# Hamiltonian Chiral Fermions and Lattice Chiral Gauge Theories

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There is no nonperturbative construction of the Standard Model.

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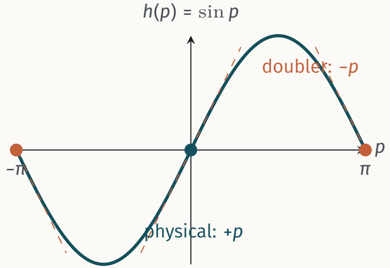
- At low energies:

$$h(q) = \sin(q) = +q + O(q^2), \quad h(\pi+q) = -q + O(q^2)$$




$\sin p$ vanishes at $p = 0$ *and* $p = \pi$, with *opposite* slopes.

Extra massless particle with **opposite chirality**!

# Two faces of the chiral fermion problem

## The “easy” problem

A **global** chiral symmetry, with a ’t Hooft anomaly. Think **QCD**.

- Perfectly fine as a global symmetry (it just cannot be gauged).
- Physical: the $\pi^0 \to \gamma\gamma$ rate.
- **Solved** in **Euclidean**: overlap / domain-wall realize it *exactly*.
- **Minkowski / Hamiltonian**: problems remain.

## The “hard” problem

A **gauged** chiral symmetry: a chiral gauge theory. Think **electroweak**.

- Anomaly cancels, then the chiral symmetry is gauged.
- No known nonperturbative lattice construction.
- We don’t yet know how to even *define* it.

’t Hooft anomaly: obstruction to gauge symmetry

# A no-go theorem

- $\sin p$ just the simplest choice: can a cleverer $h(p)$ avoid the doubler?
- **Any** $H = \sum_p \psi_p^\dagger\, h(p)\, \psi_p$ with

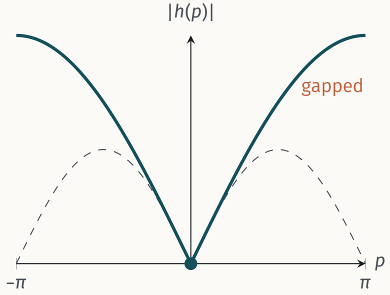
  **locality** $\Rightarrow$ $h(p)$ analytic, $2\pi$-periodic

  **right continuum limit** $\Rightarrow$ $h(p) \sim +p$ near $p = 0$
- Periodic + rises through zero $\Rightarrow$ **must come back down**: $h(p) \sim -p$ somewhere, a doubler of opposite chirality. Topology: chiralities cancel over the Brillouin zone

**Nielsen–Ninomiya** [Nielsen, Ninomiya 1981]: no lattice fermion with all of
*locality* • *no doublers* • correct continuum limit • exact $\{D, \gamma_5\} = 0$

# Wilson's fix and its cost

- **Wilson**: give up exact chiral symmetry [Wilson 1977]
- **Wilson term**: momentum-dependent mass $F(p) = \sum_\mu (1 - \cos p_\mu)$; $F(0) = 0$, $F(\pi) \sim 1$
- Physical mode untouched; doublers massive $\sim 1/a \Rightarrow$ **decouple**
- Price: $\{D, \gamma_5\} \neq 0$ **explicitly**, restored only as $a \to 0$




Doubler at $p = \pi$ is lifted; the physical mode at $p = 0$ stays gapless.

Wilson trades the doublers for a **broken** chiral symmetry.

# Is there a better way?

There are infinitely many ways to discretize a fermion.
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# Is there a better way?

There are infinitely many ways to discretize a fermion.
Wilson's is just one, and it sacrifices chiral symmetry completely.

So: **can we keep an *exact* chiral symmetry on the lattice, with no doublers, at finite lattice spacing?**

Remarkably, yes. The key is to ask for a *smarter* version of $\{D, \gamma_5\} = 0$.

