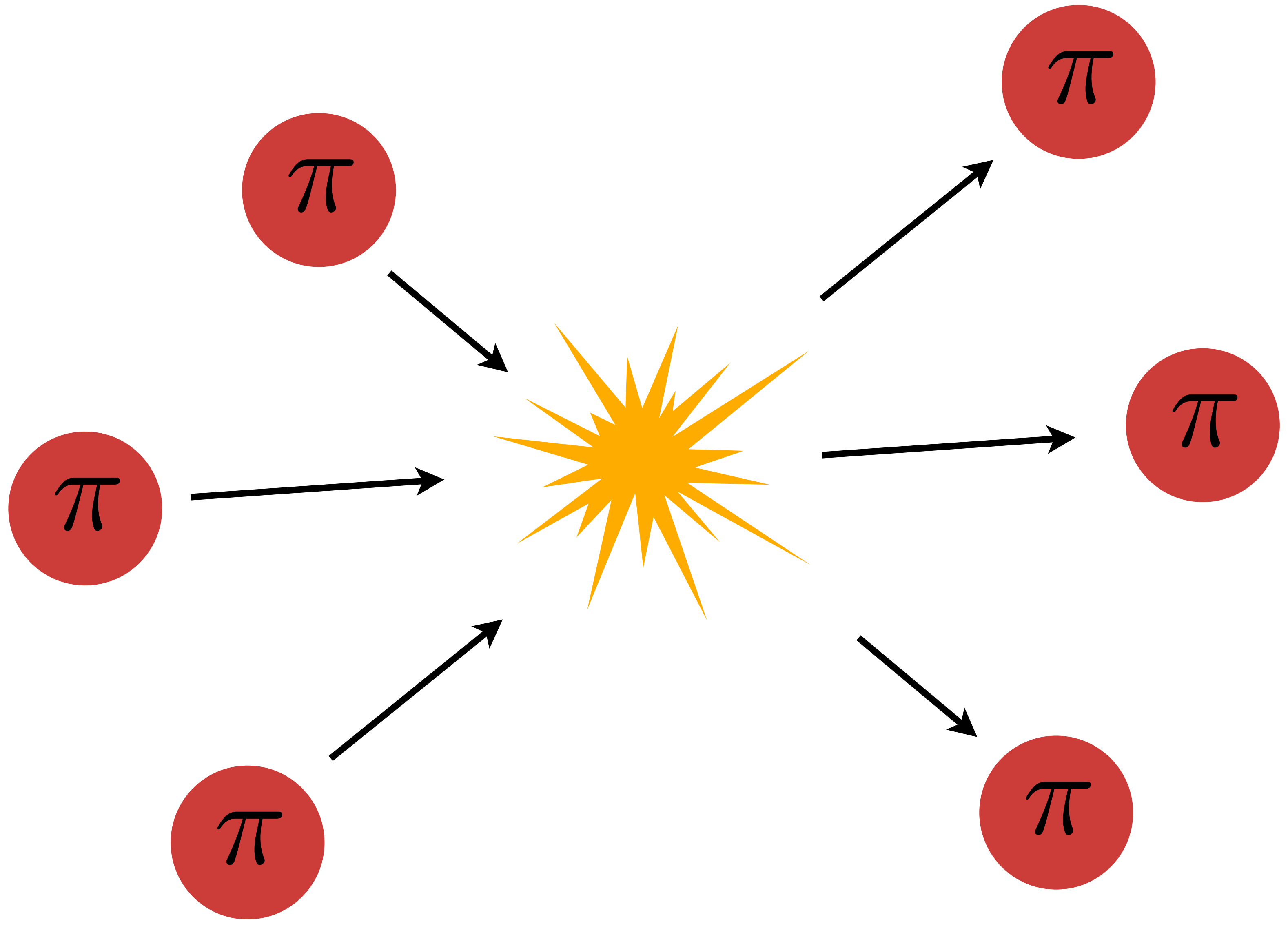
# Isotensor Three-Pion Scattering from Lattice QCD





