Phase diagram of the single-flavor Gross–Neveu–Wilson model from the Grassmann corner transfer matrix renormalization group

Jian-Gang Kong [a], Shinichiro Akiyama [b], [c], Tao Shi [d], Z. Y. Xie [a]

[a] Renmin University of China,

[b] CCS, University of Tsukuba, [c] The University of Tokyo

[d] Chinese Academy of Science

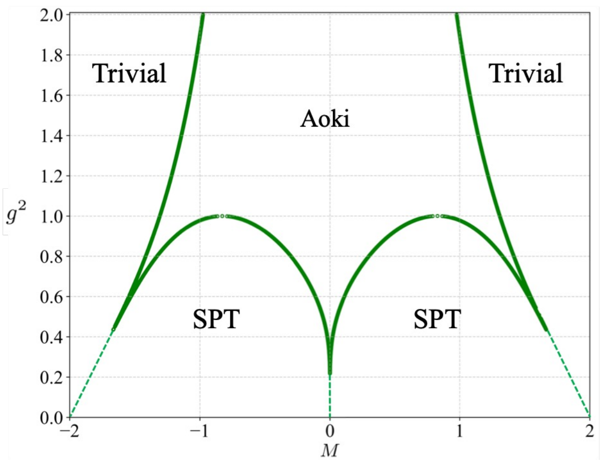
Jian-Gang Kong, SA, Tao Shi, Z. Y. Xie, PRD113(2026)114504

2026.7.30 @ Lattice 2026, University of Maryland

Gross–Neveu model w/ Wilson fermions

- **A toy model of QCD w/ Wilson fermions**
  - Massless pseudoscalar mesons can emerge due to the parity-flavor symmetry breaking in the lattice QCD w/ Wilson fermions

    Aoki, PRD30(1984)2653
    Sharpe–Singleton, PRD58(1998)074501
  - Precise determination of the boundary of Aoki phase in lattice QCD is vital to take the continuum limit
- **Aoki phase in the large-N_f Gross–Neveu model w/ Wilson fermions**
  - $$S=-\frac{1}{2}\sum_{f=1}^{N_f}\sum_{n\in\Lambda_2}\sum_{\nu=1,2}\left[\bar{\psi}^{(f)}(n)(r\mathbb{1}-\gamma_\nu)\psi^{(f)}(n+\hat{\nu})+\bar{\psi}^{(f)}(n+\hat{\nu})(r\mathbb{1}+\gamma_\nu)\psi^{(f)}(n)\right]$$
    $$+M\sum_{f,n}\bar{\psi}^{(f)}(n)\psi^{(f)}(n)-\frac{g_\sigma^2}{2N_f}\sum_n\left(\sum_f\bar{\psi}^{(f)}(n)\psi^{(f)}(n)\right)^2-\frac{g_\pi^2}{2N_f}\sum_n\left(\sum_f\bar{\psi}^{(f)}(n)\mathrm{i}\gamma_5\psi^{(f)}(n)\right)^2$$



- **'t Hooft anomaly matching condition requires the Aoki phase at M=0, even for finite N_f**

  Misumi–Tanizaki, PTEP2020(2020)033B03
- (Cold-atom) quantum simulations have increased interest in the GN model

  Bermudez–Tirrito–Rizzi–Lewenstein–Hands, Ann. Phys. 399(2018)149
  Asaduzzaman–Catterall–Toga–Meurice, PRD106(2022)114515
- **Phase diagram w/ N_f=1?**

Grassmann tensor network

- **Path integrals involving fermions can be formulated as Grassmann TN**

Gu–Wen–Verstraete, arXiv: 1004.2563
Shimizu–Kuramashi, PRD90(2014)074503
Takeda–Yoshimura, PTEP2015(2015)043B01
SA–Kadoh, JHEP10(2021)188

  - Locality of the original theory is preserved in its Grassmann TN representation
  - Exact contraction is generally intractable, whereas approximate contraction is possible
    - Any TN algorithm can be extended to Grassmann TN Cf. Kong–Zhu–Xie, Ann. Phys. 491(2026)170511

Cf. Talk by E. Itou on Tuesday

- An indispensable tool for lattice field theory
- Tensor RG on Grassmann TN has been applied to investigate the Aoki phase in the N_f=1 Schwinger model Shimizu–Kuramashi, PRD97(2018)034502

OPEN ACCESS
IOP Publishing Journal of Physics: Condensed Matter
J. Phys.: Condens. Matter 36 (2024) 343002 (31pp) https://doi.org/10.1088/1361-648X/ad4760

Topical Review

Tensor renormalization group for fermions

Shinichiro Akiyama[1,2], Yannick Meurice[3,*] and Ryo Sakai[4]

[1] Center for Computational Sciences, University of Tsukuba, Tsukuba, Ibaraki 305-8577, Japan
[2] Graduate School of Science, The University of Tokyo, Bunkyo-ku, Tokyo 113-0033, Japan
[3] Department of Physics and Astronomy, The University of Iowa, Iowa City, IA 52242, United States of America
[4] Jij Inc., Bunkyo-ku, Tokyo 113-0031, Japan

E-mail: yannick-meurice@uiowa.edu and akiyama@ccs.tsukuba.ac.jp

Received 6 February 2024, revised 2 April 2024
Accepted for publication 3 May 2024
Published 28 May 2024

CrossMark

Abstract
We review the basic ideas of the tensor renormalization group method and show how they can be applied for lattice field theory models involving relativistic fermions and Grassmann variables in arbitrary dimensions. We discuss recent progress for entanglement filtering, loop optimization, bond-weighting techniques and matrix product decompositions for Grassmann tensor networks. The new methods are tested with two-dimensional Wilson–Majorana fermions and multi-flavor Gross–Neveu models. We show that the methods can also be applied to the fermionic Hubbard model in 1+1 and 2+1 dimensions.