# The Ginsparg–Wilson relation (1982)

- Nielsen–Ninomiya forbids exact $\{D, \gamma_5\} = 0$: what is the **mildest** relaxation?
- RHS **local**, vanishing in the continuum:

## The Ginsparg–Wilson relation

$$\{D, \gamma_5\} \;=\; a\, D\, \gamma_5\, D$$

- Breaking: $O(a)$, **local**

Dormant ~ 15 years: no $D$ satisfying it. Late 90s: **overlap** and **domain wall**

Ginsparg, Wilson 1982

# A genuine lattice chiral symmetry (Lüscher)

- GW relation looks like a cheat: but it hides an **exact** symmetry
- **Modified chirality**:

$$\hat{\gamma}_5 = \gamma_5(\mathbb{1} - aD), \qquad \hat{\gamma}_5 \xrightarrow{a \to 0} \gamma_5$$

- GW $\iff$ action invariant under

$$\delta\psi = \hat{\gamma}_5\, \psi, \qquad \delta\bar{\psi} = \bar{\psi}\, \gamma_5$$

**Exact** chiral symmetry at finite $a$, generated by $\hat{\gamma}_5$:
no fine-tuning, correct anomaly, no doublers

Lüscher 1998

# Solution 1: the overlap operator

- Explicit GW solution, built from the Wilson operator $H_{\mathrm{w}}$:

## Overlap Dirac operator

$$D_{\mathrm{ov}} = \mathbb{1} + \gamma^5\,\varepsilon(H_{\mathrm{w}}), \qquad \varepsilon(X) = \frac{X}{\sqrt{X^\dagger X}}, \qquad H_{\mathrm{w}} = \gamma^5 D_w$$

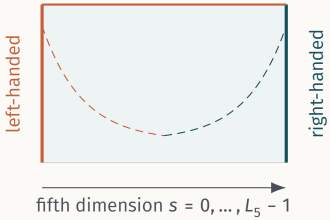
- **Matrix sign function** the whole story: $\varepsilon(H_{\mathrm{w}})^2 = \mathbb{1} \Rightarrow$ GW holds
- **Price**: $\varepsilon(H_{\mathrm{w}})$ *nonlocal*, couples all sites to all sites

In practice $\varepsilon(H_{\mathrm{w}})$ is always *approximated*: hold that thought

Neuberger 1998

# Solution 2: domain-wall fermions

- **Extra dimension**, extent $L_5$
- $(d{+}1)$-dim Wilson fermion + mass *defect* $\Rightarrow$ **chiral zero modes on the boundaries**:
  - left wall $\rightarrow$ left-handed
  - right wall $\rightarrow$ right-handed
- $L_5 \rightarrow \infty$: walls decouple $\Rightarrow$ **exact** chiral symmetry, local in $d$ dim
- Price: **extra qubits/sites**




Boundary modes, exponentially localized.

Kaplan 1992, 1993

## The two are secretly the same operator

- Integrate out the bulk $\Rightarrow$ **boundary** theory = overlap fermion, exactly
- Finite $L_5$: approximate sign function

$$\varepsilon(H_\mathrm{w}) \longrightarrow \tanh(L_5\, H_\mathrm{w}),$$

  exact as $L_5 \to \infty$

- Fifth dimension = **a tool to approximate $\varepsilon(H_\mathrm{w})$**

**Overlap** = exact sign function, non-local, few sites.
**Domain-wall** = approximate sign function via $L_5$, local, many sites.

Narayanan, Neuberger 1993 • Kikukawa, Noguchi 1999

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A quantum computer evolves a **Hamiltonian** in real time.

**What happens to the exact chiral symmetry there?**

# For a quantum computer: go Hamiltonian

- Quantum computer: real-time evolution, $|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$
- Need a **Hamiltonian**, not a Euclidean action:

$$S = \sum \psi D \psi \longrightarrow H = \sum \psi^\dagger h \psi$$

- Wilson Hamiltonian (ungauged, single-particle):

$$h_{\mathrm{w}} = \sum_i i\,\Gamma^i \delta_i + \Gamma^0\left(m - \tfrac{r}{2}\Delta\right)$$

  **local**, **gapped** (smallest |eigenvalue| = $m$)

- **Domain wall**: Wilson in $d$+1 dim ⇒ honest local Hamiltonian $h_{\mathrm{dw}}$

Domain wall: local, obvious. Overlap Hamiltonian: what *is* it?