**Andrew W. Jackura**

William & Mary

Department of Physics


**The 43rd International Symposium on Lattice Field Theory**

July 30, 2026

University of Maryland, College Park

Exotic Hadrons Topical Collaboration


awjackura@wm.edu

ajackura.github.io


William & Mary

CHARTERED 1693

# Key Take Aways

First calculation of isotensor $3\pi$ scattering amplitude

- Strongly coupled with sub-channel $\rho$ resonances
- Complete end-to-end analysis from energy levels to partial wave amplitudes
- Access to observables familiar with experimental analyses, e.g., Dalitz plots

**Relevant Literature**

Briceño, Hansen, **Jackura**, Thomas, Edwards

**Isotensor $\pi\pi\pi$ scattering with a $\rho$ resonant subsystem from QCD**

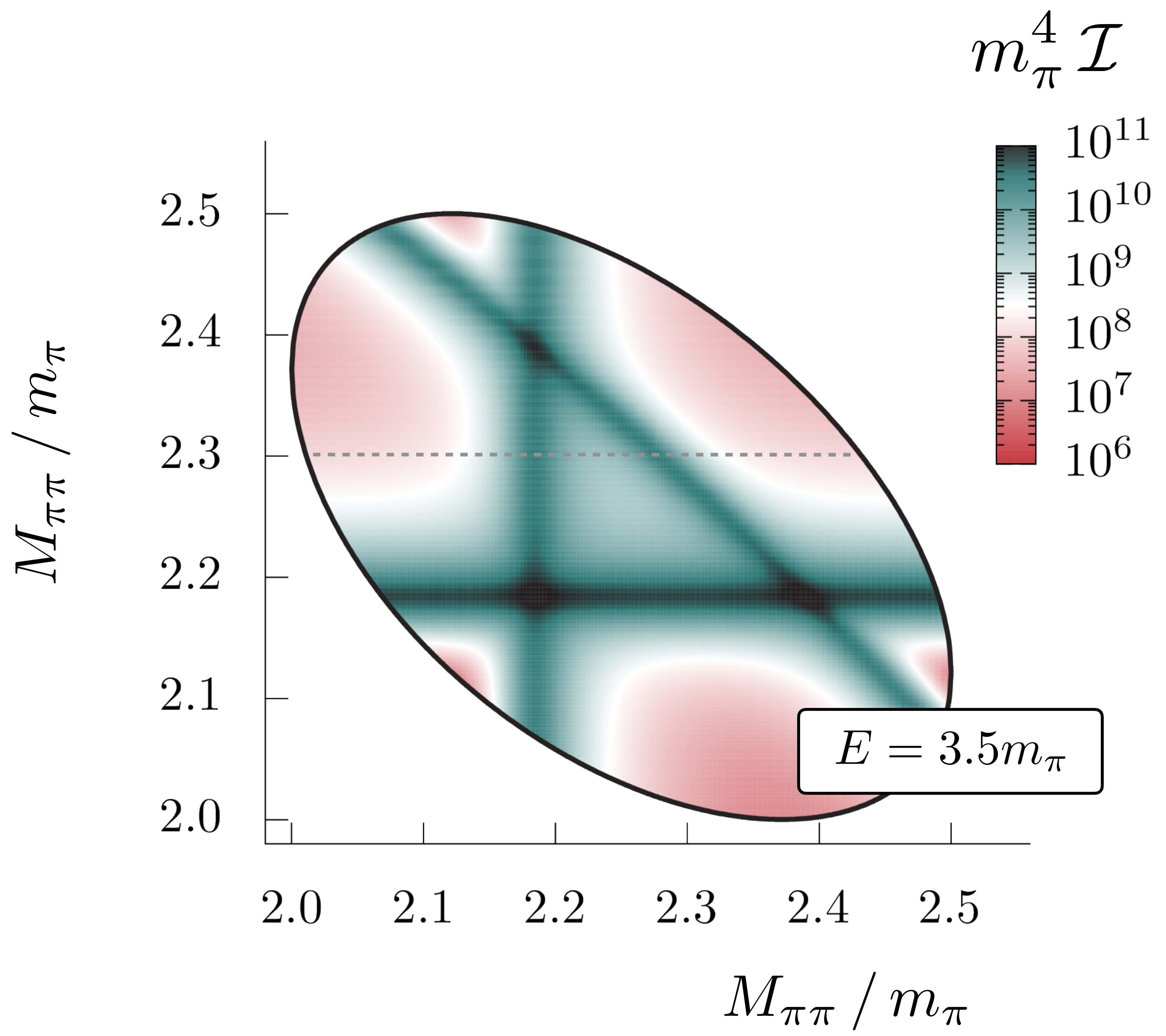
arXiv:2510.24894 (accepted to PRL)