Keywords: tensor networks, lattice gauge theory, relativistic lattice fermions, Fermi Hubbard model, Grassmann path integrals, sign problems

Contents

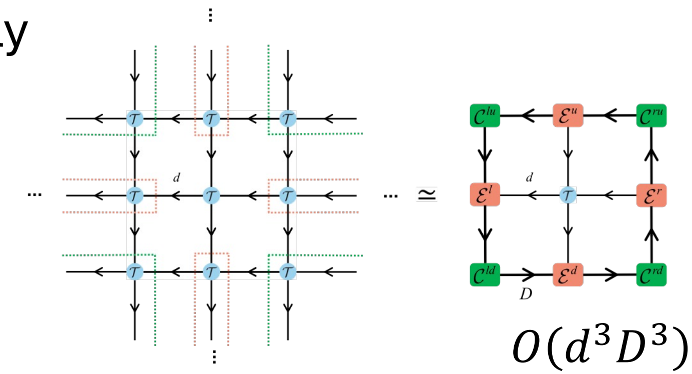
* Author to whom any correspondence should be addressed.

SA–Meurice–Sakai,
Journal of Physics: Condensed Matter 36 (2024) 343002

Corner Transfer Matrix RG

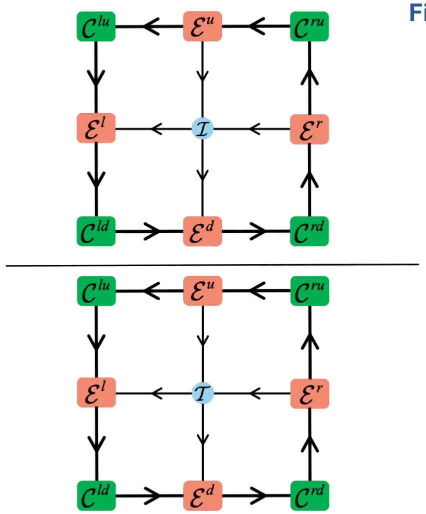
Baxter, J. Math. Phys. 9(1968)650, JPS19(1978)461
Nishino–Okunishi, JPSJ65(1996)891, JPSJ66(1997)3040

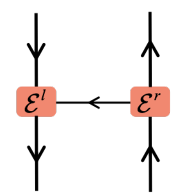
- **We determine the phase diagram for the N_f=1 model**
  - Formulating the path integral as a Grassmann TN, we use the corner transfer matrix RG (CTMRG) to evaluate it approximately
  - Thermodynamic quantities are directly available using effective environment tensors
  - Correlation length, entanglement entropy and spectrum can also be computed from transfer matrices



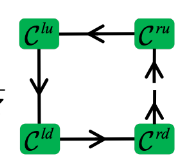
$O(d^3D^3)$

Corboz–Rice–Troyer, PRL113(2014)046402
Fishman–Venderstraeten–Zauner–Stauber–Haegeman–Verstraete, PRB98(2018)235148

$\langle\bar{\psi}(n)\mathrm{i}\gamma_5\psi(n)\rangle \simeq$ 



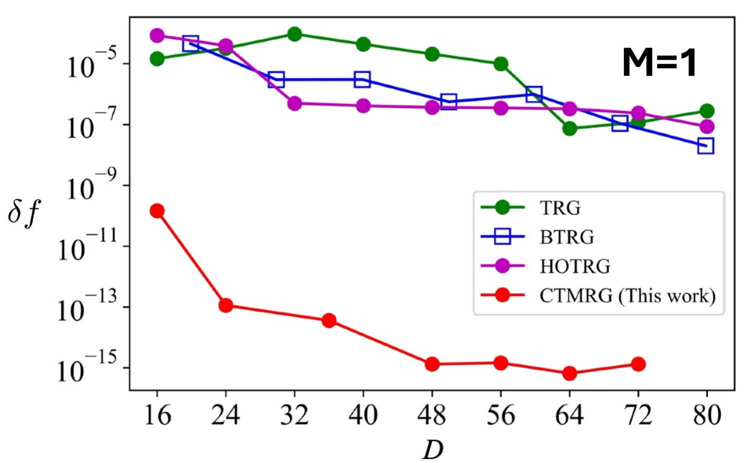
$\Rightarrow$ $$\frac{1}{\xi_D} = \ln\frac{\lambda_1}{\lambda_2}$$

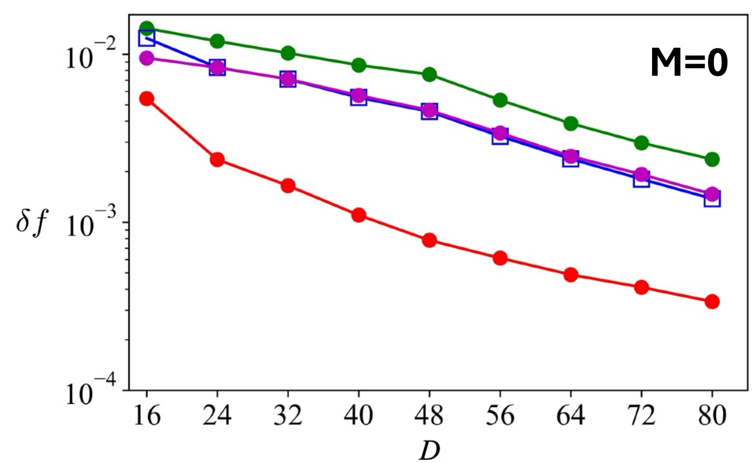
$\rho_D = \frac{1}{Z}$ 

$\Rightarrow$ $$S_D = -\mathrm{Tr}\left(\rho_D \log\rho_D\right) = -\sum_{i=1}^{D}\theta_i\log\theta_i$$

