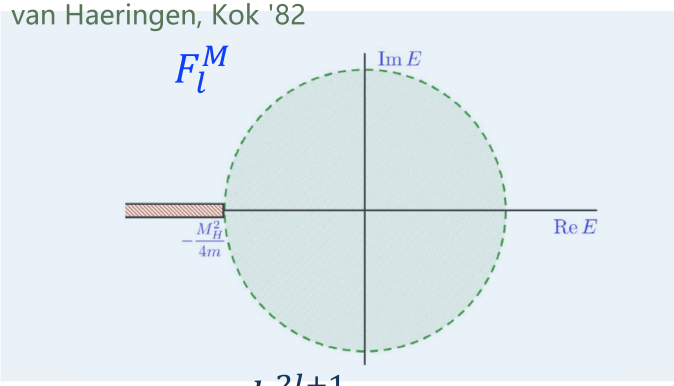


van Haeringen, Kok '82



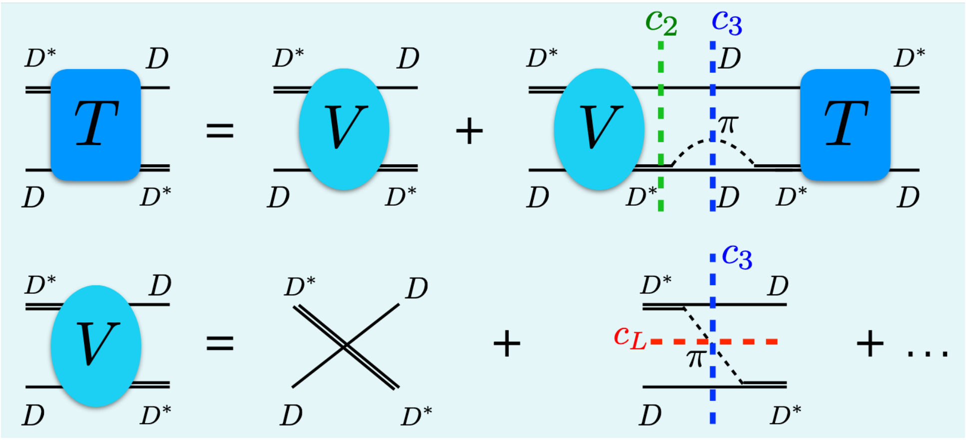
$$F_l = -\frac{1}{a} + \frac{1}{2} r k^2 + \cdots$$

Effective range expansion (ERE)

$$F_l^M \equiv M_l(q) + \frac{k^{2l+1}}{|f_l(k)|^2}\left[\cot(\delta_l - \sigma_l) - i\right]$$

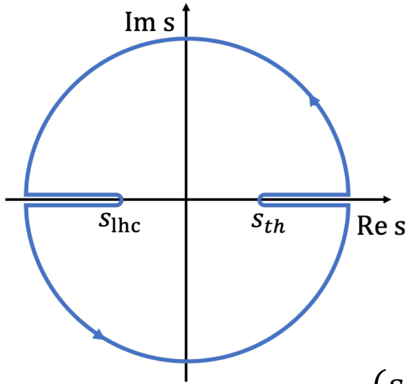
Modified Effective range expansion (MERE)

Lippmann-Schwinger equation



E.g. M.-L. Du, et al., PRL**131**, 131903 (2023)

N/D representation



$$\boxed{T_l' = \frac{N_l(s)}{D_l(s)}}$$

Contains left-hand cut

Contains right-hand cut

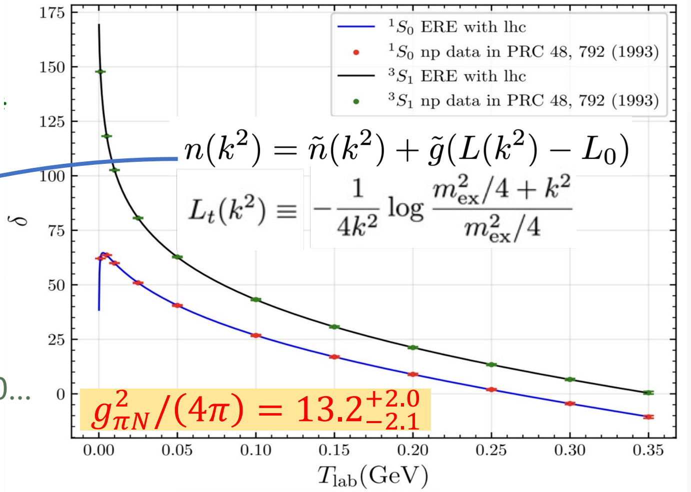
$$\mathrm{Im}\, D_l = -\rho(s)\, N_l p^{2l}, \quad s > s_{\rm th}$$

$$\mathrm{Im}\, N_l = \mathrm{Im}\, T_l' D_l, \qquad s < s_{\rm lhc}$$

$$D_l(s) = P_D(s) - \frac{(s-s_0)^n}{\pi}\int_{s_{\rm th}}^{\infty} d\,s' \frac{p(s')^{2l}\rho(s')N_l(s')}{(s'-s)(s'-s_0)^n}$$

$$N_l(s) = P_N(s) + \frac{(s-s_0)^{n-l}}{\pi}\int_{-\infty}^{s_{\rm lhc}} d\,s' \frac{\mathrm{Im}\, T_l(s')D_l'(s')}{p(s')^{2l}(s'-s_0)^{n-l}(s'-s)}$$

Chew & Mandelstam, '60; Bjorken, '60...

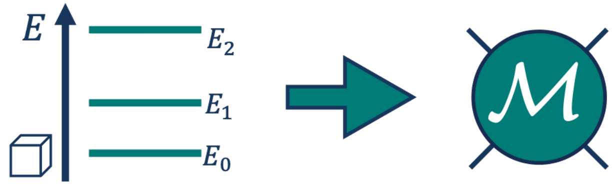


M.-L. Du, F.-K. Guo, and B. Wu, PRL**135**, 011903 (2025)
B.-Y. Liu, B. Wu, J.-W. Fu et al, arXiv:2602.03115.

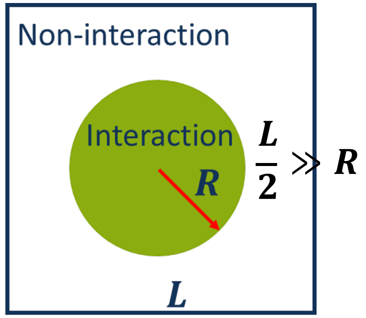
lhc problem IN Lüscher formula

- Lüscher's formula: Lüscher '90

$$\boxed{\det\left[\underbrace{G_F^{-1}(L,E^{FV})}_{\text{Kinematical term}} - \underbrace{K(E^{FV})}_{\text{K-matrix in IFV}}\right] = 0}$$



- Valid for the short-range (compared to the box) interaction



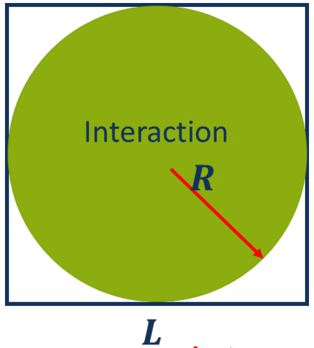
Asymptotic behavior: for $r > R$

$$(\nabla^2 + k^2)\psi_k(r) = 0,$$

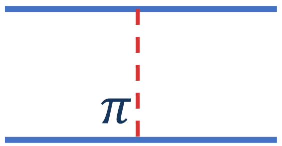
$$\psi_k(r) \sim \frac{e^{i\delta(k)}\sin[kr+\delta(k)]}{kr}.$$

Periodic Boundary condition:

$$\boldsymbol{p} = \frac{2\pi}{L}\boldsymbol{n}, \qquad \boldsymbol{n} \in Z^3$$



Long-range interaction vs small box ???



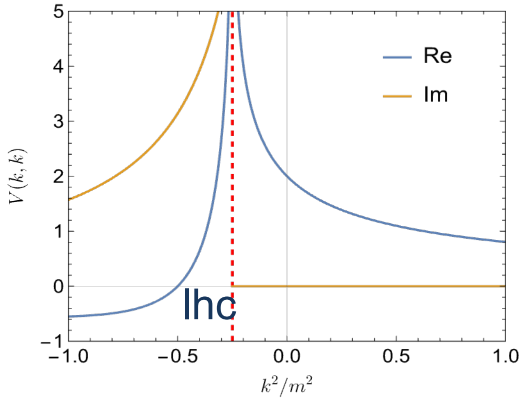
NN , D^*D systems…

▶ **Exponentially suppressed effect**: $\sim e^{-m_\pi L}$

- lhc problem: $k^2 < -\frac{m^2}{4}$

$$\det\left[\underbrace{G_F^{-1}(L,E)}_{\text{Real}} - \underbrace{K(E)}_{\text{With nonvanishing imaginary part}}\right] \cancel{=} 0$$

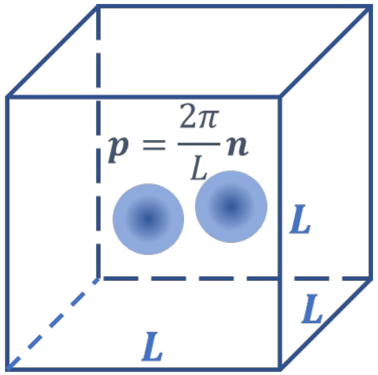
$$K = V + VG^{\mathcal{P}}K$$



Origin of the lhc problem

- From infinite volume (IFV) to finite volume (FV)

$$\int \frac{d^3q}{(2\pi)^3} \to \frac{1}{L^3} \sum_{\vec{q}=2\pi\vec{n}/L}$$



- Poisson summation formula

$$\left(\frac{1}{L^3} \sum_{\vec{q}=2\pi\vec{n}/L} - \int \frac{d^3q}{(2\pi)^3} \right) f(\vec{q}) = \sum_{\vec{n}\neq 0, \vec{n}\in\mathbb{Z}^3} \int \frac{d^3q}{(2\pi)^3} f(\vec{q}) e^{iL\vec{q}\cdot\vec{n}}$$

► E.g.

Smooth: $f(\vec{q}) = \frac{1}{\vec{q}^2 + a^2} \to \frac{e^{-anL}}{4\pi L n}$ exponentially suppressed

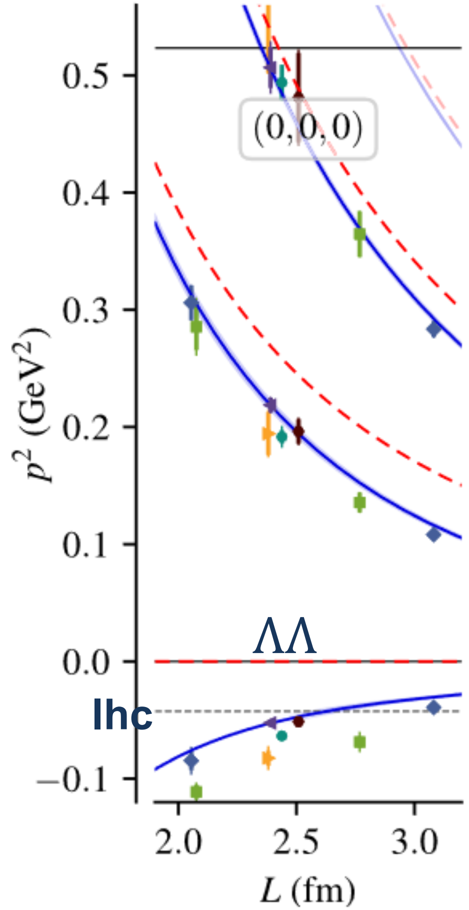
Singularity: $f(\vec{q}) = \frac{1}{\vec{q}^2 - a^2 + i\epsilon} \to \frac{e^{-ianL}}{4\pi L n}$ power-law effect

- Lüscher QCs: only taking care of the right-hand cut (unitary cut)

Left-hand cut problems in lattice QCD

H-dibaryons $(udsuds)$

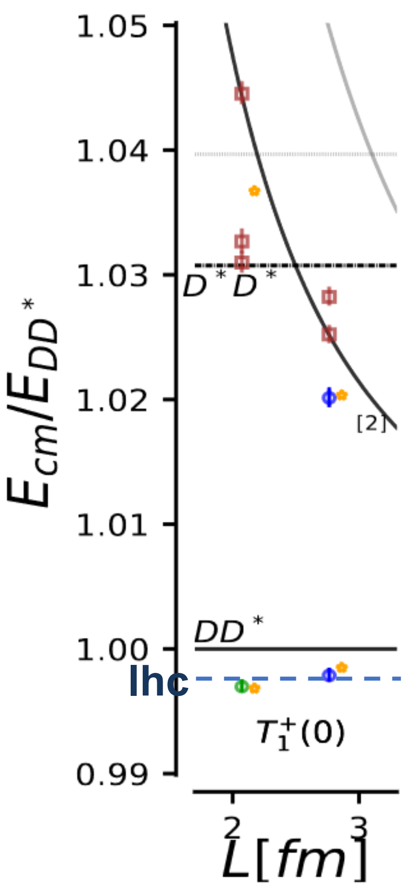
$m_\pi = m_K \approx 420$ MeV



H-dibaryons $(udsuds)$

DD^* (T_{cc})