**Raúl Briceño**
UC Berkeley

**Maxwell Hansen**
U Edinburgh

**Robert Edwards**
Jefferson Lab

**Christopher Thomas**
Cambridge



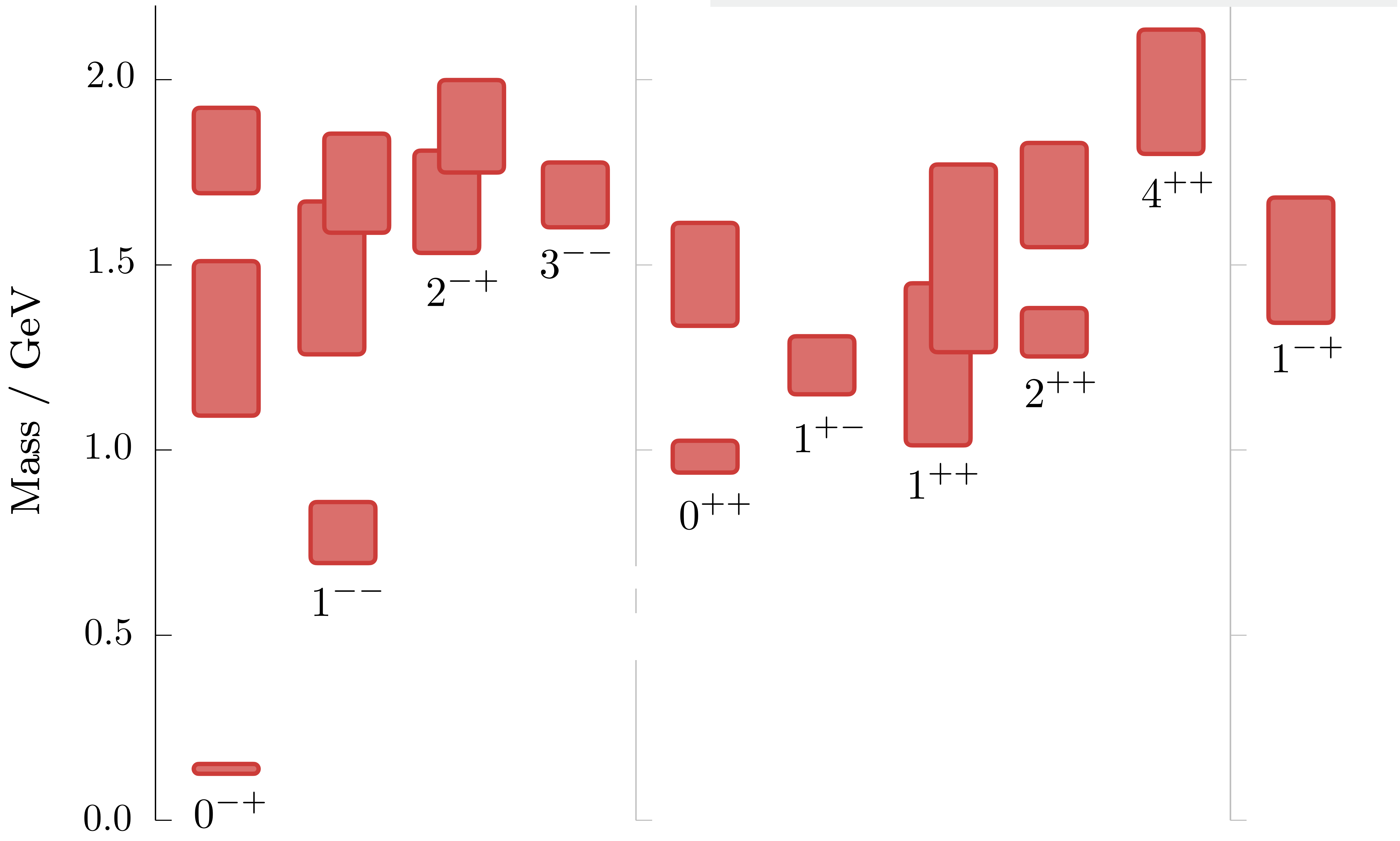
# Hadron Spectroscopy

**Goal:** Understand the excited hadron spectrum from QCD

- Establish masses, lifetimes, couplings, etc. in a rigorous manner
- Requires ascertaining **reaction amplitudes** — key objects which contain *all* information
- Have ability to study hadron physics in manner commensurate with experimental colleagues