# The overlap Hamiltonian (Creutz–Horvath–Neuberger 2002)

- Single-particle ansatz, built from the Wilson $h_{\mathrm{W}}$:

## Overlap Hamiltonian

$$h_{\mathrm{OV}} = \Gamma^0 D_{\mathrm{OV}} = \Gamma^0 + \varepsilon(h_{\mathrm{W}}), \qquad \varepsilon(h_{\mathrm{W}}) = h_{\mathrm{W}} / \sqrt{h_{\mathrm{W}}^\dagger h_{\mathrm{W}}}.$$

- $0 < m < 2r$: a **single massless Dirac fermion**, no doublers
- **Exact** chiral symmetry, from a **modified chirality**:

$$Q_5 = \psi^\dagger \, \hat{\gamma}_5 \, \psi, \qquad \hat{\gamma}_5 = \tfrac{1}{2}\gamma_5\big(1 + \Gamma^0 \varepsilon(h_{\mathrm{W}})\big), \qquad [\, \hat{\gamma}_5, \, h_{\mathrm{OV}} \,] = 0$$

- $\hat{\gamma}_5 \to \gamma_5$ in the continuum; finite $a$: the GW-deformed chirality

**Exact** lattice chiral symmetry, at the price of the nonlocal $\varepsilon(h_{\mathrm{W}})$

# Why the overlap Hamiltonian is subtle

- Overlap born **Euclidean**: a kernel in $\bar{\psi} D_{\mathrm{ov}} \psi$; Hamiltonian: Hermitian generator of real time. **No canonical recipe** between the two
- Domain wall: local in $(d{+}1) \Rightarrow$ transfer matrix $\Rightarrow$ Hamiltonian for free. Overlap analogue: **open, active question** [ Clancy 2024 • HS 2025 • Chattejee, Pace, Shao 2024 • Gioia, Thorngren 2025, • Misumi 2025 ]
- One proposal, $h_{\mathrm{ov}} = \Gamma^0 + \varepsilon(h_{\mathrm{w}})$: extra dimension traded for **all-to-all** interactions, no ultralocality [Creutz, Horvath, Neuberger 2002]
- Chiral charge **dynamical**: $\hat{\gamma}_5 = \frac{1}{2}\gamma_5(1 + \Gamma^0 \varepsilon(h_{\mathrm{w}}))$ *depends on the Hamiltonian*, unlike the fixed $\gamma_5$

# Compact vs. noncompact

## Euclidean

$$\psi \to e^{i\theta\hat{\gamma}_5}\psi, \quad \bar{\psi} \to \bar{\psi}\, e^{i\theta\gamma_5}$$

$\hat{\gamma}_5 = \gamma_5(1 - aD)$, $\hat{\gamma}_5^2 = 1$: eigenvalues $\pm 1$

$\theta$ a $2\pi$-periodic angle $\Rightarrow$ **compact** $U(1)$

Anomaly in the measure: $\mathrm{Tr}\,\hat{\gamma}_5 = \text{index} \in \mathbb{Z}$

## Hamiltonian

$$Q_5 = \psi^\dagger\, \hat{\gamma}_5\, \psi, \quad \hat{\gamma}_5 = \tfrac{1}{2}\gamma_5(1 + \Gamma^0 \varepsilon(h_{\mathrm{w}})))$$

$\hat{\gamma}_5^2 \neq 1$: spectrum *not* $\pm 1$

$e^{i\theta Q_5}$ **never closes up** $\Rightarrow$ **noncompact** $\mathbb{R}$

The compact Euclidean chiral $U(1)$ becomes a **noncompact** $\mathbb{R}$ in the Hamiltonian

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First we need a few tools.

## What a quantum computer actually does for us

- $n$ qubits: state in $\mathbb{C}^{2^n}$, **exponentially large**
- Machine applies **unitary** gates; target:

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle$$

- $H$: a $2^n \times 2^n$ matrix never written down, only *acted* with

### The core challenge

$H$ **Hermitian, not unitary**; the machine applies only *unitaries*.

How to apply a function of $H$: $e^{-iHt}$, or $\varepsilon(H)$?