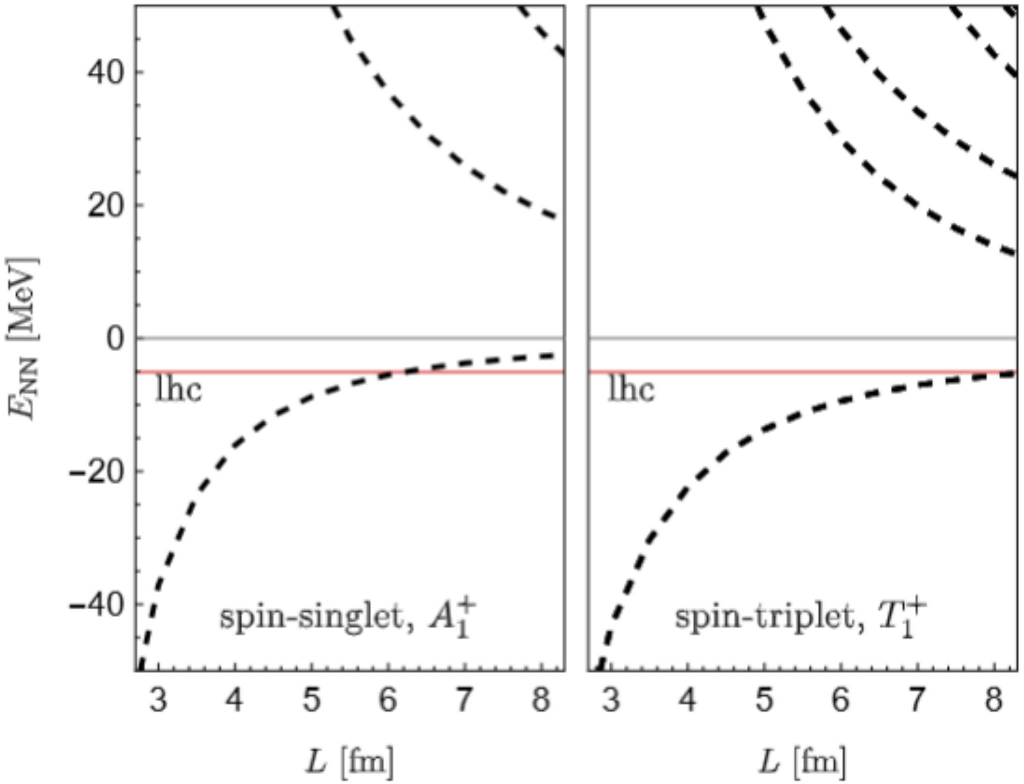
$m_\pi = 280$ MeV



DD^* (T_{cc})

NN

$m_\pi = 137$ MeV



Evaluate using Chiral nuclear force

$L > 8$ fm to get rid of the lhc problem

J.R. Green, et al. PRL127 (2021) 24, 242003
M.Padmanath, et al., PRL129 (2022) 3, 032002

Several methods

Mentioned by previous plenary talks:
F. Erben @ lattice2024, S.Sharpe@lattice2025, J.Green @lattice2025…

- Plane wave basis + Chiral effective field theory
  - LM and E. Epelbaum, JHEP 10, 051 (2021); LM, V. Baru, E. Epelbaum et al., PRD109, L071506 (2024).
  - S. Prelovsek, E. Ortiz-Pacheco, S. Collins et al., PRD112, 014507 (2025).
  - **[NPHF]** K. Yu, G.-J. Wang, J.-J. Wu, and Z. Yang, JHEP 04, 108 (2025); K. Zhang, K. Yu, Y. Geng et al., arXiv:2605.16977 [hep-lat]
  - LM @ lattice2021, LM@ lattice2022, I. Vujmilović @lattice2024, M. Padmanath@ lattice2025

Jorge Baeza Ballesteros, poster

Miguel Salg, Wed. 9:20

- Relativistic left-hand-cut formalism
  - A. Raposo and M. Hansen, *JHEP* 08 (2024) 075; A. B. Raposo, R. A. Briceño, M. T. Hansen et al., JHEP 06, 186 (2025).
  - N. Pitanga Lachini and R. A. Briceño, arXiv:2607.19009; A. Baião-Raposo, R. A. Briceño, N. Lang et al., arXiv:2607.24648
  - A. Raposo @ lattice2022, 2023, 2024
- Modified effective range expansion (MERE)
  - R. Bubna, H-W. Hammer, F. Müller et al., *JHEP* 05, 168 (2024); R. Bubna, H.-W. Hammer, B.-L. Hoid,et al. JHEP 10, 197 (2025)
  - A. Rusetsky @ lattice2024
- Using QC3: relativistic field theory (RFT)
  - M. Hansen, F. Romero-López and S. *Sharpe, JHEP 06 (2024) 051; S. M. Dawid, F. Romero-López, and S. R. Sharpe, JHEP 01, 060 (2025).*
  - S.Sharpe @ lattice2023, M.Islam @ lattice2023, S. Dawid@ lattice2024, A. Raposo @ lattice2025

Stephen Sharpe, Thu. 16:30

- Using QC3: finite-volume unitarity (FVU)

M. Mai and M. Doring, EPJA53 (2017) 12, 240

- HAL QCD approach
  - Lyu et al, PhysRevLett.131.161901
  - S.Aoki@ lattice2024

Maxim Mai, Wed. 11:50
Jinzi Wu, Fri. 14:40
Ivan Vujmilovic, Fri. 14:00

Plenary: **R. Perry**, Thu. 9:30
M. Padmanath, Thu. 9:00
Parallel: **A. Stump**, Thu. 16:50
Juan, Fri. 17:10

- N/D representation
  - S. M. Dawid, A. W. Jackura et al., PLB 864, 139442 (2025);
  - A. Rodas, L. Qiu, C. Fernández-Ramírez et al., arXiv:2605.22957

Sebastian Dawid, Wed. 9:00
Lin Qiu, Fri. 16:50

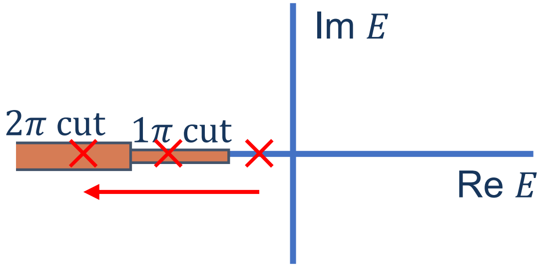
No lhc problem in Schrödinger Eq.

Plane wave QCs

- Schrödinger Eq. in the IFV to get the **bound state** solutions

$$\frac{\boldsymbol{p}^2}{2\mu}\psi(\boldsymbol{p}) + \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} V(\boldsymbol{p},\boldsymbol{p}')\psi(\boldsymbol{p}) = E\psi(\boldsymbol{p})$$

Off-shell, for $\mathrm{E} < 0$



  - ▶ Works well even below the left-hand cut
  - ▶ For the $p, p' > 0$, no lhc in potential

$$V_{l=0}(p,p') = \int_{-1}^{1} dz \frac{1}{p^2 + p'^2 - 2pp'z + m^2} = -\frac{1}{2pp'} \log\left(\frac{(p-p')^2 + m^2}{(p+p')^2 + m^2}\right)$$

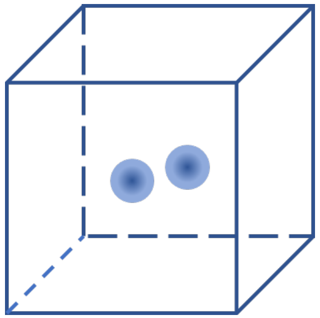
HAL QCD

- No lhc problem: S.Aoki@ lattice2024
  - ▶ using Schrödinger-like Eq.
- LO local potential:
  - ▶ easy to extract V_{long}

Lyu et al, PhysRevLett.131.161901

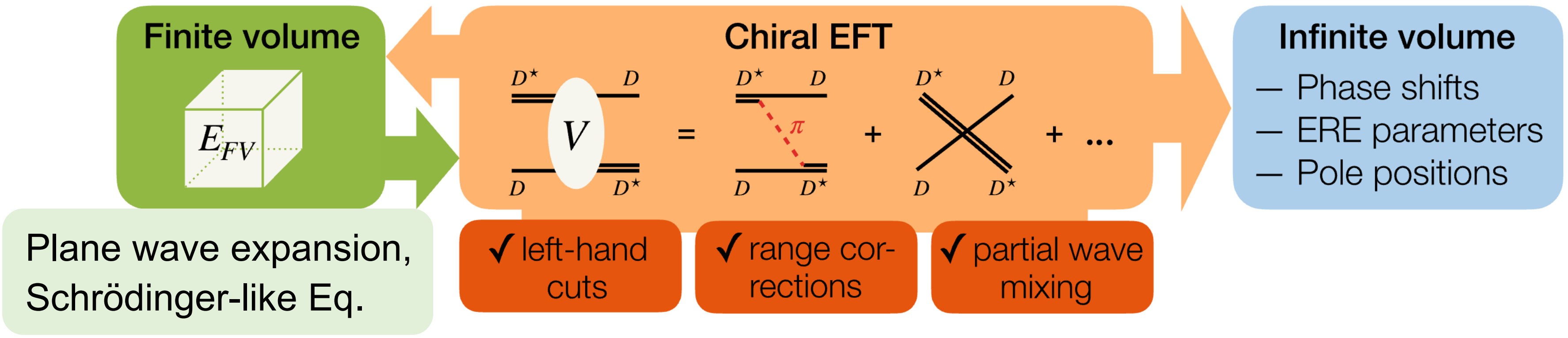
- FV energy levels are "bound states" trapped by the box

$$\int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \to \frac{1}{L^3}\sum_{\boldsymbol{p}_n}$$



- Basic idea:
  - ▶ using V to connected FV and IFV
  - ▶ FV effect: Schrödinger-like Eq.

Plane wave basis + Chiral EFT



- Infinite volume: Lippmann-Schwinger-like Eq.

$$T(\boldsymbol{p},\boldsymbol{p}') = V(\boldsymbol{p},\boldsymbol{p}') + \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} V(\boldsymbol{p},\boldsymbol{q}) \frac{1}{2w_1 w_2} \frac{(w_1+w_2)}{P_0^2 - (w_1+w_2)^2 + i\epsilon} T(\boldsymbol{q},\boldsymbol{p}')$$

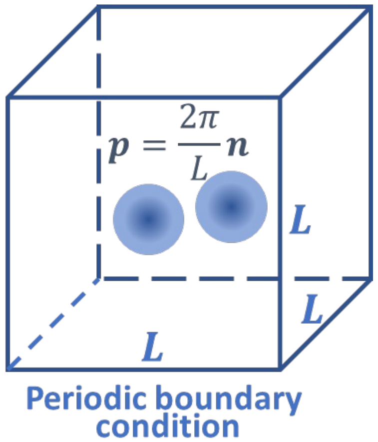
- Finite volume: plane wave (with discrete momentum) basis

$$\mathbb{T} = \mathbb{V} + \mathbb{V}.\mathbb{G}.\mathbb{T} \rightarrow \det(\mathbb{H} - \lambda\mathbb{I}) = 0 \rightarrow \mathbb{H}\boldsymbol{v} = \lambda\boldsymbol{v}$$

Matrix Eq. To get FV energy levels

$$\mathbb{H} \Rightarrow \mathrm{diag}\{\mathbb{H}_{\Gamma_i}, \mathbb{H}_{\Gamma_j}, \ldots\} \Rightarrow \quad \mathbb{H}_\Gamma \boldsymbol{v} = \lambda \boldsymbol{v}$$

Γ: irreps.



- Additional advantage: a natural treatment of partial-wave mixing
  - ▶ $\{l, m\}$ are not good quantum numbers to label states

LM and E. Epelbaum, JHEP 10, 051 (2021)
LM, V. Baru, E. Epelbaum et al., PRD109, L071506 (2024).

Applications: DD^* (T_{cc})

- **Lüscher QCs**

χ^2/dof=3.7/5, $E_{\rm pole}^{^3S_1} = -9.9^{+3.6}_{-7.2}$ MeV

$a_{^3S_1} = 1.04(29)$fm, $r_{^3S_1} = 0.96^{+0.18}_{-0.20}$fm

$a_{^3P_0} = 0.076^{+0.008}_{-0.009}\text{fm}^3$, $r_{^3P_0} = 6.9(2.1)\text{fm}^{-1}$

Four parameters:

$a_{^3S_1}, r_{^3S_1}, a_{^3P_0}, r_{^3P_0}$

Pole:

Virtual state

- **Plane wave + ChEFT**

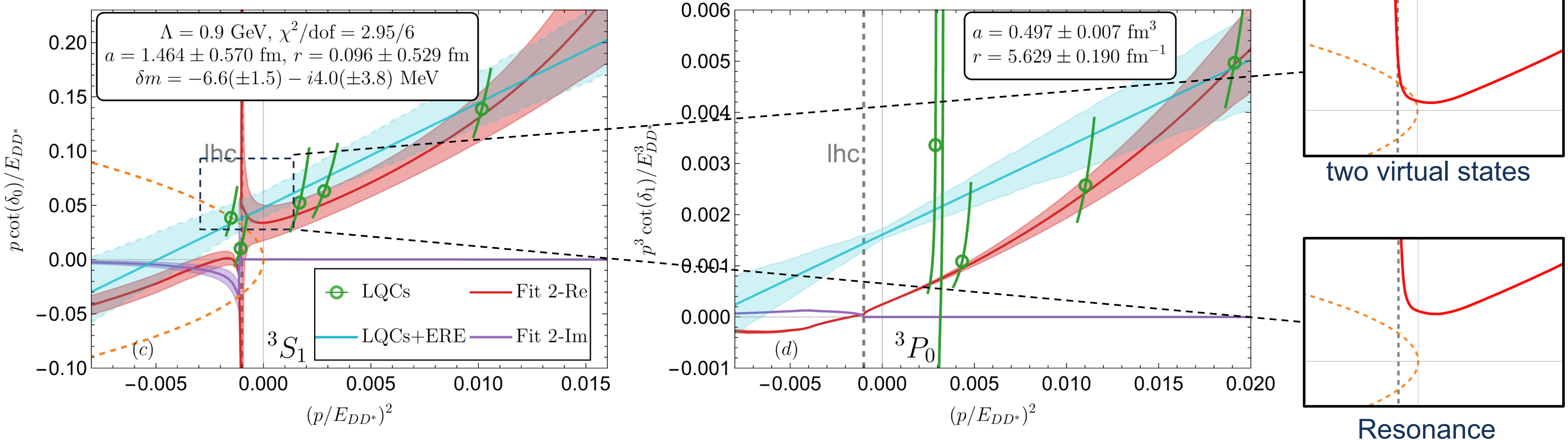
Three parameters:

LO and NLO $^3S_1$, NLO $^3P_0$ LECs

Pole:

More likely resonance

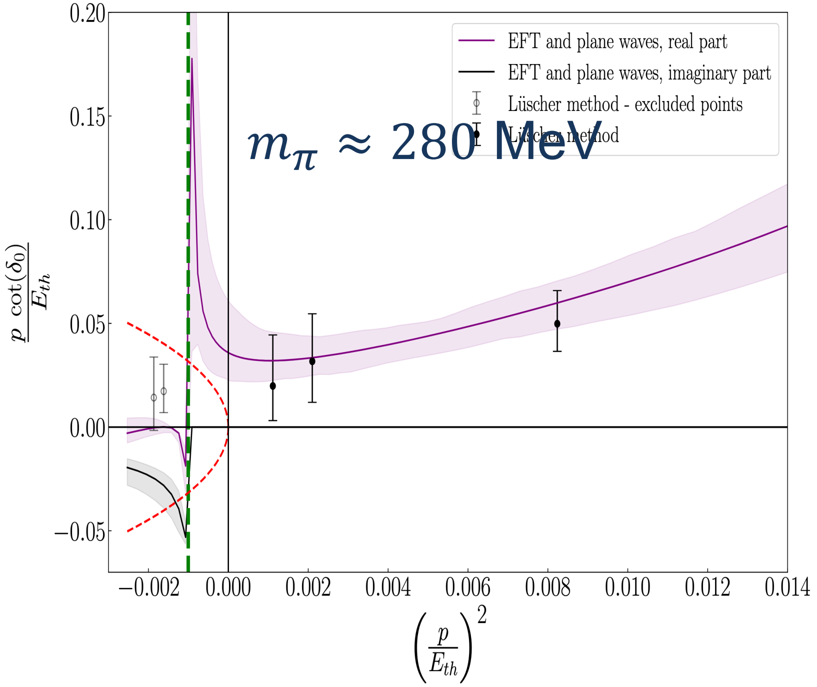
$m_\pi \approx 280$ MeV



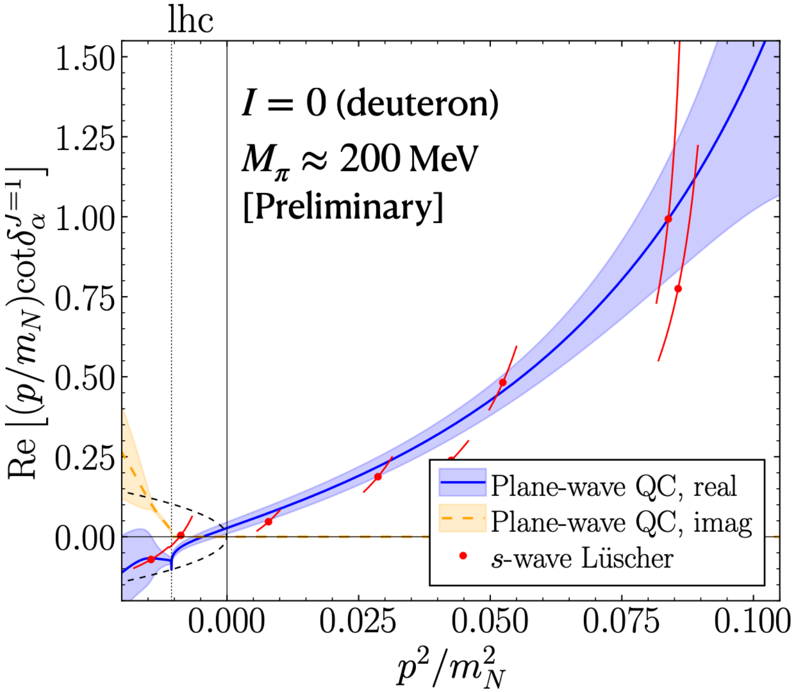
LM, V. Baru, E. Epelbaum et al., PRD109, L071506 (2024).

Other applications

- T_{cc} with diquark-antidiquark operator
- [BaSc] NN with m_π ≈710, 280 and 200 MeV
- [CLQCD] NN with $m_\pi \approx$ 290 MeV



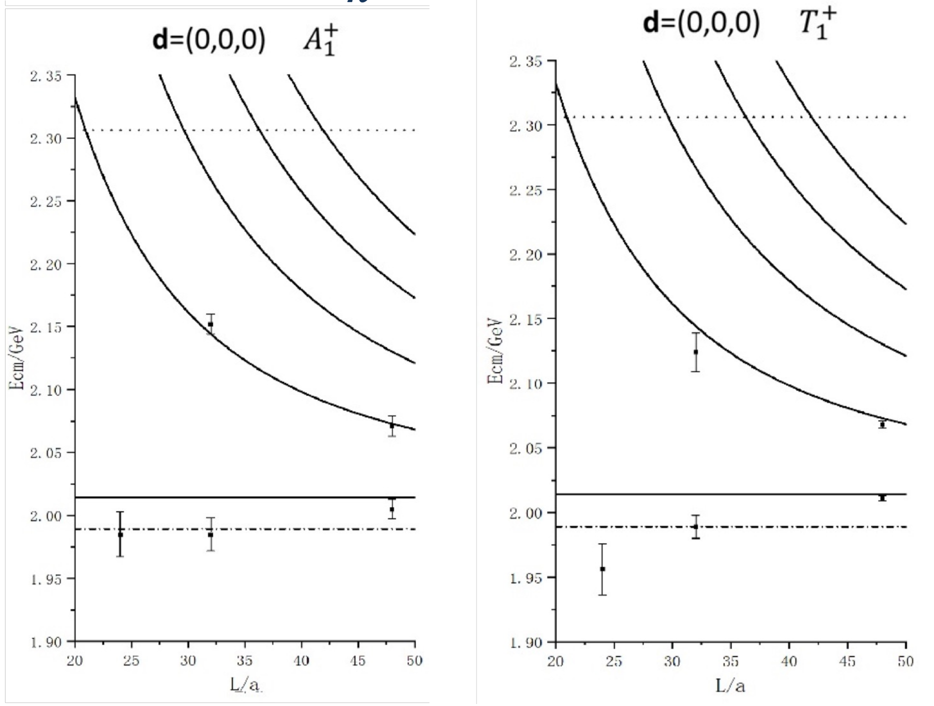
S. Prelovsek, E. Ortiz-Pacheco, S. Collins et al. PRD112, 014507 (2025).



Significant lhc effects around threshold

Miguel Salg, Wed. 9:20

$m_\pi \approx 292$ MeV



CLQCD, arXiv:2605.16977 [hep-lat]

Analysis: Nonperturbative Hamiltonian framework

Model dependence ?

- Question: (modified) Lüscher formula **vs** FV plane wave method + EFTs

Model-independent *Model-dependent?*

Our answer：

FV Plane-wave method

EFT (parametrization of kernel)

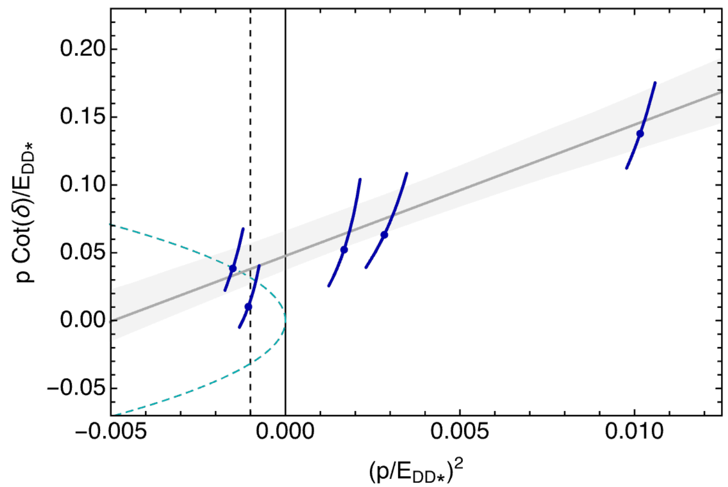
vs

(modified) Lüscher QCs

Parametrization of T-matrix

framework-dependent

also framework-dependent



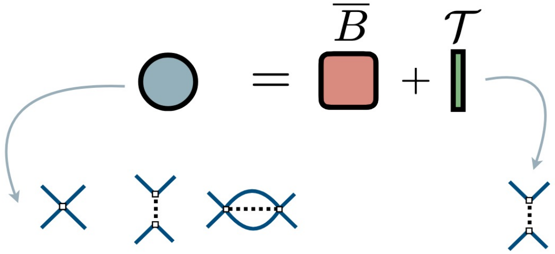
EFT: Systemic calculation, controllable truncation error

Relativistic lhc formalism

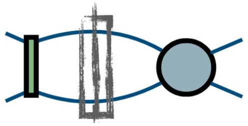
Relativistic lhc

Resolution:

separate the t-channel exchange from the Bethe Salpeter kernel



change the details of the cut to keep T partly off-shell



modify index space to reach a new quantisation condition

$$\ell, m \longrightarrow |\boldsymbol{k}_{\rm cm}|, \ell, m$$

$E_n(L)$ can now constrain modified K matrix + the cut

$$\overline{\mathcal{K}}(s) \qquad g^2 \log[\cdots]$$

Known integral equations relate this to the standard K (or the amplitude)

$$\det_{\boldsymbol{k}_{\rm cm}\ell m}\left[S(P,L)^{-1}+\xi^{\dagger}\,\overline{\mathcal{K}}^{\rm os}(P)\,\xi+2g^2\mathcal{T}(P)\right]=0$$

$S(P,L) =$ Matrix of known geometric functions

$\mathcal{T} =$ Matrix of known off-shell logs

e.g. $\mathcal{T}_{|\boldsymbol{k}_{\rm cm}|00,|\boldsymbol{k}'_{\rm cm}|00} = -\dfrac{1}{|\boldsymbol{k}_{\rm cm}||\boldsymbol{k}'_{\rm cm}|}\log\left[\dfrac{2\omega_{\boldsymbol{k}_{\rm cm}}\omega_{\boldsymbol{k}'_{\rm cm}} - 2|\boldsymbol{k}_{\rm cm}||\boldsymbol{k}'_{\rm cm}| - 2M_N^2 + m_\pi^2 - i\epsilon}{2\omega_{\boldsymbol{k}_{\rm cm}}\omega_{\boldsymbol{k}'_{\rm cm}} + 2|\boldsymbol{k}_{\rm cm}||\boldsymbol{k}'_{\rm cm}| - 2M_N^2 + m_\pi^2 - i\epsilon}\right]$

$\xi_{\boldsymbol{k}_{\rm cm}} = 1 \longleftrightarrow \xi = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$

$g = NN\pi$ coupling

$\overline{\mathcal{K}}^{\rm os}(P) =$ Diagonal matrix of Lorentz scalars

A. Raposo and M. Hansen, *JHEP* 08 (2024) 075; M. Hansen's talk @ CD2024

Development and applications

Developments:

$$\det_{aJm_J\ell S}\left[\mathcal{M}_0^{-1}+F+\mathcal{C}_L\right]=0$$

A. B. Raposo, R. A. Briceño, M. T. Hansen,et al., JHEP 06, 186 (2025).

- Reformulate equivalently
- F: widely used Lüscher function
- No discrete momentum index in det, hidden in $\mathcal{C}_L$
- (partially) bypassing the integral equations

A. Raposo @ lattice2024

Application 1

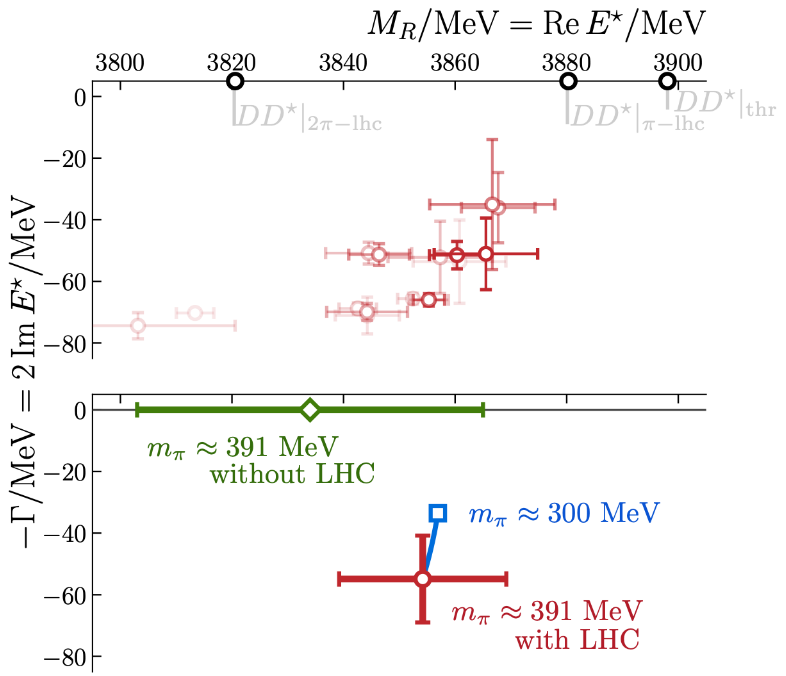
- Reanalyze data about T_{cc} — Hadron Spectrum, PRD111, 034511 (2025)
- Pole shifts with imaginary part nonvanishing