$$\mathcal{L}_{\mathrm{QCD}} = \sum_f \bar{\psi}_f \left( i\not{D} - m_f \right) \psi_f - \frac{1}{2} \mathrm{tr} \left( \mathbf{G}_{\mu\nu} \mathbf{G}^{\mu\nu} \right)$$

observed light isovector mesons

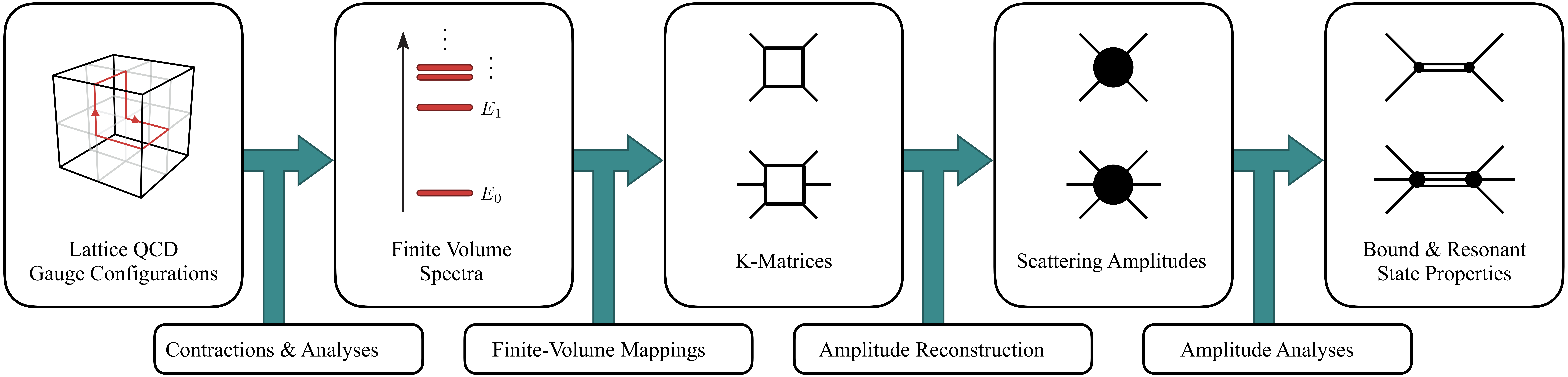


# Hadron Spectroscopy

**Goal:** Understand the excited hadron spectrum from QCD

- Establish masses, lifetimes, couplings, etc. in a rigorous manner
- Requires ascertaining **reaction amplitudes** — key objects which contain *all* information
- Have ability to study hadron physics in manner commensurate with experimental colleagues