Benchmarking in the free theory

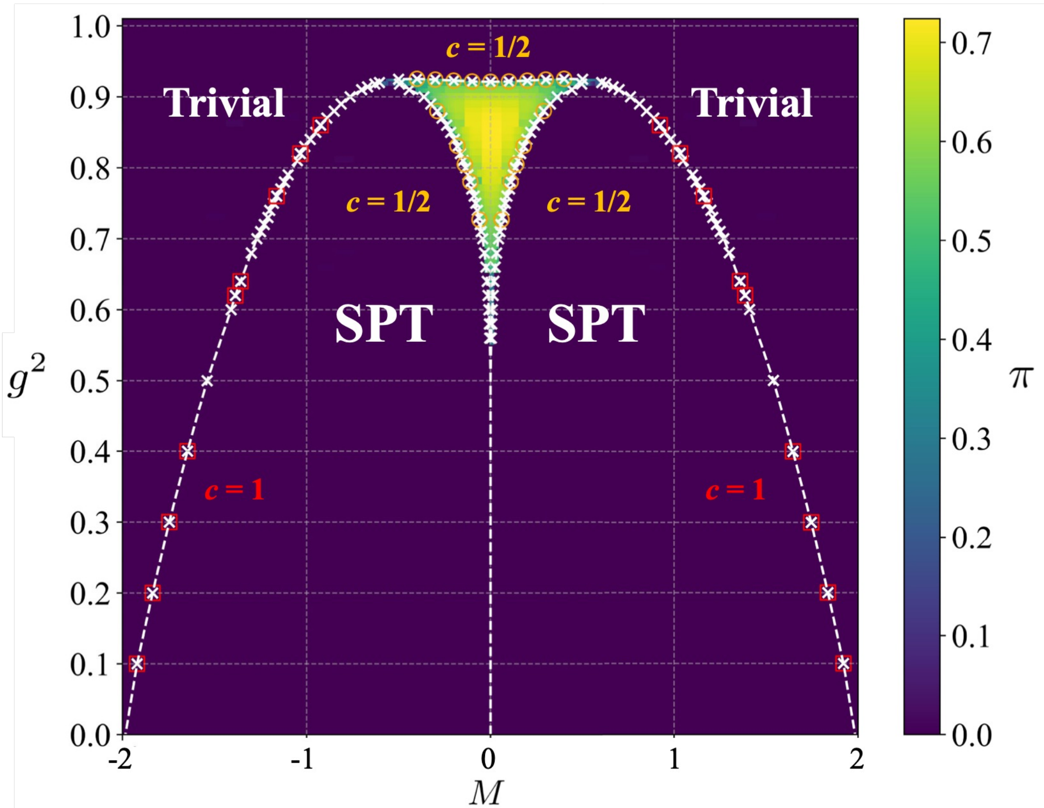
- **At a fixed bond dimension, CTMRG performs better than tensor RG**
  - Away from criticality, CTMRG reproduces the exact solution to nearly double-precision accuracy





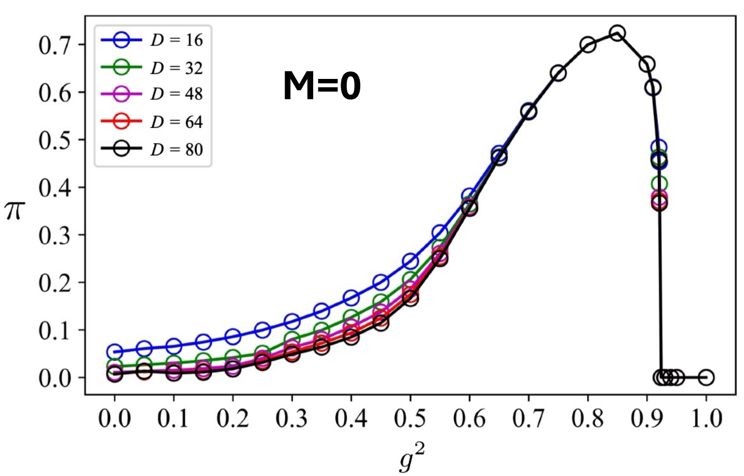
Phase diagram of the N_f=1model on a lattice

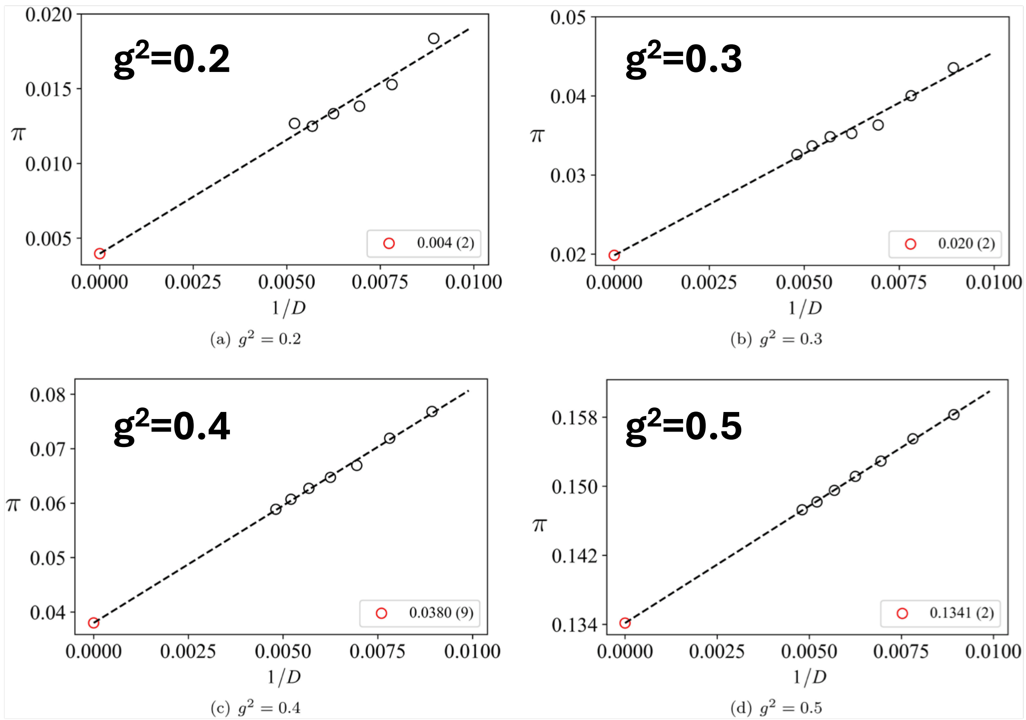
- As in the large-N_f, the model describes three phases
  - **In contrast to the large-N_f phase diagram, the Aoki phase does not persist in the strong-coupling regime when N_f=1**
  - In the following, we describe how we obtained this phase diagram