N. Pitanga Lachini and R. A. Briceño, arXiv:2607.19009

Application 2

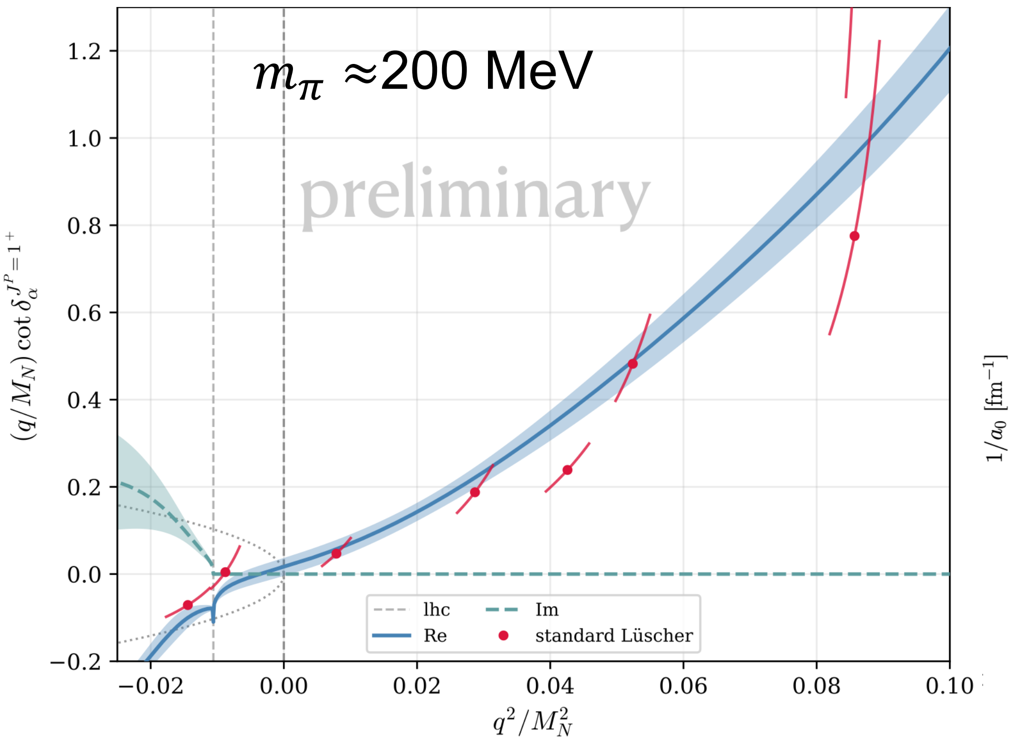
- Electroweak $2+J\to 2$ transitions

Baião-Raposo, R. A. Briceño, N. Lang et al., arXiv:2607.24648

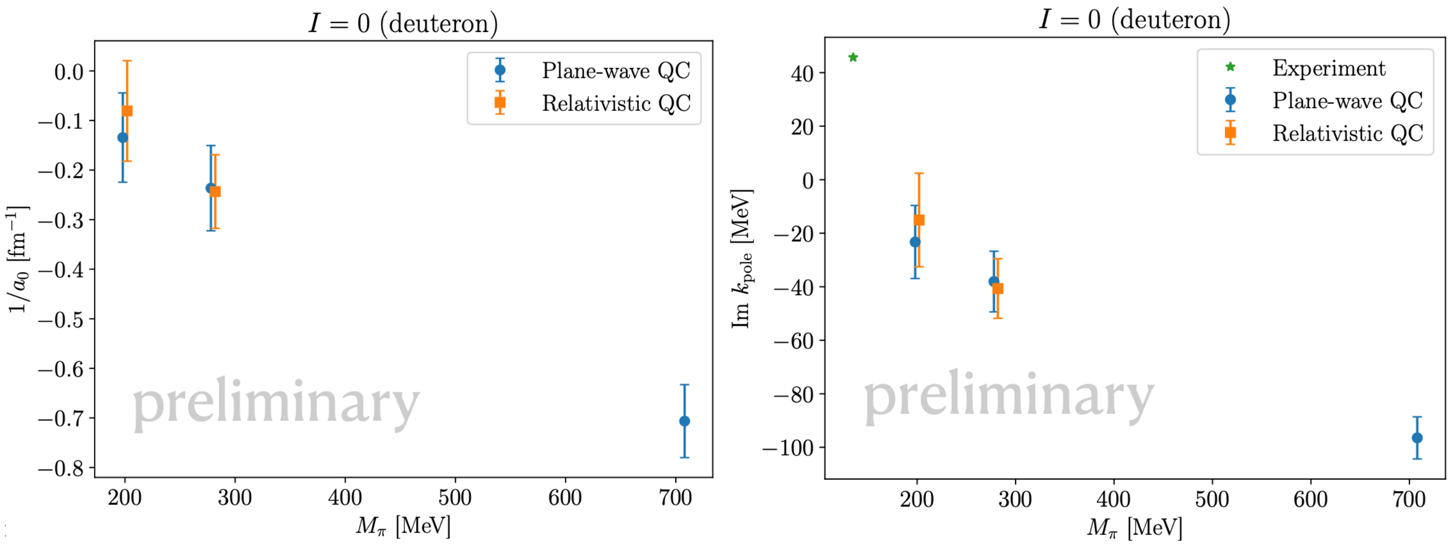


Application 3

- [BaSc] NN with m_π ≈710, 280, 200 MeV



Significant lhc effects around threshold



Comparison between relativistic QC and Plane-wave QC

Miguel Salg, Wed. 9:20

Finite volume Modified ERE

FV-MERE

- Two potential formalism

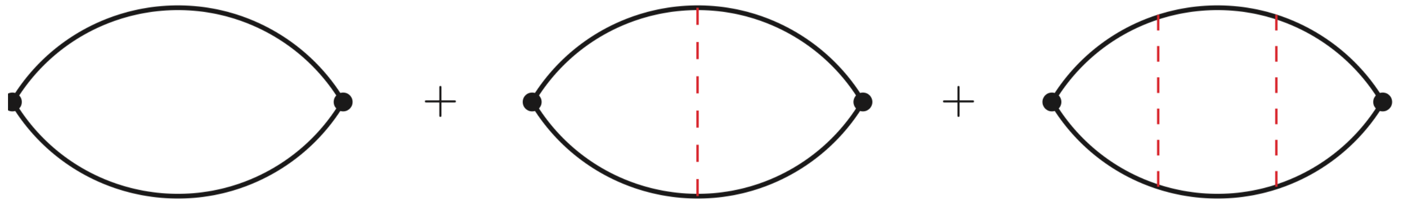
$$V(r) = \underbrace{V_L(r)}_{\text{known, local}} + \underbrace{V_S(r)}_{\text{unknown}}$$

- Modified ERE functions: only related to short-range interaction

$$F_l^M \equiv M_l(q) + \frac{k^{2l+1}}{|f_l(k)|^2}[\cot(\delta_l - \sigma_l) - i]$$

- Modified QCs $\boxed{Det\left[F_l^M - H\right] = 0}$
- Lüscher function: modified by the long-range interaction

Lüscher zeta-function

$$H =$$  $$+ \cdots$$

$$H_{\ell m,\ell' m'}(q_0) = \frac{4\pi}{L^6}\sum_{\boldsymbol{p},\boldsymbol{q}} \mathscr{Y}^*_{\ell m}(\boldsymbol{p})\langle \boldsymbol{p}| G_L(\hat{q}_0^2)|\boldsymbol{q}\rangle \mathscr{Y}_{\ell' m'}(\boldsymbol{q})$$

$$G_L(q_0^2) = G_0 + G_0 V_L G_0 + G_0 V_L G_0 V_L G_0 + \cdots$$

R. Bubna, H-W. Hammer, F. Müller et al., *JHEP* 05, 168 (2024)
R. Bubna, H.-W. Hammer, B.-L. Hoid et al. JHEP 10, 197 (2025)

Finite volume N/D representation

FV-N/D

- N/D representation in finite volume:

$$\mathcal{M}_L(s,\boldsymbol{P}) = \mathcal{N}_L(s,\boldsymbol{P})\mathcal{D}_L^{-1}(s,\boldsymbol{P})$$

$$\mathrm{Im}\,\mathcal{D}_L(s,\boldsymbol{P}) = -\frac{1}{L^3}\sum_{\boldsymbol{k}}\frac{2s}{E}\frac{\delta(s-E^\star(\boldsymbol{k})^2)}{2\omega_1(\boldsymbol{k})2\omega_2(\boldsymbol{P}-\boldsymbol{k})}\mathcal{N}_L(s,\boldsymbol{P})$$

Discrete spectrum in FV

kinematic normalization factor

$$\mathcal{D}_L(s,\boldsymbol{P}) = \mathbb{1} + \frac{1}{L^3}\sum_{\boldsymbol{k}}\frac{\mathcal{L}(\boldsymbol{k},\boldsymbol{P})}{s-E^\star(\boldsymbol{k})^2}\mathcal{N}(E^\star(\boldsymbol{k})^2)$$

$\omega_1(\boldsymbol{k}^\star)+\omega_2(\boldsymbol{k}^\star)$

Incorporate the lhc and short-range para.

Quantization condition $\det_{\ell m_\ell}\mathcal{D}_L = 0$

Finite-volume energy levels

$\{E_n\}$

IV numerator function

$\mathcal{N}_\ell(s)$

$$\mathcal{D}_\ell(s) = 1 - \int_{s_{\rm thr}}^{\infty}\frac{\mathrm{d}s'}{\pi}\frac{\rho(s')}{s'-s-i\epsilon}\mathcal{N}_\ell(s')$$

$$\mathcal{M}_\ell(s) = \mathcal{N}_\ell(s)\,\mathcal{D}_\ell^{-1}(s)$$

IFV and FV match up to exponentially suppressed effect

Sebastian Dawid, Wed. 9:00

S. M. Dawid, A. W. Jackura,et al., PLB 864, 139442 (2025)

Applications

- Reanalyze H-dibaryon data from J.R. Green et al. PRL127 (2021) 24, 242003
  - ▶ $m_\pi = m_K \approx 420$ MeV
  - ▶ original analysis: energy-restricted ERE

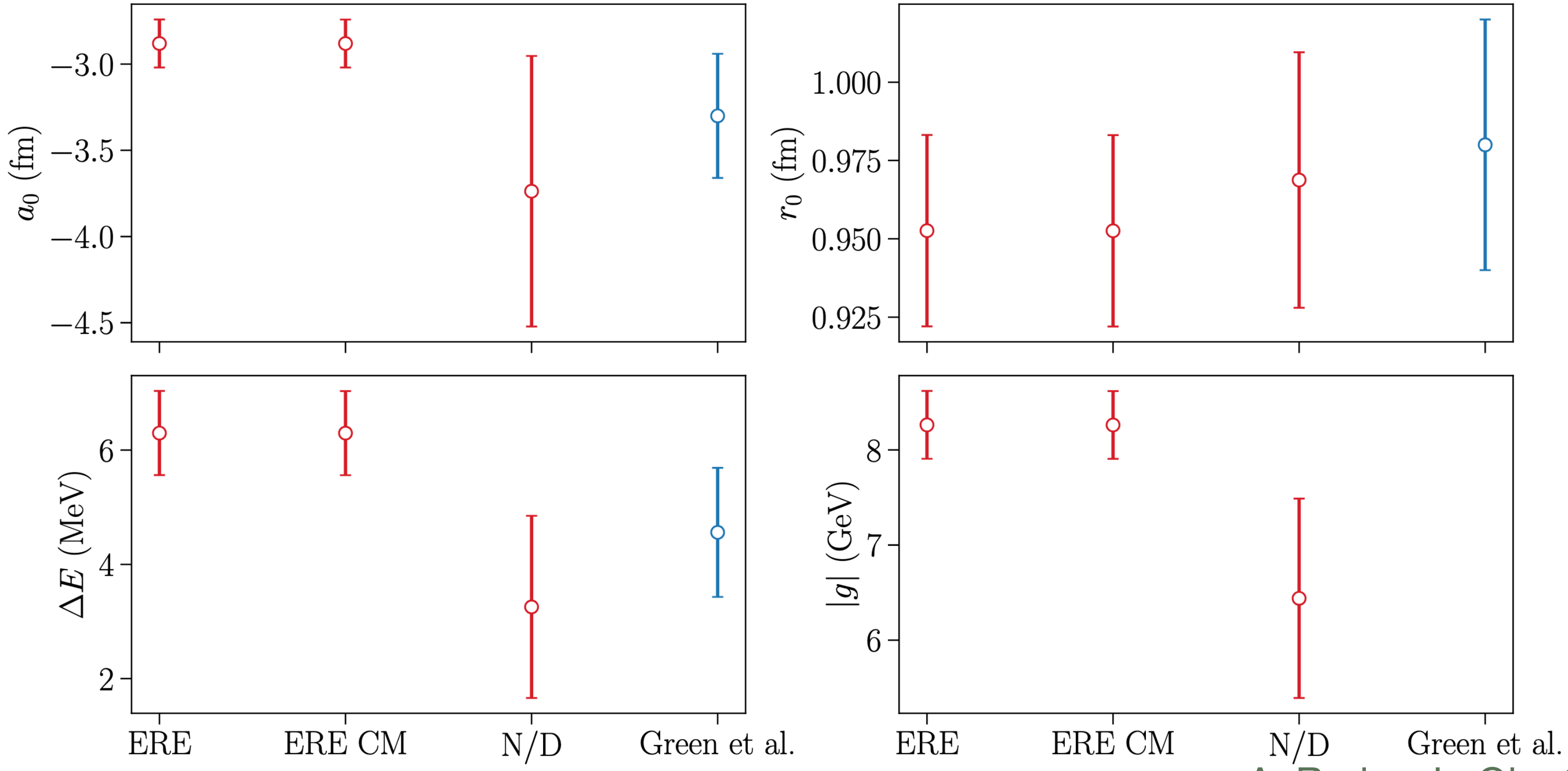
    E_{FV} **below lhc branch point is not used**

$$N(s) = n_0 + g_0^2\, L_t(s).$$

$$L_t(s) \equiv \frac{1}{2}\int_{-1}^{+1} \frac{d\cos\theta}{t - m_\pi^2} = -\frac{1}{s - s_{\mathrm{thr}}} \log \frac{s - s_{\mathrm{thr}} + m_\pi^2}{m_\pi^2}.$$