**Lüscher Methodology**




M. Lüscher
Commun.Math.Phys. **105**, 153 (1986)
Nucl.Phys. **B354**, 531 (1991)

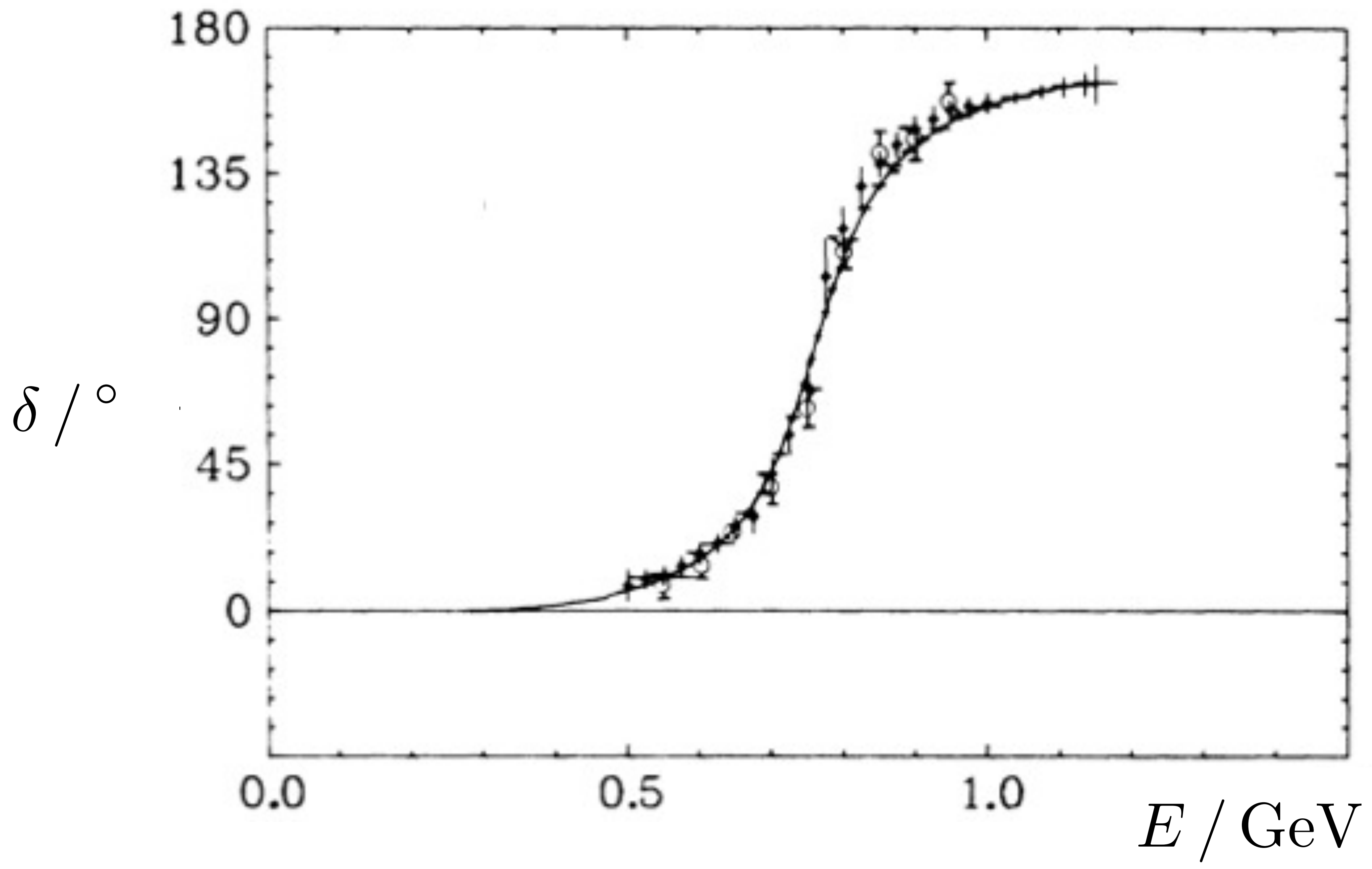
Many many others…

# Hadron Spectroscopy

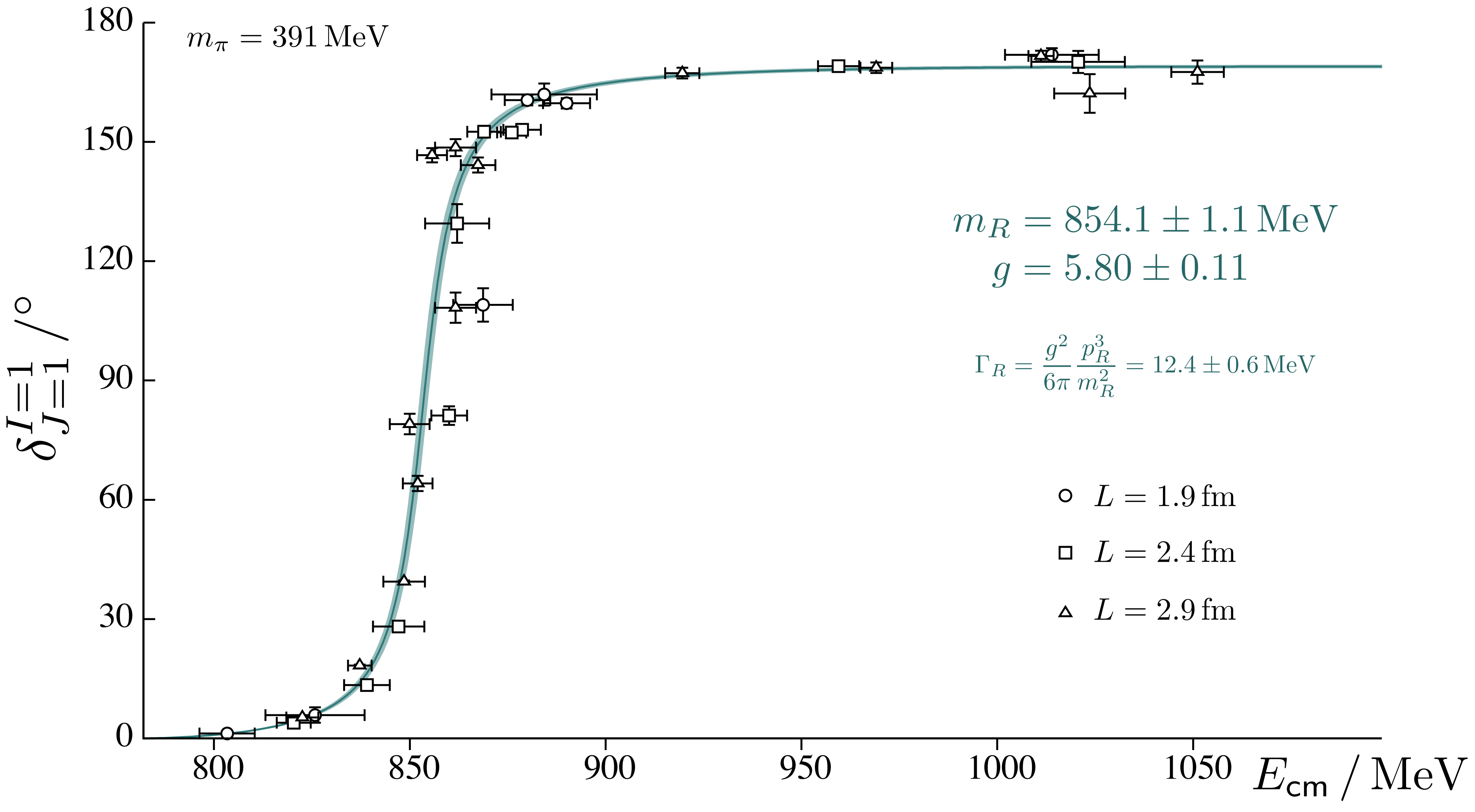
**Goal:** Understand the excited hadron spectrum from QCD

- Establish masses, lifetimes, couplings, etc. in a rigorous manner
- Requires ascertaining **reaction amplitudes** — key objects which contain *all* information
- Have ability to study hadron physics in manner commensurate with experimental colleagues