# First instinct: just Trotterize?

- Standard recipe: split $H = \sum_k H_k$,

$$e^{-iHt} \approx \left(\prod_k e^{-iH_k t/n}\right)^n$$

- **Local** *H*: shallow, local circuits, few ancillas

## Falls short for exactly-chiral fermions

- **Precision costly**: error *polynomial* in $1/\epsilon$, vs optimal $t + \log\frac{1}{\epsilon}$
- **Nothing to split**: $\varepsilon(h_{\mathrm{w}})$ *all-to-all*
- **Breaks the symmetry**: each step adds violation, no single knob

## Why quantum signal processing?

One method fixes all three at once: **QSP**

- **Optimal**: cost ~ $t + \log \frac{1}{\epsilon}$, **exponentially** better than Trotter, matches the lower bound
- **Functions of $H$, directly**: *any* polynomial $p(H)$, incl. $\varepsilon(h_{\mathrm{w}})$: nonlocality a **feature**, not an obstacle
- **One knob**: degree $M$ dials the GW violation $\epsilon_e$

One framework: $e^{-iHt}$ *and* $\varepsilon(h_{\mathrm{w}})$, optimal, controlled chiral symmetry

# Idea 1: hide $H$ inside a unitary ("block encoding")

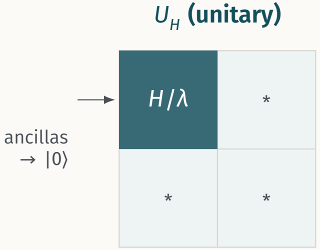
- Embed $H$ (rescaled) as the **top-left corner** of a unitary $U_H$, via **ancillas**:

$$U_H = \begin{pmatrix} H/\lambda & * \\ * & * \end{pmatrix}$$

- Ancillas in $|0\rangle$, apply $U_H$, ancillas out in $|0\rangle$:

$$\langle 0^a | \, U_H \, | 0^a \rangle \;=\; H/\lambda$$

- *Local* Wilson $h_{\mathrm{w}}$: cheap, few unitaries (shifts × $\gamma$-matrices), cost $O(Q)$




Low, Chuang 2019 • Gilyén et al. 2019

## Idea 2: quantum signal processing (QSP)

- **Interleave** $U_H$ with ancilla phase rotations $\phi_1, \dots, \phi_M$:

$$U_H \quad \to \quad R(\phi_M)\, U_H\, R(\phi_{M-1}) \cdots U_H\, R(\phi_1)$$

- Magic: the circuit block-encodes a **polynomial** of $H$:

$$\langle 0|\,(\text{circuit})\,|0\rangle \;=\; p(H), \qquad \deg p = M$$

- Phases $\{\phi_k\} \Rightarrow$ *any* polynomial ($|p| \leq 1$)

A **"polynomial-of-$H$ machine"**:
degree-$M$ polynomial = $M$ uses of $U_H$

Low, Chuang 2019 • Gilyén et al. 2019

# Two polynomials we want

## Time evolution

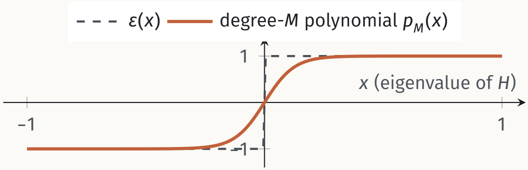
$e^{-iHt} \approx \sum_{k=0}^{M} c_k\, T_k(H)$, degree $M = O(\lambda t + \log \frac{1}{\epsilon})$.

**Asymptotically optimal** simulation.

## The sign function

$\varepsilon(H) \approx p_M(H)$, degree $M = O(\kappa^{-1} \log \frac{1}{\epsilon})$ ($\kappa$ = gap).