Aoki phase at M=0

- When g^2 is large, the finite-D effect is well suppressed
- **Finite-D effect is enhanced when g^2 is small**
  - The infinite-D limit w/ assuming π=a/D+C finds a finite pseudoscalar condensate

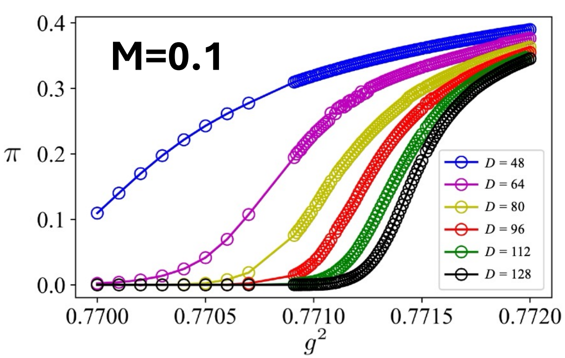


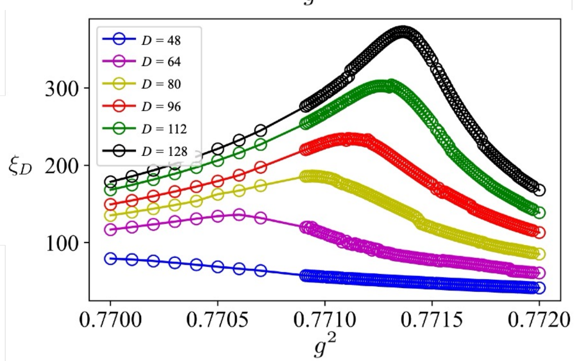


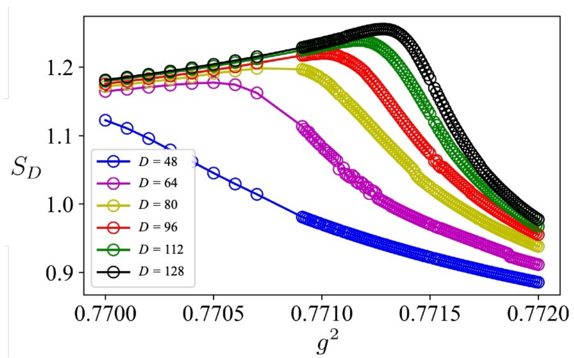
Phase boundaries of Aoki phase

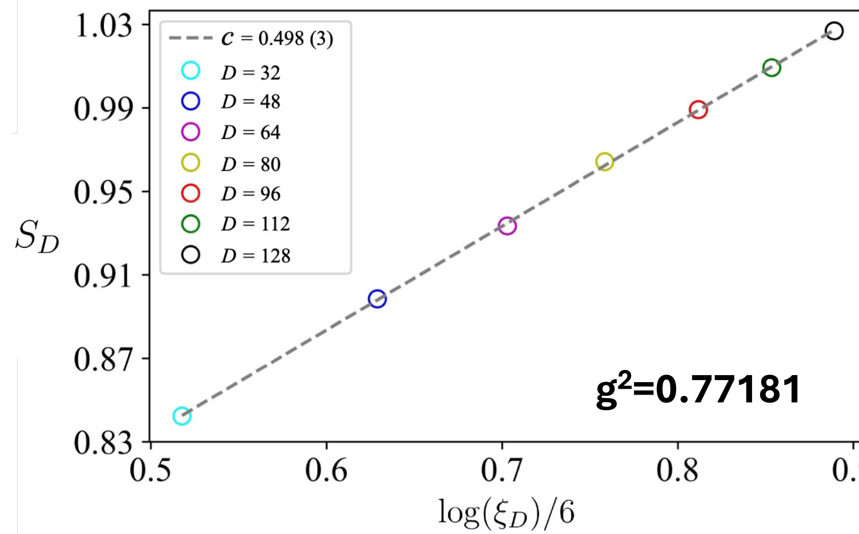
Cf. Tagliacozzo–Oliveira–Iblisdir–Latorre, PRB78(2008)024410
Pollmann–Mukherjee–Turner–Moore, PRL102(2009)255701

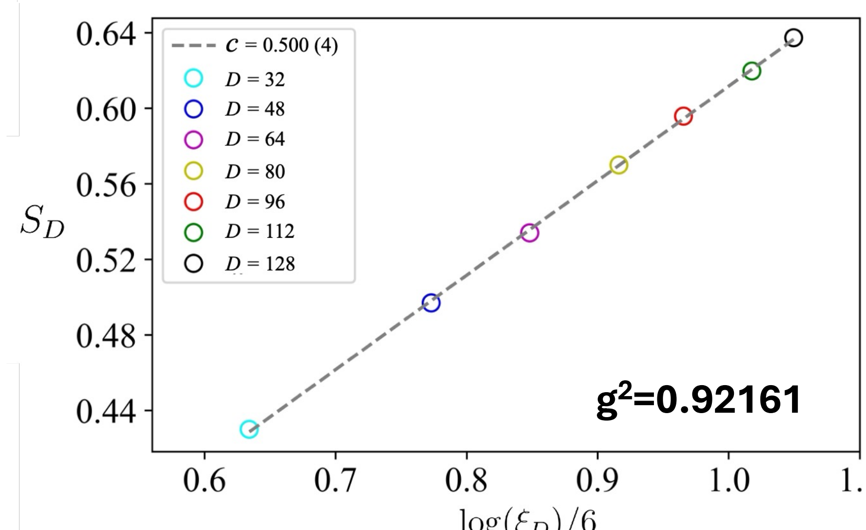
- **Finite-entanglement scaling finds the Ising criticality**