Incorporate the lhc

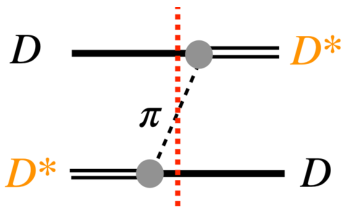
Lin Qiu, Fri. 16:50



Non-negligible contribution from lhc to the extracted pole properties

A. Rodas, L. Qiu, C. Fernández-Ramírez et al., arXiv:2605.22957

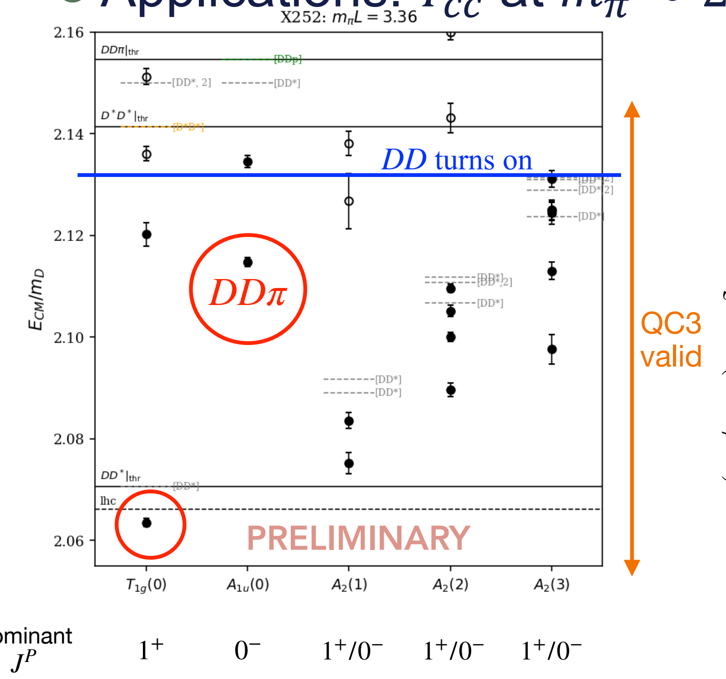
Using 3-body quantization conditions

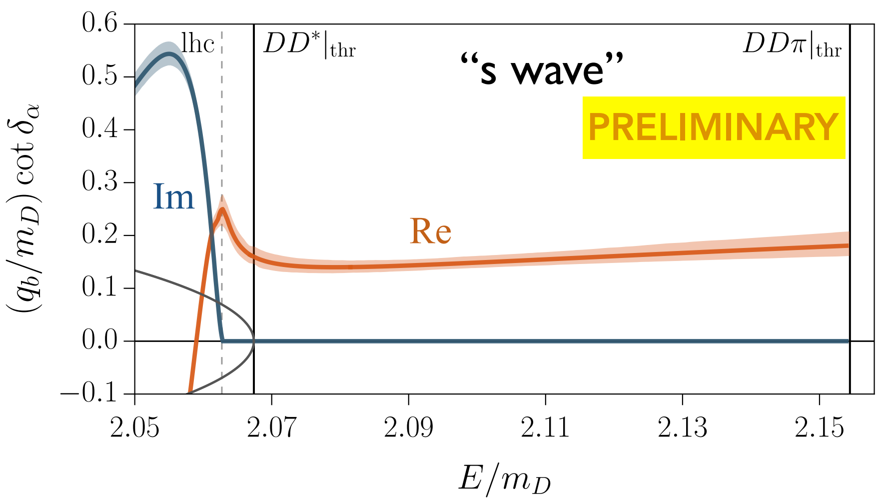


Stephen Sharpe, Thu. 16:30

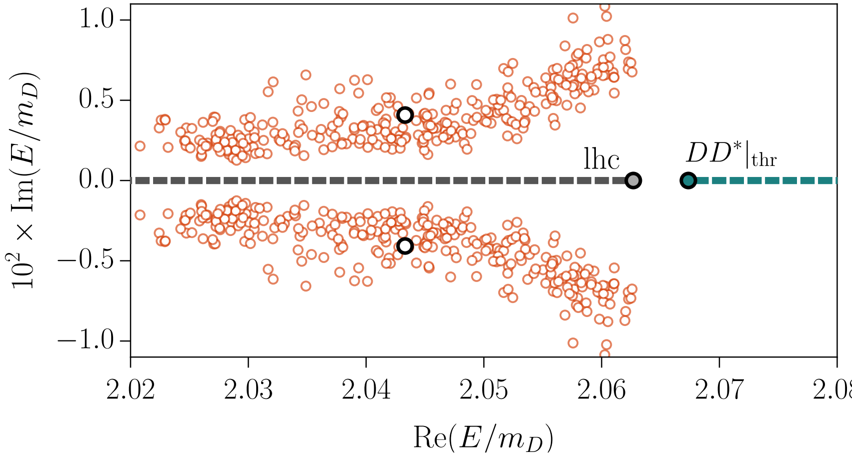
- Left-hand cut closely related to the three-body effect
- QC3: relativistic field theory (RFT)
  - ▶ D^* as the bound state of $D\pi$, incorporating the lhc naturally

- Applications: T_{cc} at $m_\pi \approx 290$ MeV





Subthreshold resonance



Using 3-body quantization conditions

- QC3: finite-volume unitarity (FVU)

M. Mai and M. Doring, EPJA53 (2017) 12, 240

$$T = V - VGT$$

V : Bethe-Salpeter kernel

$$V^\infty = V^L + \mathcal{O}(e^{-m_\pi L})$$

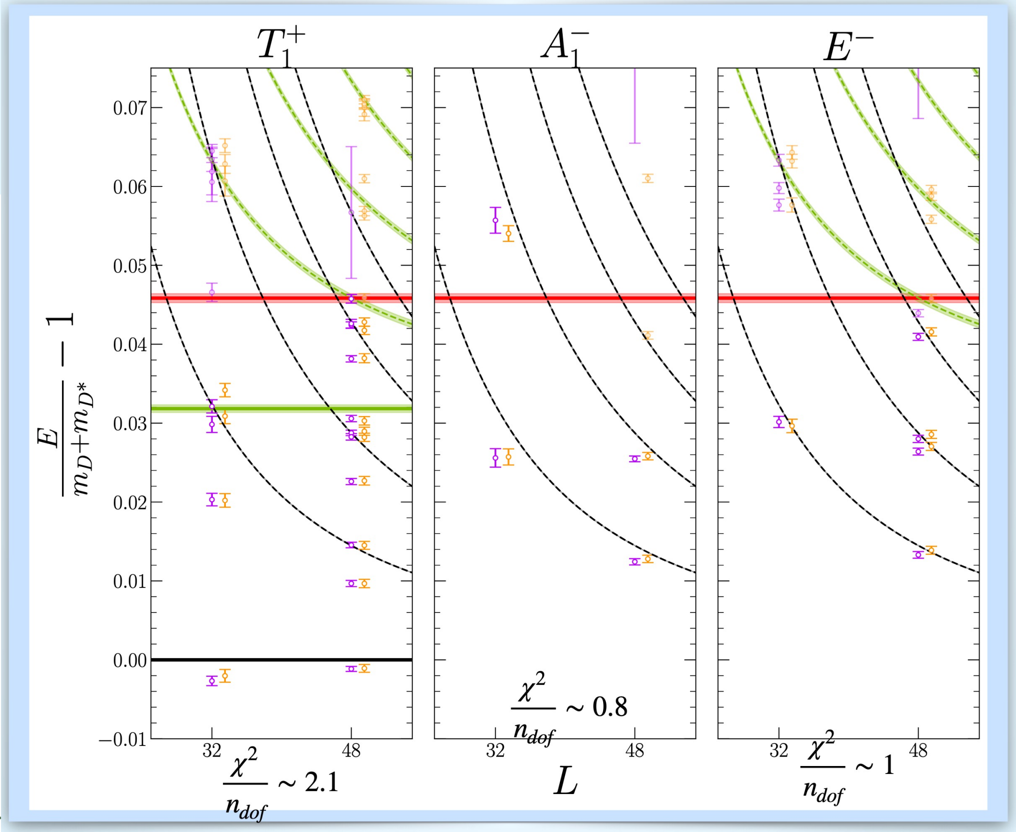
G : two-body loop function

$$\equiv \frac{1}{k^2 - m_1^2 + i\epsilon}\frac{1}{(P-k)^2 - m_2^2 + i\epsilon}$$

$$\boxed{\det\left(2L^3\omega_1(p^2)\left((P-k)^2 - m_2^2\right)\Big|_{k^0=\omega_1(p^2)} - V\right) = 0}$$

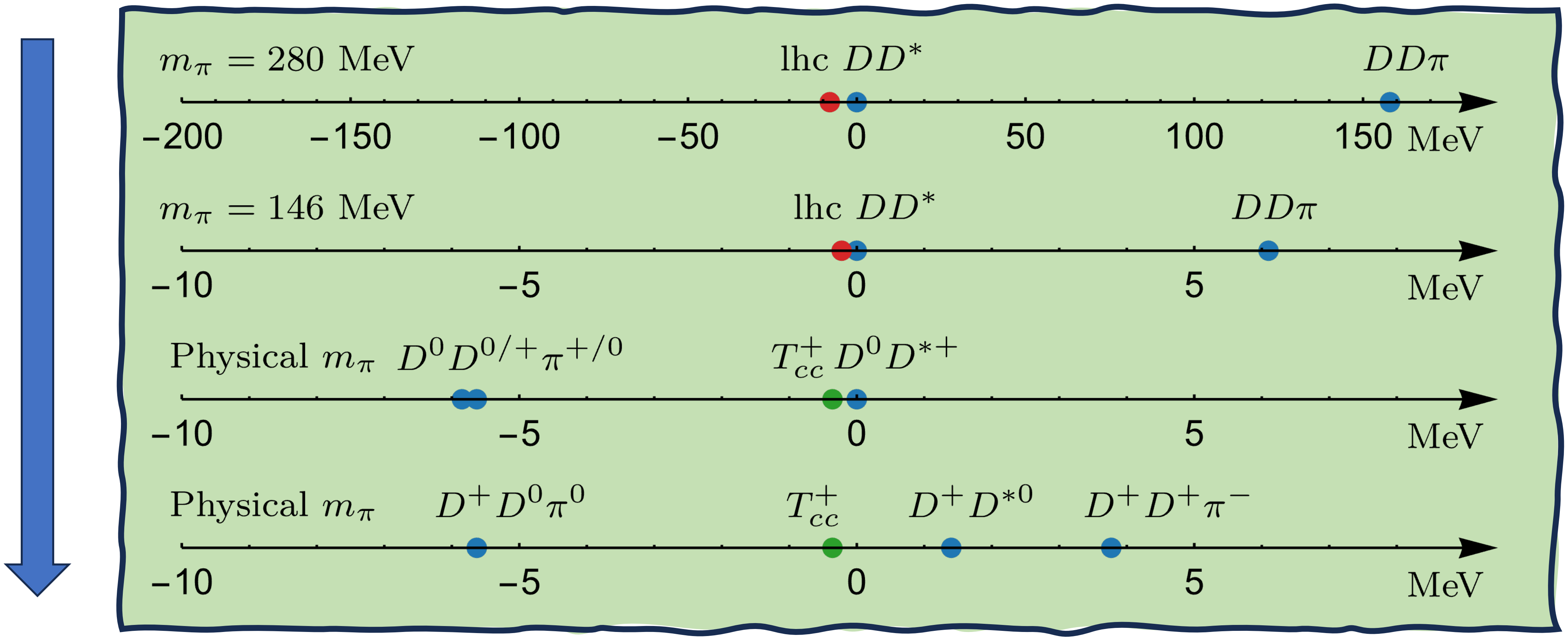
helicity plane-wave basis: $|D(\vec{p})D^*(-\vec{p},\lambda)\rangle,\ |D^*(\vec{p},\lambda_1)D^*(-\vec{p},\lambda_2)\rangle$

- Application: T_{cc} at $m_\pi \approx 305$ MeV



Maxim Mai, Wed. 11:50
Jinzi Wu, Fri. 14:40
Ivan Vujmilovic, Fri. 14:00

Towards physical T_{cc}

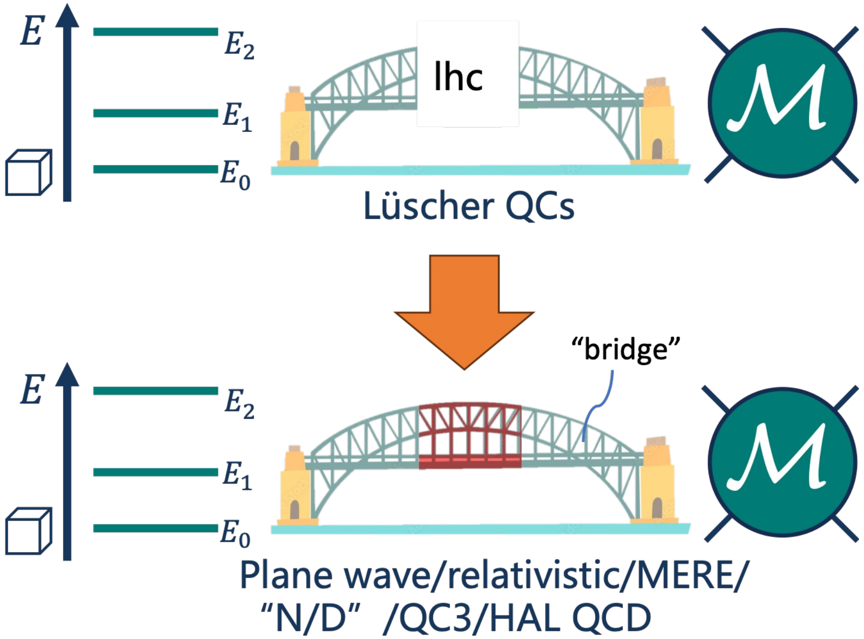


Towards physical T_{cc} : QC3 formalisms remains valid for smaller pion mass

More general talks on 3-body frontier:
M. Hansen, R. Mukherjee, H.-B. Yan, A. Jackura, J. Sitison, N. Chambers, …
S. Sharpe, Plenary talk @lattice2025