**Exactly what overlap fermions need.**



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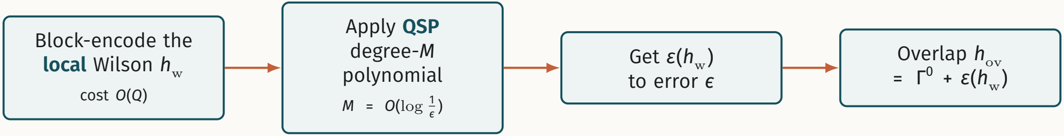
**Can we assemble them into a real algorithm for chiral fermions?**

# The recipe for overlap fermions




- Nonlocal $\varepsilon(h_{\mathrm{w}})$ **never built by hand**: QSP manufactures it from the cheap local $h_{\mathrm{w}}$
- Time evolution $e^{-iH_{\mathrm{ov}}t}$: standard QSP
- Gauge fields: shifts → *covariant* shifts $A_i \to A_i\, U_{(x,i)}$; scaling unchanged

# From one particle to the many-body Hamiltonian

- So far: **single-particle** $h$, a $Q \times Q$ matrix. Promote to

$$q = \lceil \log_2 Q \rceil = O(\log Q) \text{ qubits}$$

All sign-function work in this **exponentially compressed** register

- Evolve the **many-body** $H$ on Fock space:

$$H = \psi^\dagger h \, \psi \qquad \text{on } Q \text{ qubits (Jordan–Wigner)}$$

- Standard wrapping of the block encoding:

$$O(Q) \text{ gates}, \quad O(\log Q) \text{ ancillas}$$

Sign function: built on $\log Q$ qubits.
Second quantization: the $O(Q)$ factor = linear system-size scaling

## What does it cost?

- $h_{\mathrm{w}}$ **gapped** (smallest |eigenvalue| = $m$) $\Rightarrow \kappa = O(1)$:

$$M = \mathcal{O}\left(\log \frac{1}{\epsilon}\right) \quad (\textit{logarithmic} \text{ in accuracy})$$

- One QSP step = one $O(Q)$ block encoding $\Rightarrow \varepsilon(h_{\mathrm{w}})$ costs

$$\mathcal{O}\left(Q \log \frac{1}{\epsilon}\right)$$

- Full real-time evolution:

$$\mathcal{O}\left(\underbrace{Q \log \frac{1}{\epsilon_e}}_{\text{sign fn}} \cdot \left(Qt + \log \frac{1}{\epsilon_t}\right)\right)$$

$Q$ = fermionic modes; $\epsilon_e$ = sign-function error, $\epsilon_t$ = time error

sign-function scaling: Low, Chuang 2017 • Gilyén et al. 2019

# The chiral symmetry is broken only *softly*

- Truncating to degree $M$ means $p_M(h_{\mathrm{w}})$ is *not exactly* a sign function:

$$\|p_M(h_{\mathrm{w}})^2 - \mathbb{1}\| \;\le\; 2\epsilon_e.$$

- One short computation (anticommutators of $\gamma$'s) turns this into a bound on the **Ginsparg–Wilson violation**:

$$\left\| \{D, \gamma_5\} - D\gamma_5 D \right\| \le 2\epsilon_e \qquad \Longrightarrow \qquad \left\| [\, \hat{\gamma}_5,\, h \,] \right\| \le 2\epsilon_e.$$

- The exact chiral symmetry is recovered **exponentially fast**: $\epsilon_e$ is what we dial with the polynomial degree $M$.

> Chiral symmetry violation is **controlled directly** by the QSP error $\epsilon_e$, and costs only $\log(1/\epsilon_e)$ to shrink.

# And domain-wall fermions?

- Domain-wall stays a **local** Hamiltonian in $d+1$ dimensions: no sign function to approximate.
- Locality is a real asset on a quantum computer
- But you pay in **memory**: every one of the $Q$ modes is copied $L_5$ times.
- The residual GW violation is set by the wall separation: $\sim e^{-cL_5}$.

Two knobs ($M$ for overlap, $L_5$ for domain-wall), both controlling the same chiral-symmetry violation. Coincidence?

local simulation: Haah et al. 2018 • Rhodes et al. 2024

We now have *two* chiral-fermion algorithms: overlap and domain-wall.