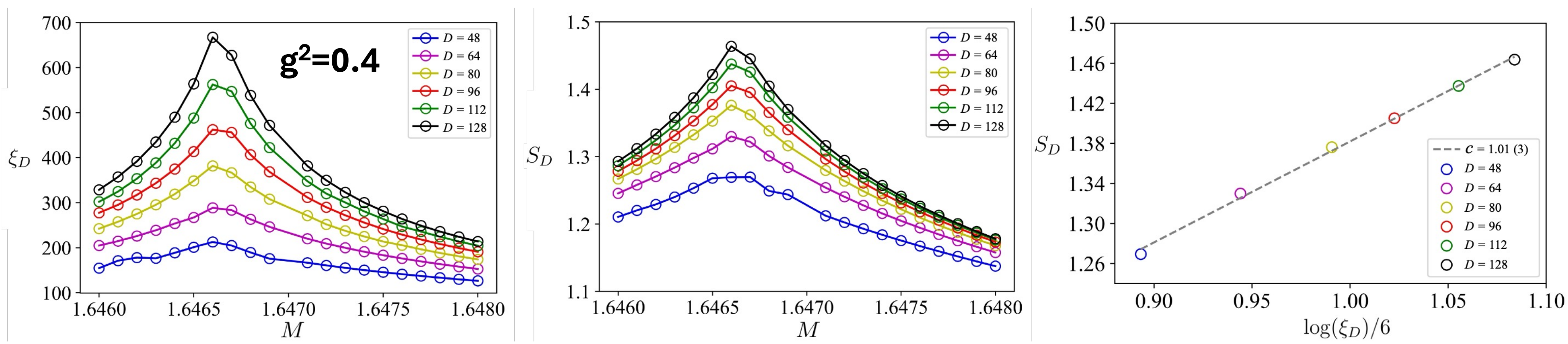
Another phase boundary

- Away from the M=0 line, the Aoki phase rapidly disappears
  - **Yet, the correlation length and entanglement entropy still show signatures of criticality w/ c=1**
  - This picture is consistent with an MPS simulation

Bermudez+, Ann. Phys. 399(2018)149

- The region inside the lobe is expected to be an SPT phase (BDI class)

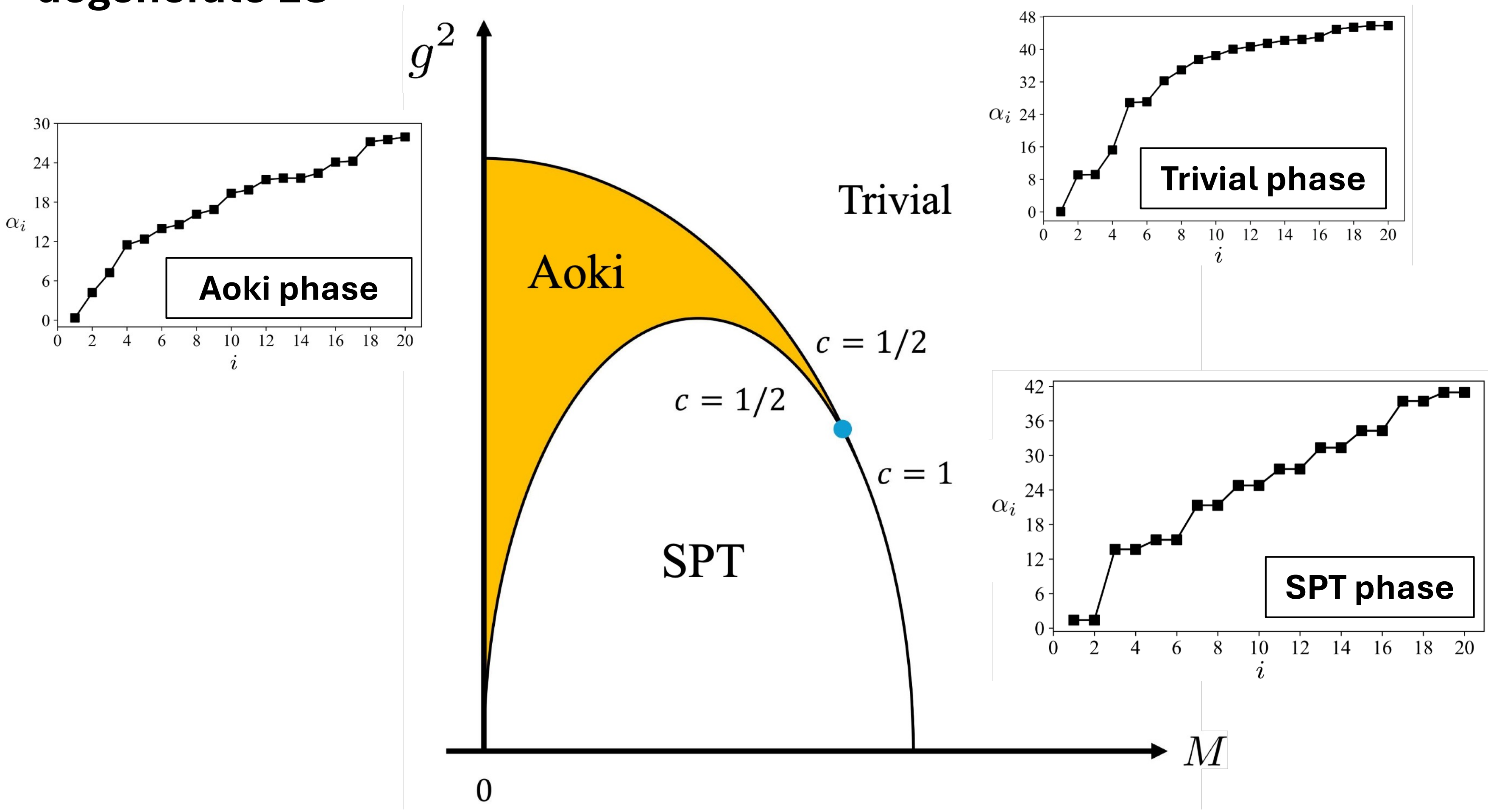
Cf. Kuno, PRB99(2019)064105



Signal of SPT

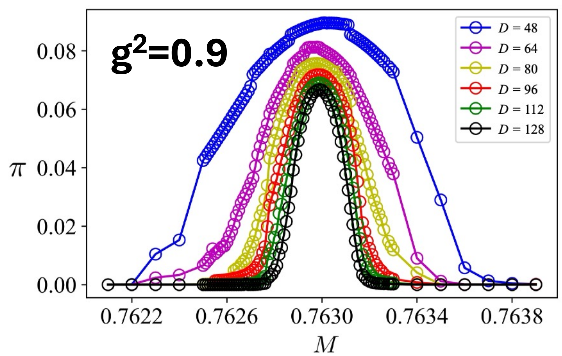
Cf. Pollmann–Turner–Berg–Oshikawa, PRB81(2010)064439
Jünemann–Piga–Ran–Lewenstein–Rizzi–Bermudez, PRX7(2017)031057