Summary and outlook

- Left-cut problems ↔ long-range interaction, e.g. one-pion-exchange
  - ▶ Infinite volume: amplitude formulae with lhc, LSE, N/D, MERE
  - ▶ Finite volume: Lüscher QCs are invalid below the branch point of lhc
- Several approach to FV lhc problems



- Applications: H-dibaryon, NN, T_{cc},…
  - ▶ Reanalyze existing data
  - ▶ New lattice simulations
  - ▶ Significant lhc effect for E_{FV} close and below the lhc branch point

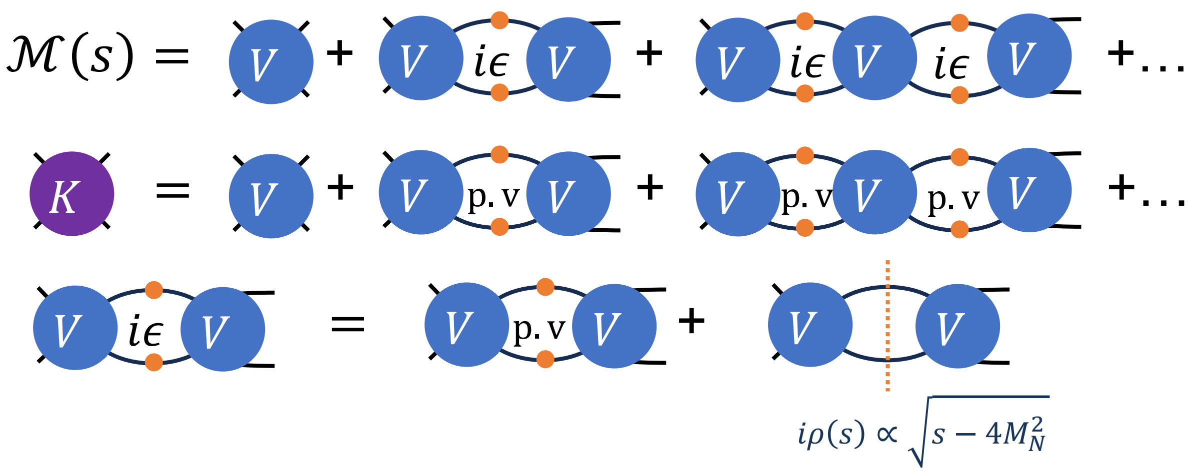
Developing formalisms

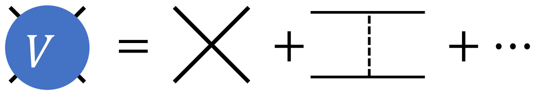
Applications, **Comparisons** and **Benchmarks**

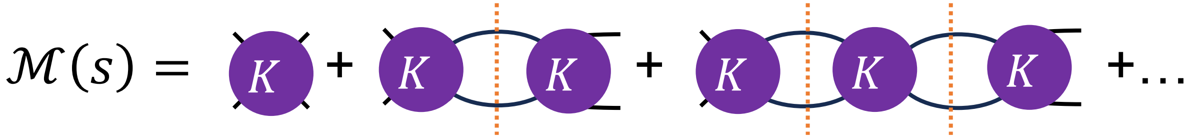
Thanks for your attentions!

backup

Infinite volume

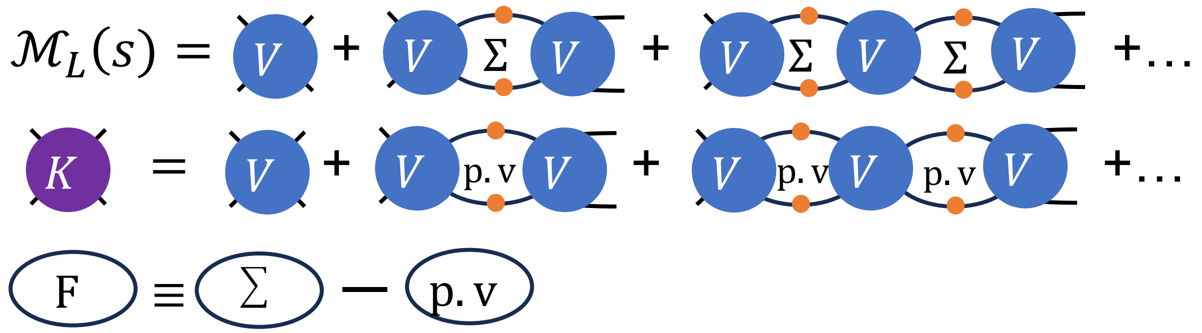




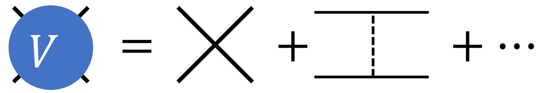


$$= K(s) + K(s)i\rho(s)K + \cdots = \frac{1}{K(s)^{-1} - i\rho(s)}$$

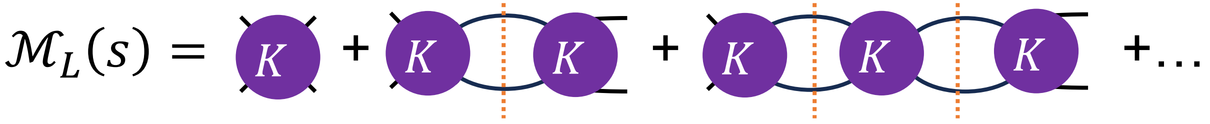
Detailed derivation of Lüscher's formula



Also taking on-shell



$$\left[\frac{1}{L^3}\sum_{\boldsymbol{k}} - \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\right] f(k) = \begin{cases} \mathcal{O}(e^{-mL}) & \text{smooth } f(k) \\ \text{power of } L & \text{otherwise} \end{cases}$$



$$= K(s) + K(s)F(L)K + \cdots = \frac{1}{K(s)^{-1} - F(L)}$$

Lüscher formula: $\det[K(s)^{-1} - F(L)] = 0$

Chiral EFT

- Symmetry from QCD
  - ▶ Chiral symmetry and its spontaneous breaking
- Weinberg power counting
  - ▶ Systemic calculation, controllable truncation error

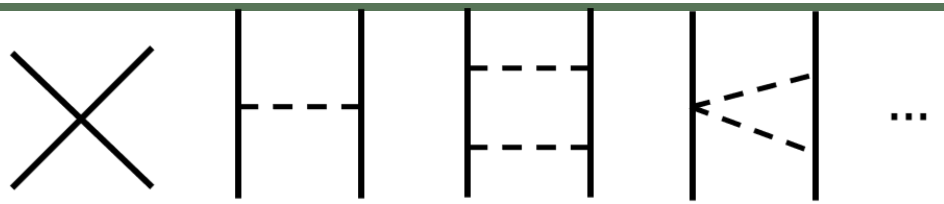
Reinert:2017usi

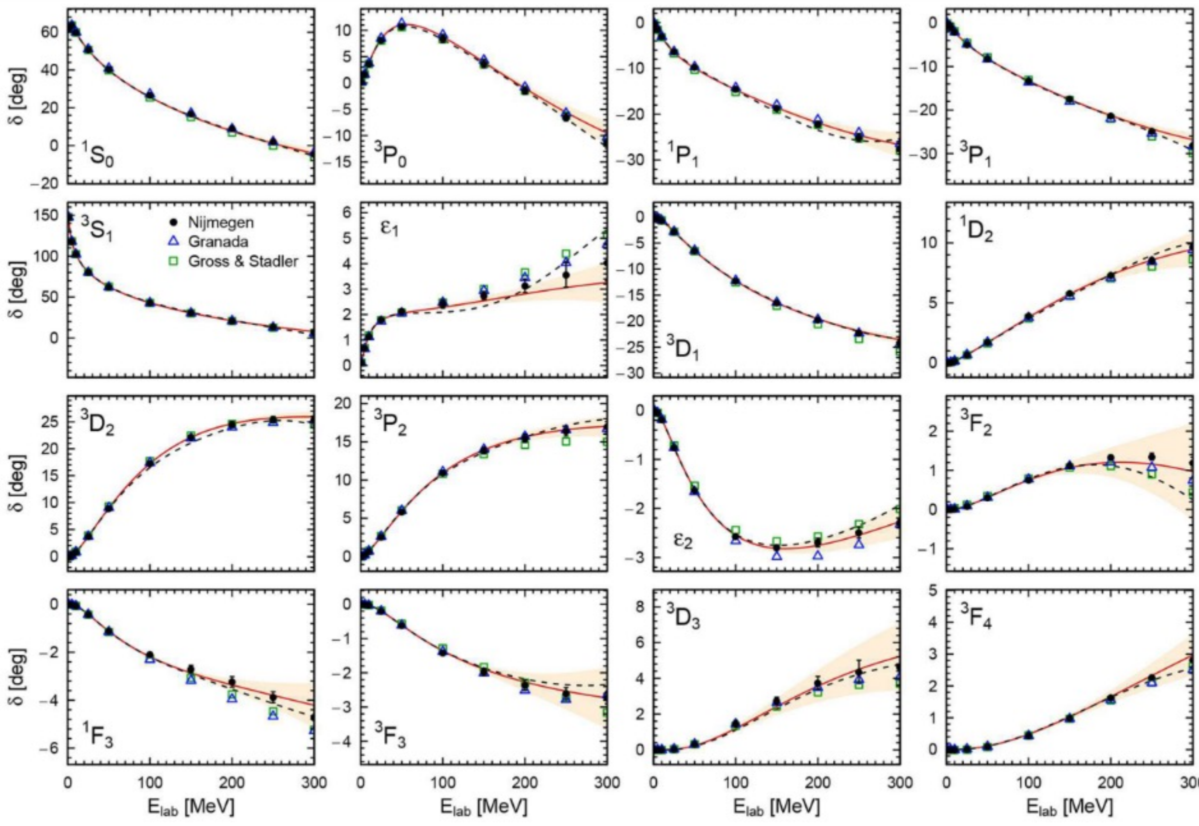
$$V(\vec{p}', \vec{p}) = V_{\text{contact}} + V_{1\pi} + V_{2\pi}$$

- Great success in the nuclear force
- Semilocal momentum-space regularization

$$V_{1\pi}(\vec{p}', \vec{p}) = -\frac{g_A^2}{4F_\pi^2}\left(\frac{\vec{\sigma}_1 \cdot \vec{q}\vec{\sigma}_2 \cdot \vec{q}}{q^2 + m_\pi^2} + C(m_\pi)\vec{\sigma}_1 \cdot \vec{\sigma}_2\right) e^{-\frac{q^2+m_\pi^2}{\Lambda^2}}$$

- Long-range interaction: $V_{1\pi}$ is known
- Short-range interaction: contact interaction
  - ▶ Unknown low energy constants (LECs)
  - ▶ fitting lattice QCD data

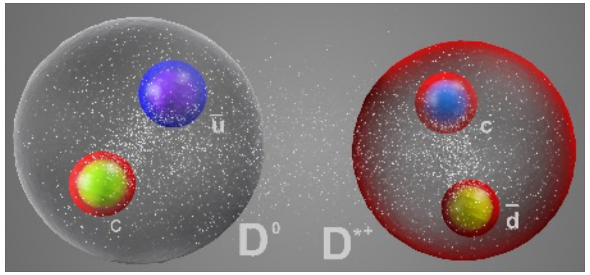


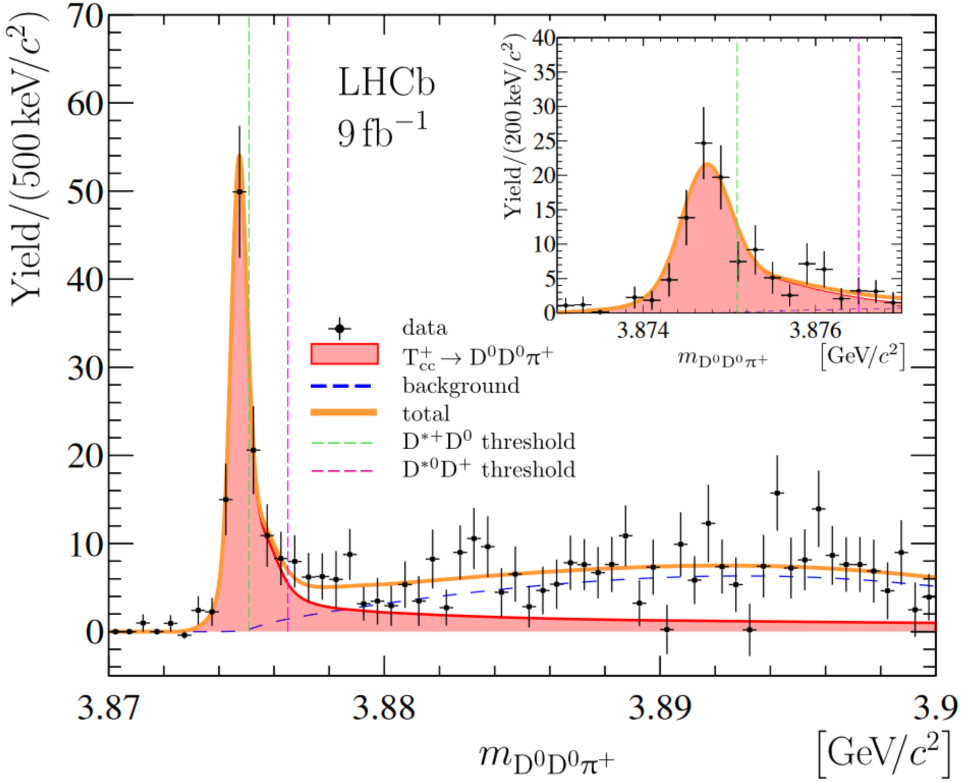


$T_{cc}(3875)^+$ state

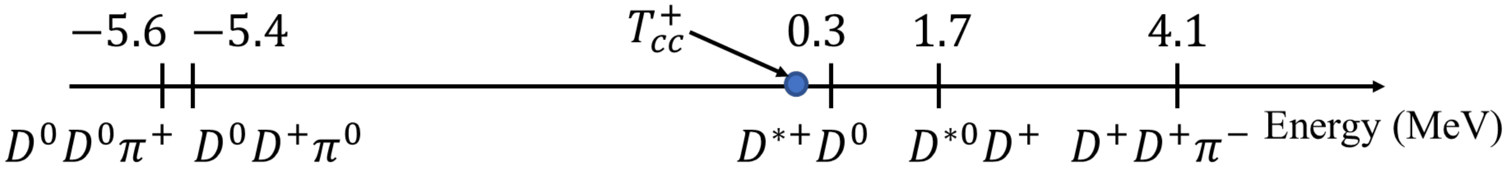
LHCb Collaboration

- $T_{cc}(3875)^+$ was observed in 3-body final states: $D^0D^0\pi^+$
- Very close to D^0D^{*+} thresholds: $\delta m_U \approx -360$keV, $\Gamma \approx 48$ keV
- Exotic hadrons: minimal quark content: $cc\bar{u}\bar{d}$
- Good candidates of D^0D^{*+} molecule