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**Which is better, and what does comparing them tell us about the physics?**

## The scoreboard

| | **Overlap** | **Wilson** | **Domain-wall** |
|---|---|---|---|
| *Qubits (memory)* | $Q$ | $Q$ | $Q\,L_5$ |
| *Gates: block-encode h* | $Q\log\frac{1}{\epsilon_e}$ | $Q$ | $Q\,L_5$ |
| *Gates: time evolution* | $Q\log\frac{1}{\epsilon_e}\,(Qt+\log\frac{1}{\epsilon_t})$ | $Qt\ \log\frac{Qt}{\epsilon_t}$ | $QL_5t\ \log\frac{QL_5t}{\epsilon_t}$ |
| *Chiral (GW) violation* | $\epsilon_e$ | $O(1)$ | $e^{-cL_5}$ |

Wilson & domain-wall scalings: Rhodes et al. 2024

- **Wilson**: cheapest, but chiral symmetry broken at $O(1)$.
- **Overlap**: fewest qubits, deeper circuits (the $\log\frac{1}{\epsilon_e}$ overhead).
- **Domain-wall**: shallow local circuits, but a factor $L_5$ in memory.

Stare at the last row.

## Reading the table: $M \leftrightarrow L_5$

- Domain-wall GW violation ~ $e^{-cL_5}$. Overlap GW violation ~ $\epsilon_e$.
- Set them equal:

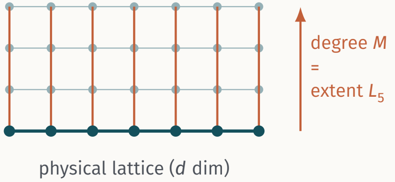
$$\boxed{\epsilon_e \sim e^{-cL_5} \qquad \Longleftrightarrow \qquad L_5 \sim \log\frac{1}{\epsilon_e} \sim M}$$

- The QSP **polynomial degree** $M$ plays *exactly* the role of the domain-wall **fifth-dimension extent** $L_5$:
  - both control the chiral-symmetry violation,
  - both add the *same* logarithmic cost,
  - both interpolate between local and fully non-local.

When QSP approximates $\varepsilon(h_{\rm w})$ with a degree-$M$ polynomial,
it is **secretly reconstructing the extra dimension** of domain-wall fermions.

# Why this is the *same* extra dimension

- $h_w$ is **nearest-neighbor** (sparsity $s \sim d$).
- A degree-$M$ polynomial $p_M(h_w)$ reaches at most $M$ sites away: sparsity $\sim d\,M$.
- So increasing $M$ literally **spreads the operator's support** by $M$ lattice spacings.
- Domain-wall: hopping $L_5$ steps along the fifth dimension spreads the boundary operator by $L_5$ spacings. **Same picture.**




$M = 1$: local Wilson. $M \to \infty$: fully non-local overlap.

> Overlap *traded the extra dimension for all-to-all interactions*;
> QSP's all-to-all polynomial **is** that dimension, grown back.

# This is the quantum version of an old story

- Classical lattice QCD has wrestled with $\varepsilon(H_{\mathrm{w}})$ for decades: [review: Blum, Shamir 2026]
  - rational (Zolotarev) approximations for overlap,
  - the Möbius / truncated domain-wall formulations.
- They all hit the *same* scaling $M = O(\kappa^{-1} \log \frac{1}{\epsilon_e})$, with the approximation order playing the role of $L_5$.
- **QSP provides the unifying quantum analog**: the polynomial degree is the single knob, the quantum analog of $L_5$

The "overlap = boundary of domain-wall" equivalence re-emerges as a statement about *quantum algorithms*.

## Can we put a chiral <u>gauge</u> theory on the lattice at all?

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# Part 2: The Hard Problem

## The hard problem: where things stand

- "Easy" problem essentially solved by **Ginsparg–Wilson** fermions (overlap / domain wall)
- **Lüscher**: modified chiral symmetry of GW fermions ⇒ nonperturbative construction of **abelian** chiral gauge theories [Lüscher 1999, 2000]
- **Nonabelian** problem: open
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- **Lots of recent activity**:
  - Kaplan–Sen disk model: domain wall + gradient flow ▸ *see Dang*
  - Symmetric mass generation ▸ *see Hasenfratz, Marinković; and Catterall's plenary tomorrow*
  - Anomalies in staggered fermions ▸ *see Aoki, Yamaoka*
  - Bosonization, in 2d and higher ▸ *see Berkovitz, Onoda*