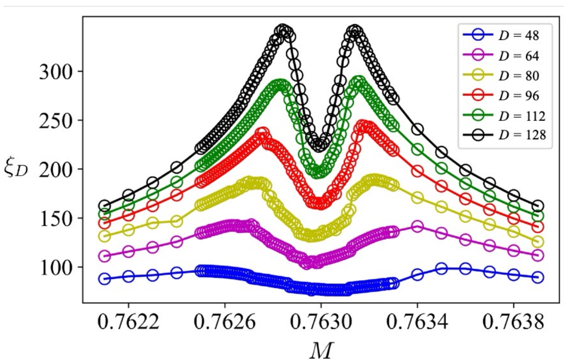
- **The ES forms two-fold degenerate pairs only inside the lobe**
  - **The topological insulating phase is indeed characterized by a two-fold degenerate ES**

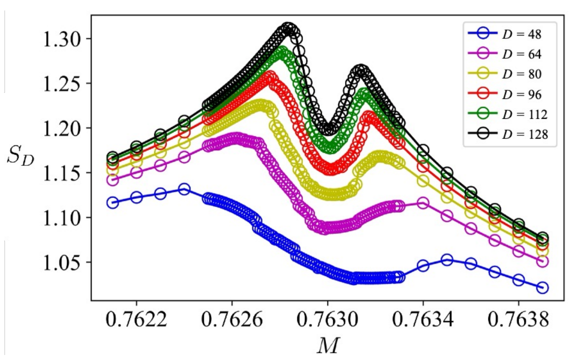


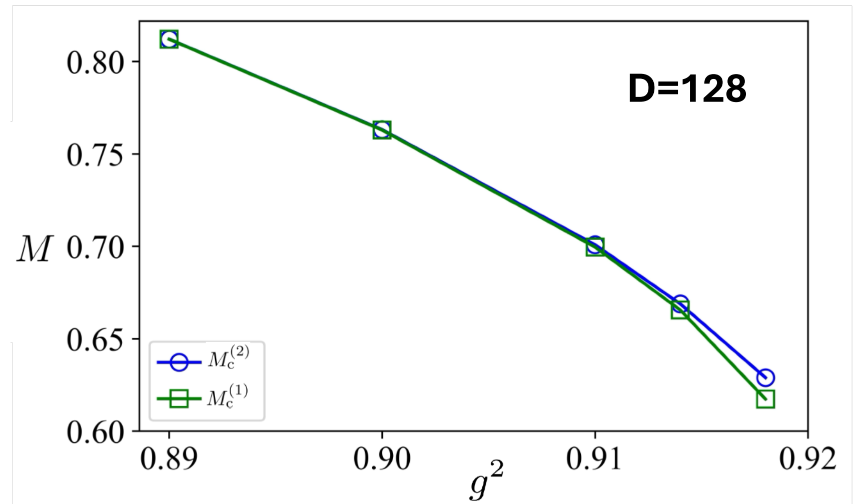
Where two Ising critical lines meet

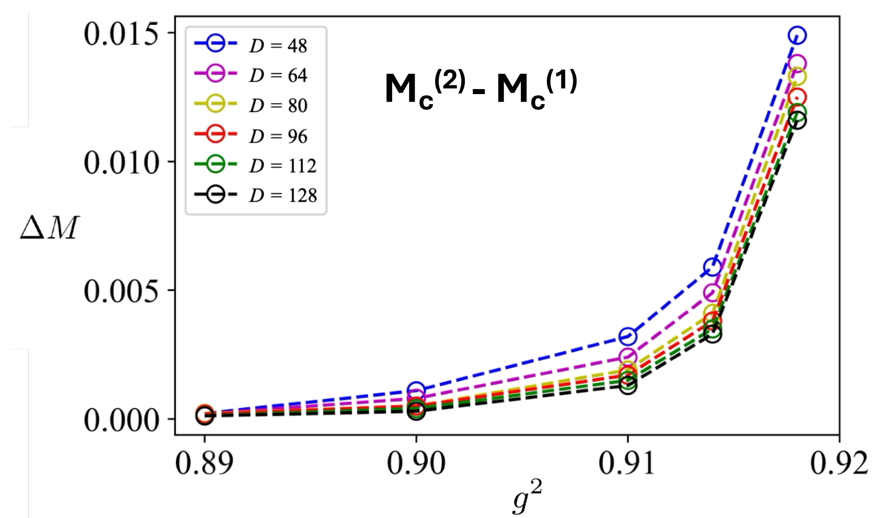
- The two peaks in ξ_D suggest two c=1/2 lines meet at (M, g^2)~($\pm$0.812,0.89)