- 3-body dynamics could be important

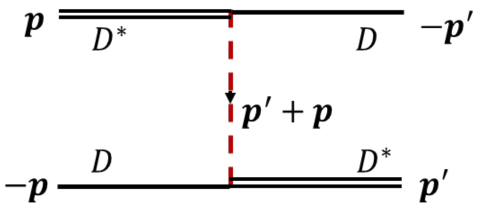


lhcb:2021vvq, lhcb:2021auc, Du:2021zzh, Meng:2021jnw…

- LQCD setting: $m_\pi \approx$ 280 MeV, $m_D \approx 1927$ MeV, $m_{D^*} \approx 2049$ MeV, $L \approx 2.07, 2.76$ fm, $a \approx 0.086$fm
- Some quick estimations

Padmanath:2022cvl

  - $m_{\rm eff}^2 = m_\pi^2 - (m_{D^*} - m_D)^2 > 0$, $m_{\rm eff} \approx 252$ MeV
  - $p_{\rm lhc}^2 \approx -\left(\frac{m_{\rm eff}}{2}\right)^2 = -(126\ \mathrm{MeV})^2$
  - $p_{\rm rhc3}^2 \approx 2\mu_{DD^*}(2m_D + m_\pi - m_D - m_{D^*}) \approx (560\ \mathrm{MeV})^2$

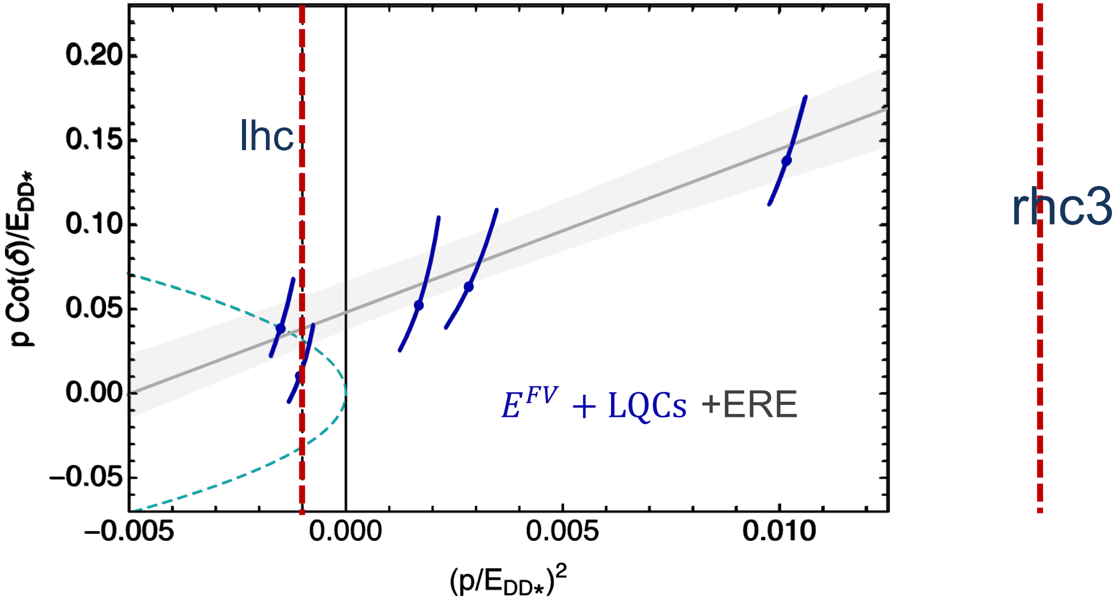


$$\boxed{\frac{-1}{\boldsymbol{k}^2 + m_\pi^2 - k_0^2 - i\epsilon} = \frac{-1}{\boldsymbol{k}^2 + m_{\rm eff}^2 - i\epsilon}}$$

- A conventional procedure to the IFV:
  - Using Lüscher formula to get phase shift
  - Use ERE to parameterize K-matrix
- Conclusion: virtual states
- Limitations
  - $m_{\rm eff}L = 2.6, 3.5$

  exponential effect can be important
  - left-hand cut

  ~~Lüscher formula~~, ~~effective range~~ expansion

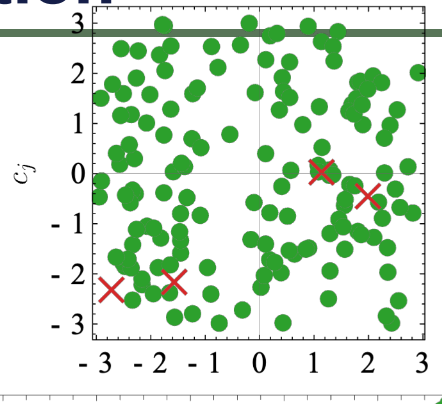


Other lattice results: Cheung:2017tnt, Junnarkar:2018twb, Chen:2022vpo, Lyu:2023xro, Whyte:2024ihh…

Towards a practical approach: eigenvector continuation

- Plane wave basis+Eigenvector continuation
  - ▶ Eigenvector continuation (EC) with subspace learning

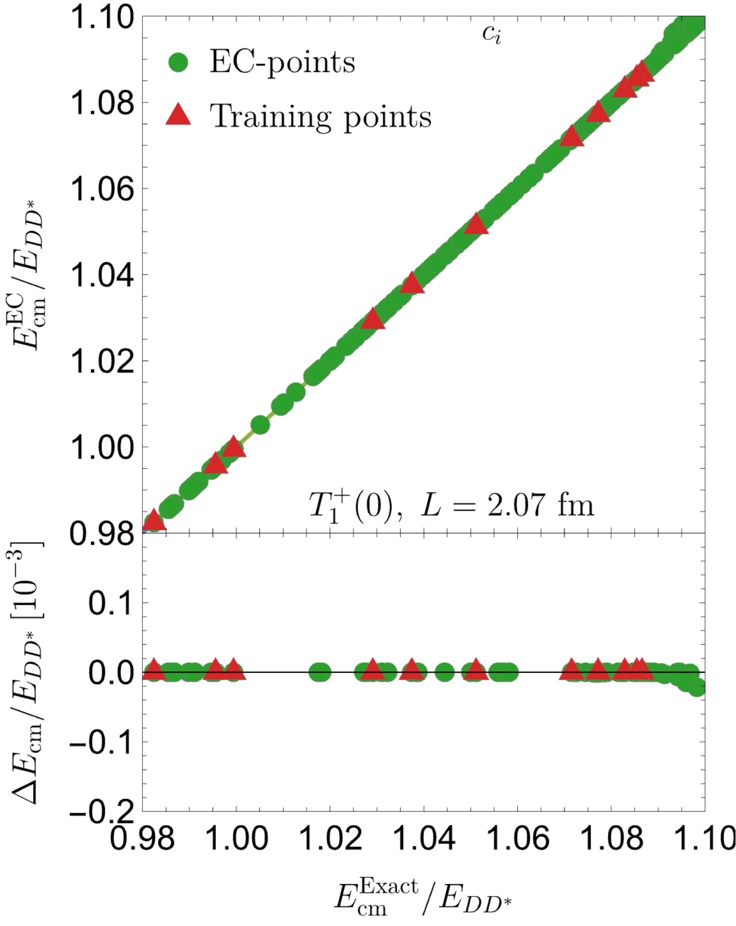
Frame:2017fah,Demol:2019yjt, Furnstahl:2020abp,Yapa:2022nnv



- To fit or quantify uncertainty: solve eigenvalue problem with different $\{c_i\}$ repeatedly
- EC basis: eigenvectors from a selection of parameter sets $\{c_i\}_1$, $\{c_i\}_2$,... (training point)
- Naturalness of LEC in EFT (~ 1) makes the EC more reliable
- dim is linear function

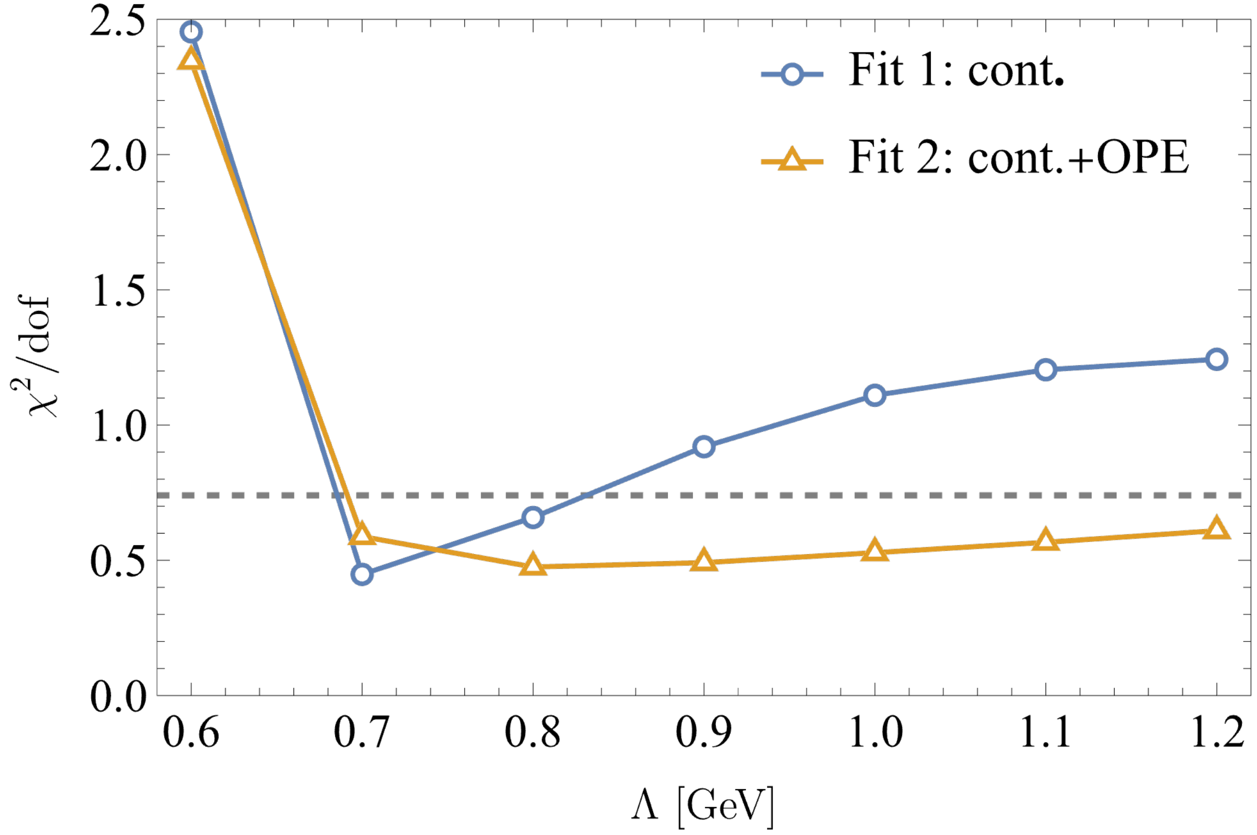
$$\dim^{EC} = \frac{p_{max}}{2\pi/L} \sim \mathcal{O}(10), \quad p_{max} \approx 0.6 \text{ GeV}$$

- The subspace learning is the one-time cost
- Make the calculation fast and accurate