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**Are we now at a point where we can finally simulate a chiral gauge theory?**

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  **Solution**: qubit regularizations ("digitizations"), inspired by quantum computing

**Let’s consider the simplest chiral gauge theory.**

# The simplest chiral gauge theory: 3-4-5-0

- The **simplest** chiral gauge theory? A $U(1)$ in 1+1d
- Gauge anomaly $\Rightarrow$ **Pythagorean triples**
- **gravitational** anomaly $\Rightarrow$ one **neutral** Weyl: the **3-4-5-0 model**

| left-movers | right-movers |
| --- | --- |
| $\longleftarrow \quad \psi^{(3)} \mapsto e^{3i\alpha}\,\psi^{(3)}$ | $\psi^{(5)} \mapsto e^{5i\alpha}\,\psi^{(5)} \quad \longrightarrow$ |
| $\longleftarrow \quad \psi^{(4)} \mapsto e^{4i\alpha}\,\psi^{(4)}$ | $\psi^{(0)} \mapsto \psi^{(0)} \quad \longrightarrow$ neutral |

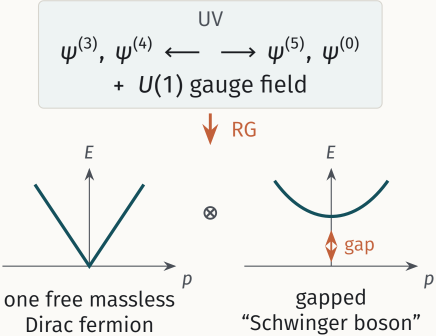
gauge anomaly: $\sum q_L^2 = \sum q_R^2$: $3^2 + 4^2 = 5^2 + 0^2$

gravitational anomaly: $n_L = n_R$: $2 = 2$

This is the UV. What is the IR of this model?

# Bosonization solves the model

- 1+1d **bosonization** solves the continuum theory: exact IR known
- For decades, **the** benchmark for lattice tests of chiral gauge theories
- But never been successfully demonstrated on the lattice with any technique



New understanding in implementing anomalous bosonic symmetries on the lattice.

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**Can we make progress with new ideas:**
**exact anomalies, tensor networks, qubit models?**

## Latticizing the bosonization

- Bosonization is **exact** in 1+1d: so latticize the **bosonic** theory, not the fermions
- Recently, an exactly solvable rotor model of 3-4-5-0, chiral gauge invariance exact at finite spacing

continuum **3-4-5-0**
four Weyl fermions
+ $U(1)$ gauge field

Berkowitz, Cherman, Jacobson 2025 • Seifnashri 2026

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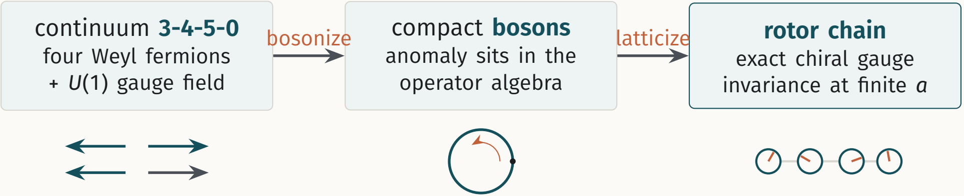


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**rotor chain**
exact chiral gauge
invariance at finite $a$

**problem:** rotors are **infinite-dimensional**:
tensor networks and quantum computers need a finite Hilbert space

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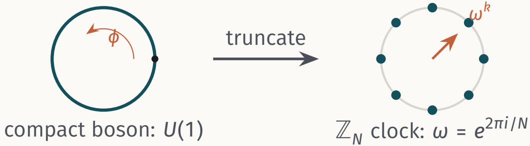
**Is there a truncation (qubit-regularization) of this chiral lattice gauge theory, with the anomaly structure preserved exactly?**

# An $\mathbb{Z}_N$ lattice chiral gauge theory

- Solution: Truncate the rotors to a **finite-dimensional** $\mathbb{Z}_N$ clock model; $N \to \infty$ targets $U(1)$