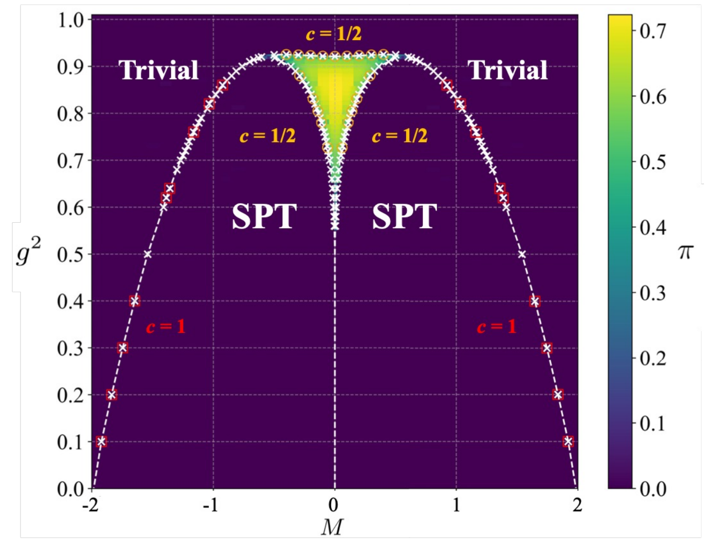




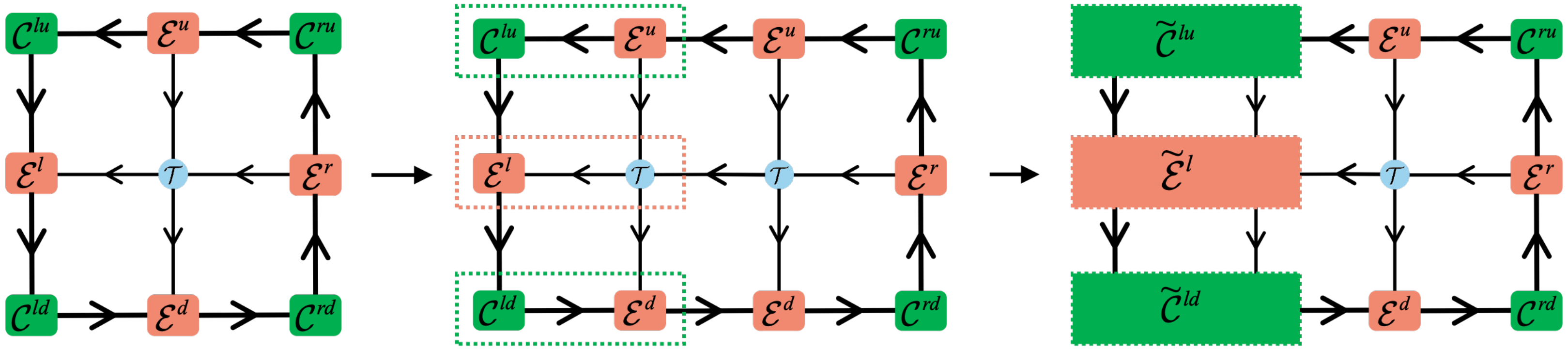
Summary & Outlook

- Our work represents the first TN study to determine the full phase diagram of the N_f=1 GN model w/ Wilson fermions within the Lagrangian formalism
  - The N_f=1 model contains the three distinct phases; the Aoki phase, the SPT phase, and trivial phase
  - **The Aoki phase is separated by the c=1/2 critical lines, and the phase boundaries btw the SPT and trivial phases are characterized by c=1.** Consistent w/ the recent MPS simulation
  - **In contrast to the large-N_f limit, the Aoki phase does not persist in the strong-coupling regime**
- Extension to the N_f=2 model is in progress
  - At finite temperature/density
  - Nested CTMRG on multi-layered Grassmann TN

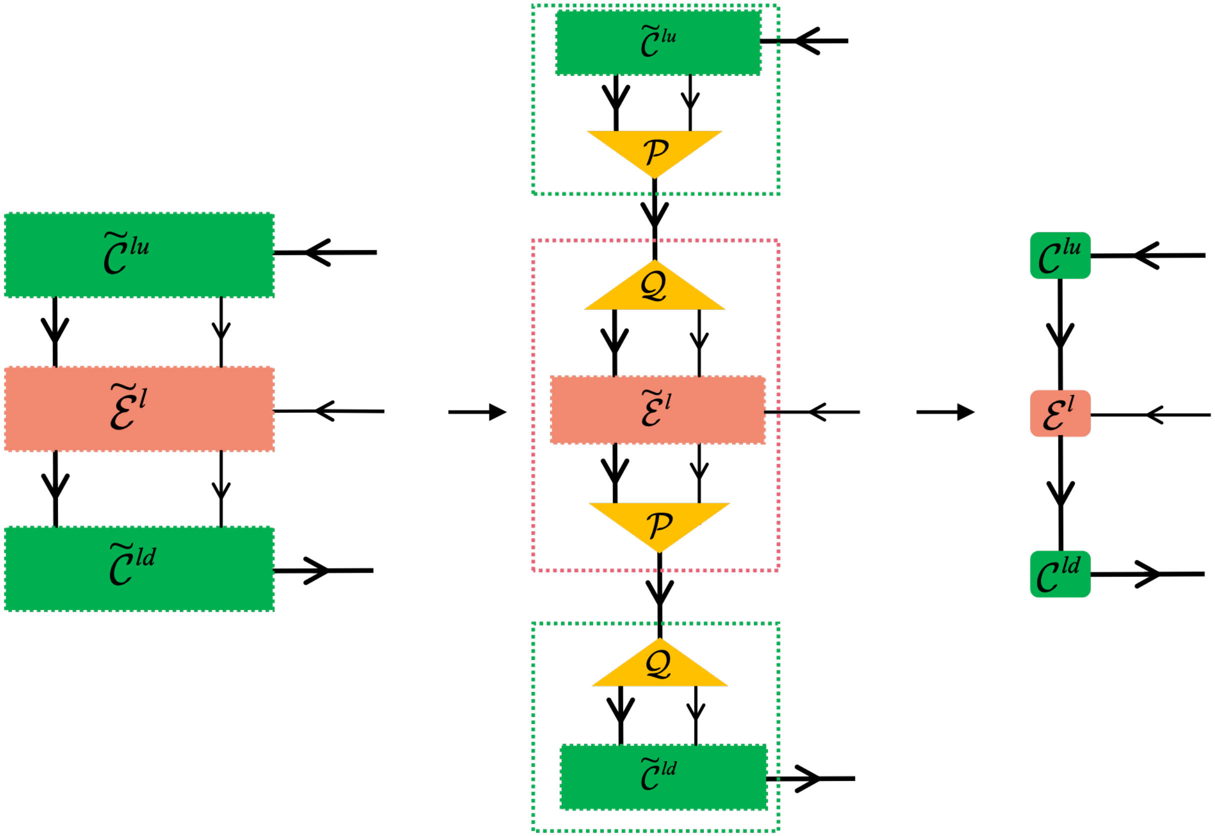
Cf. Xie+, PRB96(2017)045128
Naumann+, PRB111(2025)235116
Chen+, PRB113(2026)155127
Cf. SA, PRD108(2023)034514