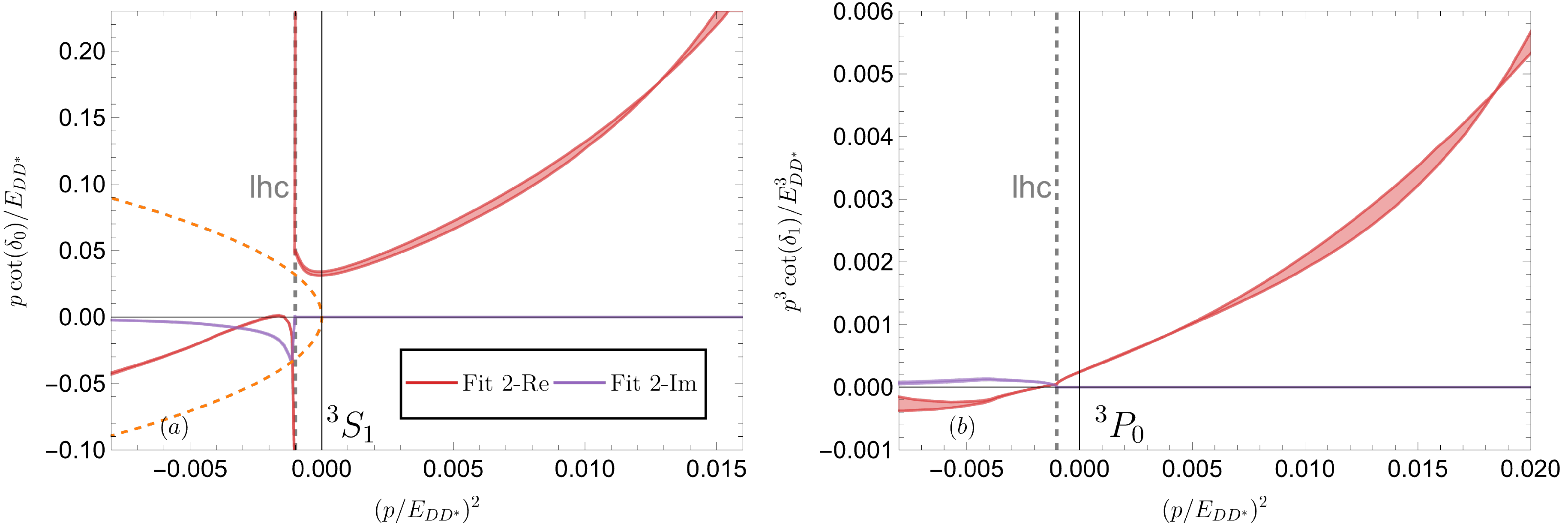
Cutoff dependence of χ^2

- 3 LECs: LO and NLO ${}^3S_1$ contact terms, NLO ${}^3P_0$
- In $V_{\rm ctc}$ fit, the P-wave dominate states control Λ-dependence of the χ^2
  - The shape of the of $k^3 \cot \delta_1$ is determined by regulator and cutoff
  - Sensitive to Λ
- The $V_{\rm ctc}$+ $V_{1\pi}$ fit is stable with Λ
- The $V_{\rm ctc}$+ $V_{1\pi}$ fit is even better than QCs

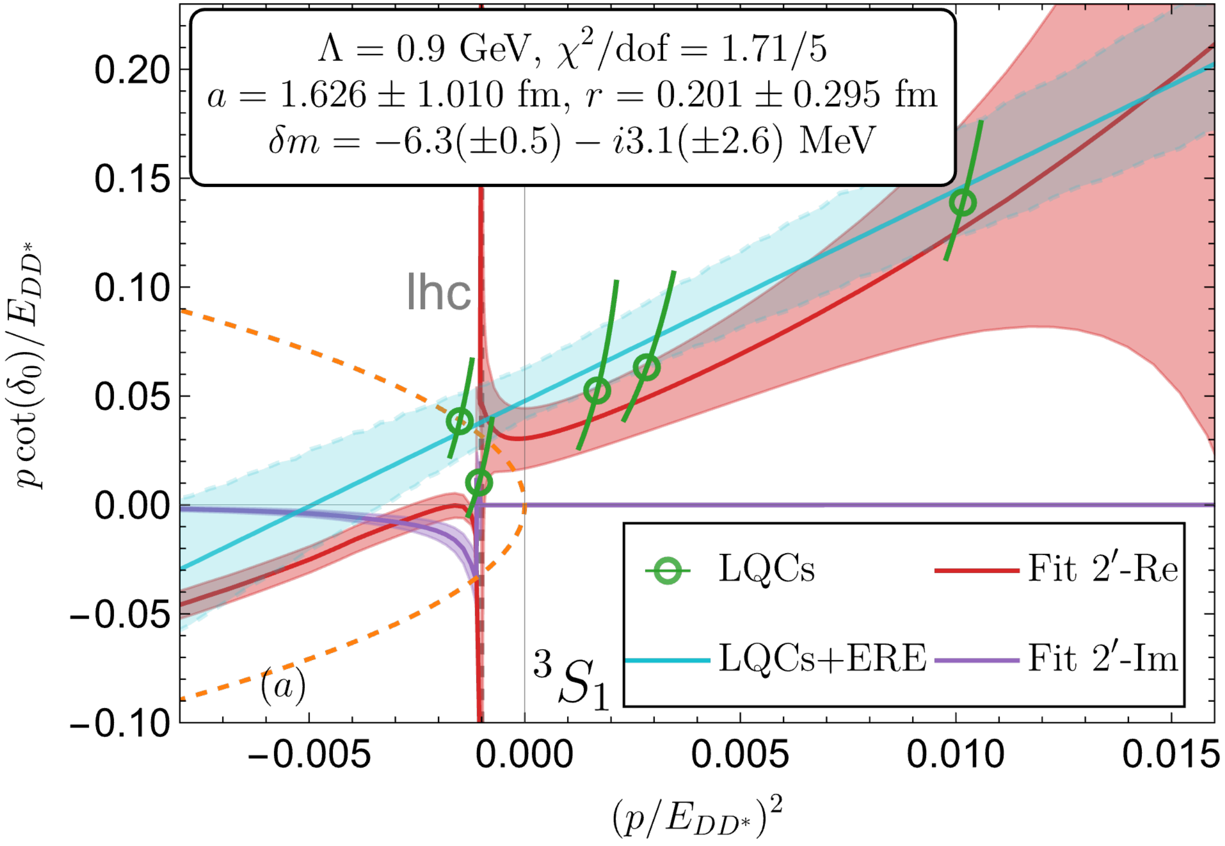


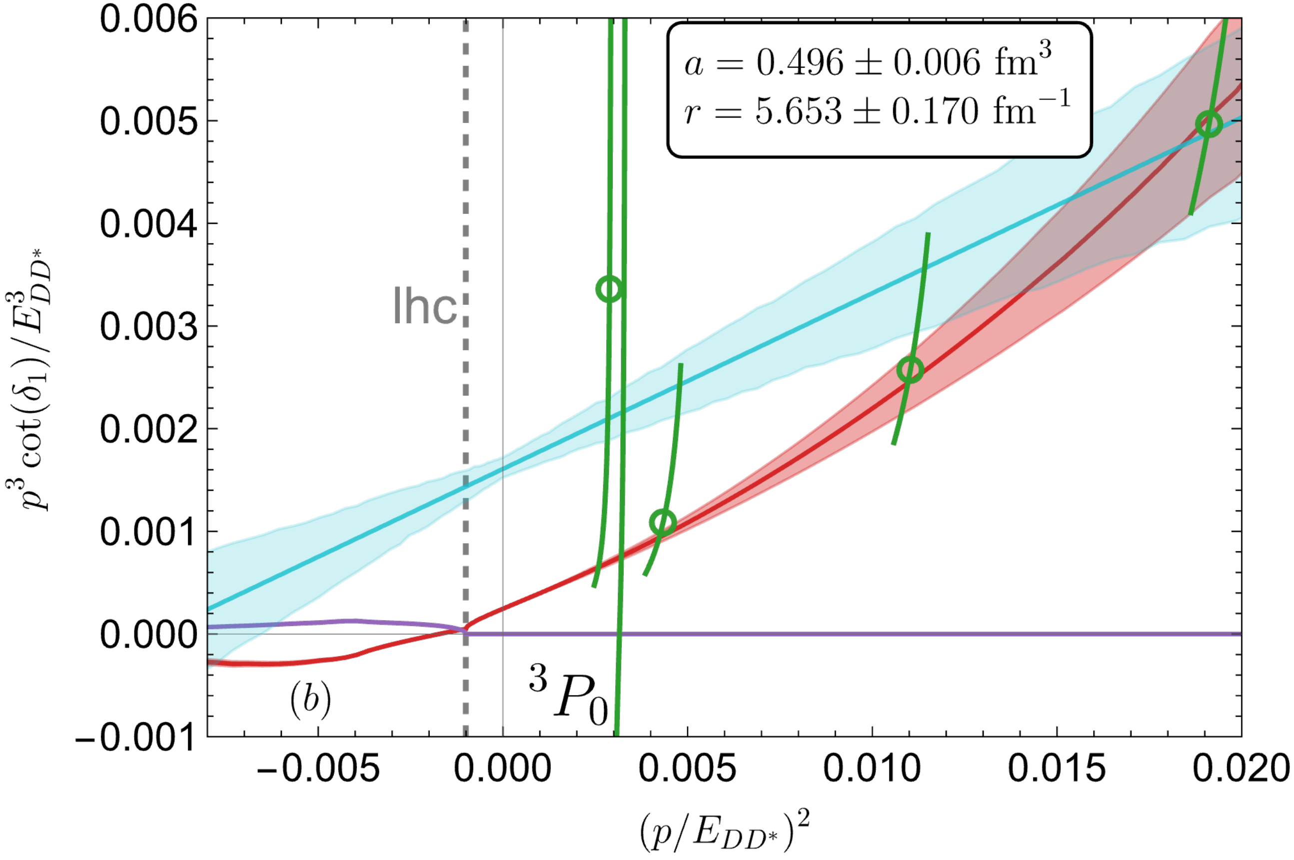
Cutoff dependence of phase shift

$\Lambda = 0.7 - 1.2$ GeV

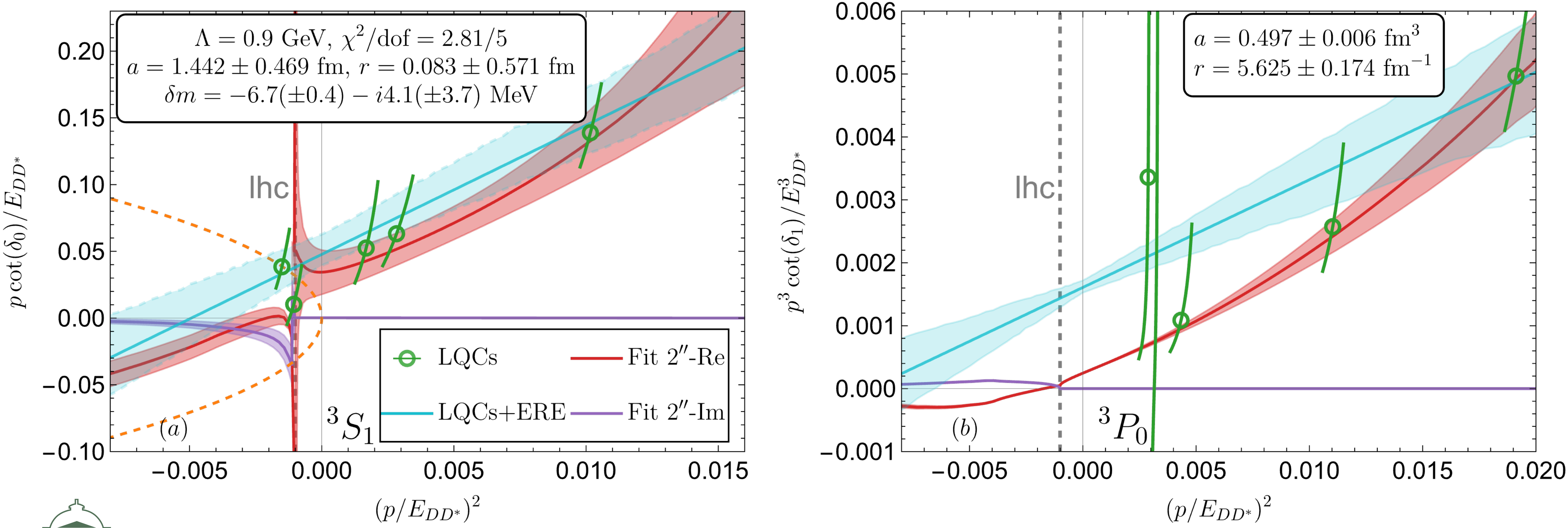


Including SD transition terms



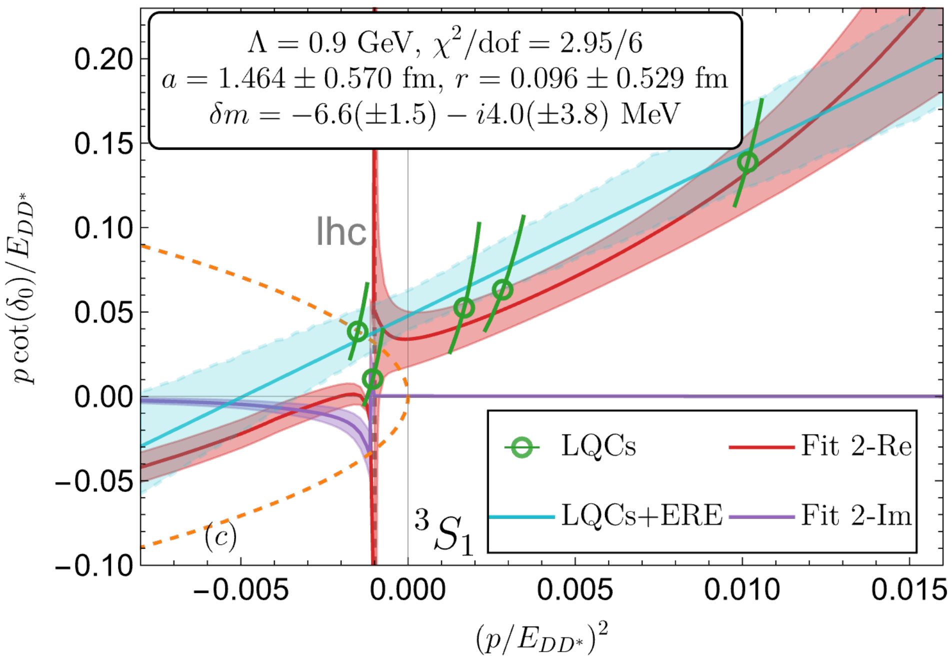


Including 3P2 term



Amplitude zero and left-hand cut

LO $$\frac{m^2}{4k^2}\log(1+\frac{4k^2}{m^2}) = 1+\frac{m^2}{\tilde{g}}$$



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Higher order $$\frac{m^2}{4k^2}\log(1+\frac{4k^2}{m^2}) = 1+\frac{\tilde{n}(k^2)m^2}{g}$$