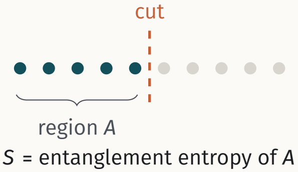

$$
\begin{array}{rcl}
e^{i\phi_j} & \longrightarrow & Z_j \quad (Z^N = \mathbb{1}) \\
e^{2\pi i L_j/N} & \longrightarrow & X_j \quad (XZ = \omega\, ZX) \\
\text{charges } q \in \mathbb{Z} & \longrightarrow & q \mod N \\
\textbf{anomaly cancellation: } \sum q_L^2 = \sum q_R^2 & \longrightarrow & \sum q_L^2 \equiv \sum q_R^2 \mod N
\end{array}
$$

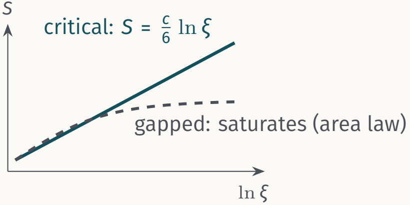
Anomaly cancellation with **no uncontrolled approximation**.

Lootens, HS 2026

# What we measure: entanglement entropy

- Bipartition the chain: MPS methods hand us the **entanglement entropy** $S$ for free
- How $S$ scales with the correlation length $\xi$ **diagnoses the IR**: **gapped** $\Rightarrow$ **area law**, $S$ saturates; **critical** $\Rightarrow$ $S$ grows with $\ln \xi$

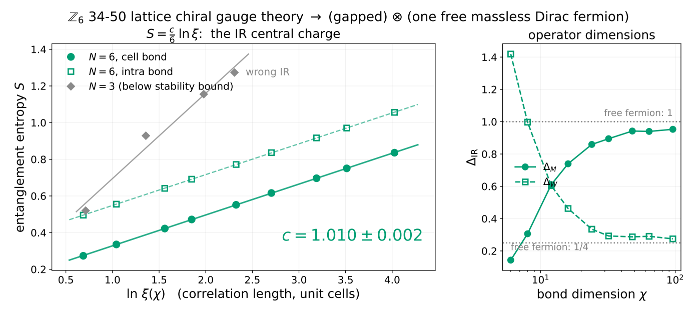



The slope of $S$ vs $\ln \xi$ is the **central charge**:

$c = 1 \iff$ one free massless Dirac fermion in the IR.

# Simulating the 3-4-5-0 model



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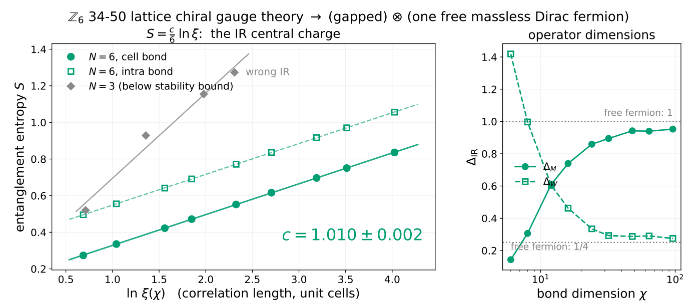


A chiral lattice gauge theory, simulated **for the first time**:
the predicted IR emerges, a free massless Dirac fermion.
**Universality: Finite *N*=6 is enough**

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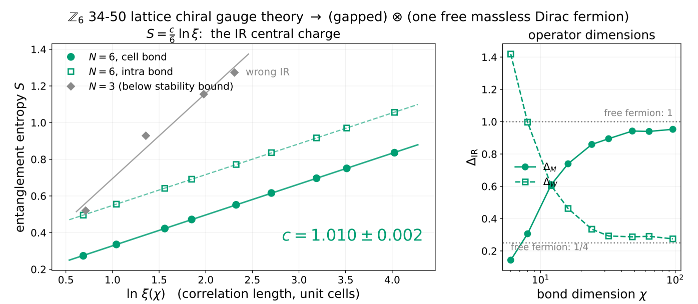


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**Thank you.**

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