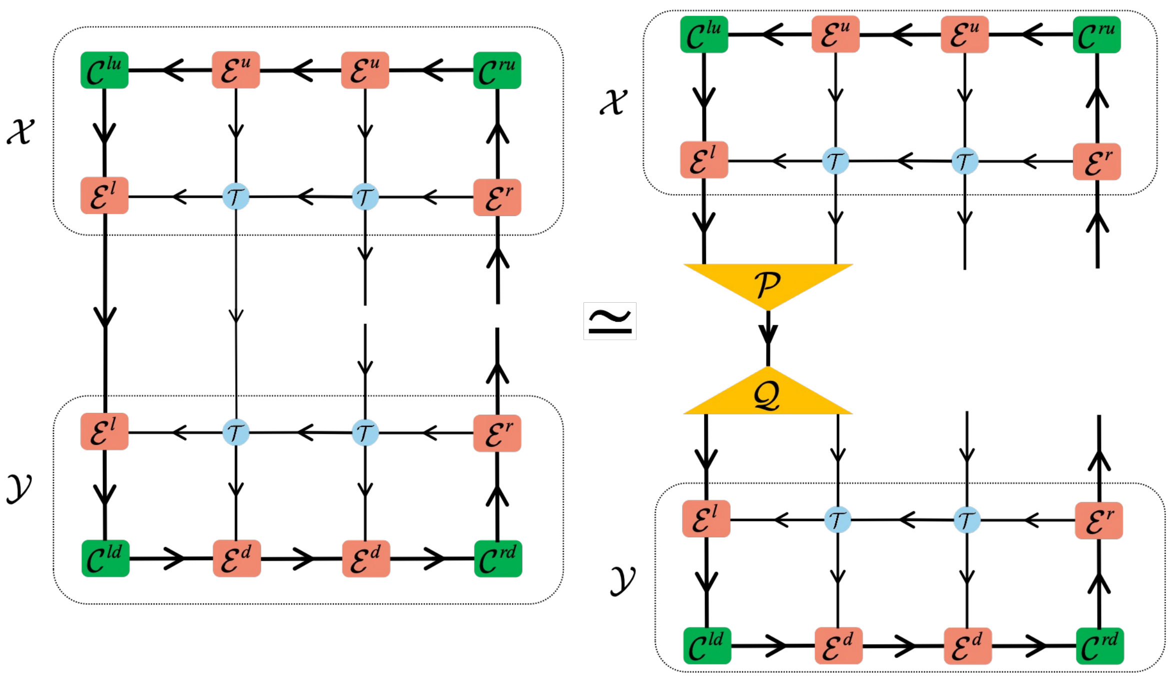
Left move in the CTMRG 1/2



Left move in the CTMRG 2/2

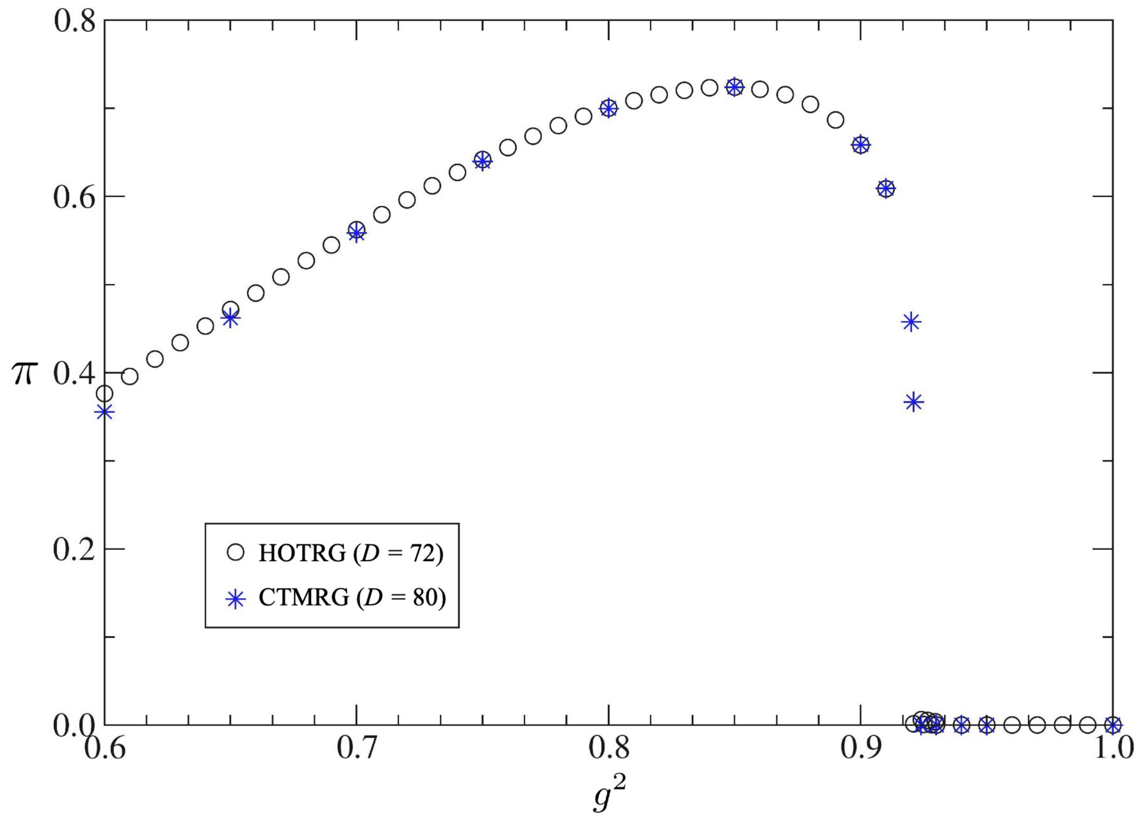


Projectors



$\mathcal{X}$
$\mathcal{Y}$
$\mathcal{C}^{lu}$
$\mathcal{E}^{u}$
$\mathcal{C}^{ru}$
$\mathcal{E}^{l}$
$\mathcal{T}$
$\mathcal{E}^{r}$
$\mathcal{C}^{ld}$
$\mathcal{E}^{d}$
$\mathcal{C}^{rd}$
$\simeq$
$\mathcal{P}$
$\mathcal{Q}$

Periodic boundary conditions



Data collapse