**Example:** $\pi\pi\ (I=1)$ Scattering




S.Protopopescu, et al
Phys.Rev.D, **7** (1973) 1279



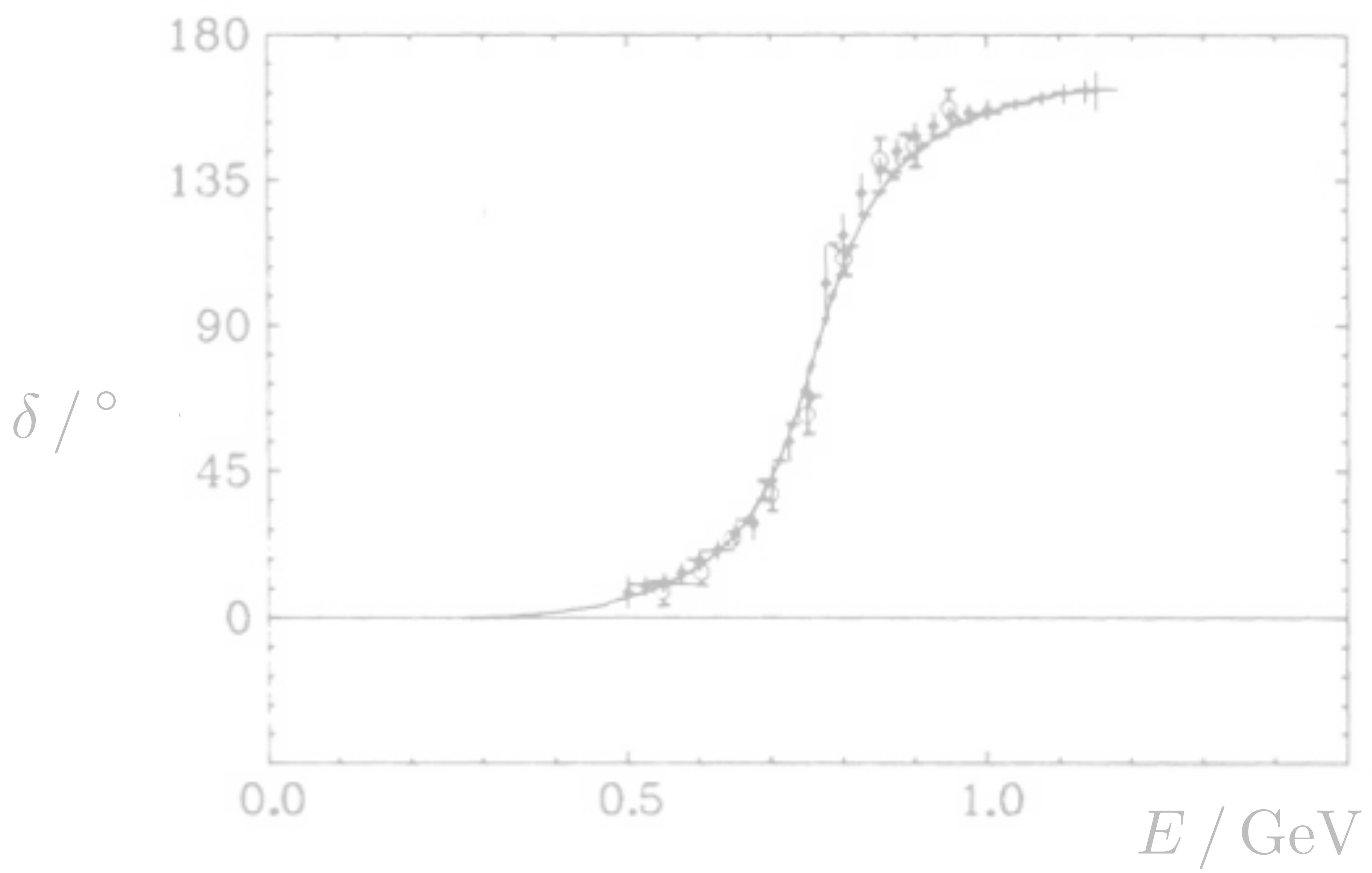

J.Dudek, R.Edwards, & C.Thomas
Phys.Rev.D, **87** (2013) 034505

# Hadron Spectroscopy

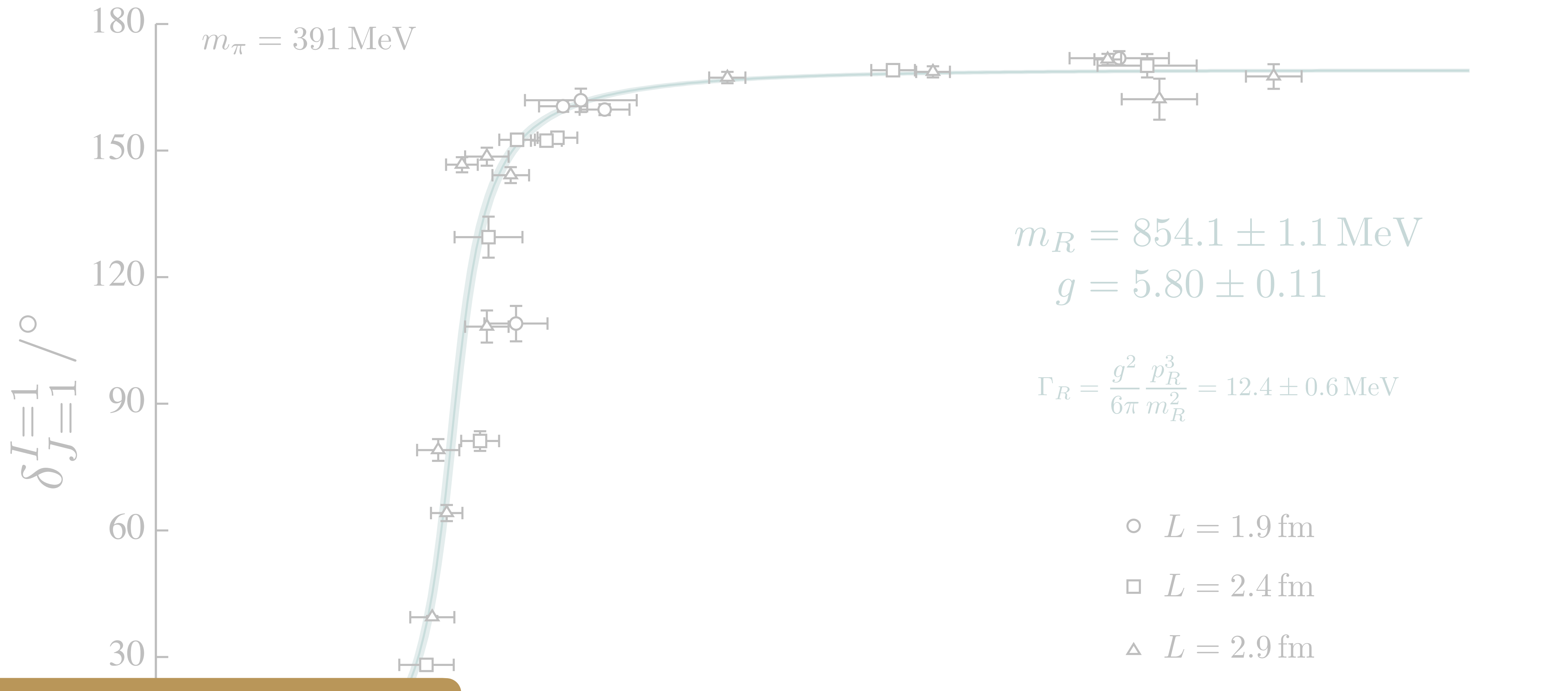
**Goal:** Understand the excited hadron spectrum from QCD

- Establish masses, lifetimes, couplings, etc. in a rigorous manner
- Requires ascertaining **reaction amplitudes** — key objects which contain *all* information
- Have ability to study hadron physics in manner commensurate with experimental colleagues

**Example:** $\pi\pi\ (I=1)$ Scattering




S.Protopopescu, et al
Phys.Rev.D, **7** (1973) 1279




**Two-Hadron Systems**

Two-hadron systems have been studied for decades, many interesting applications and physics learned — *no time to discuss here*

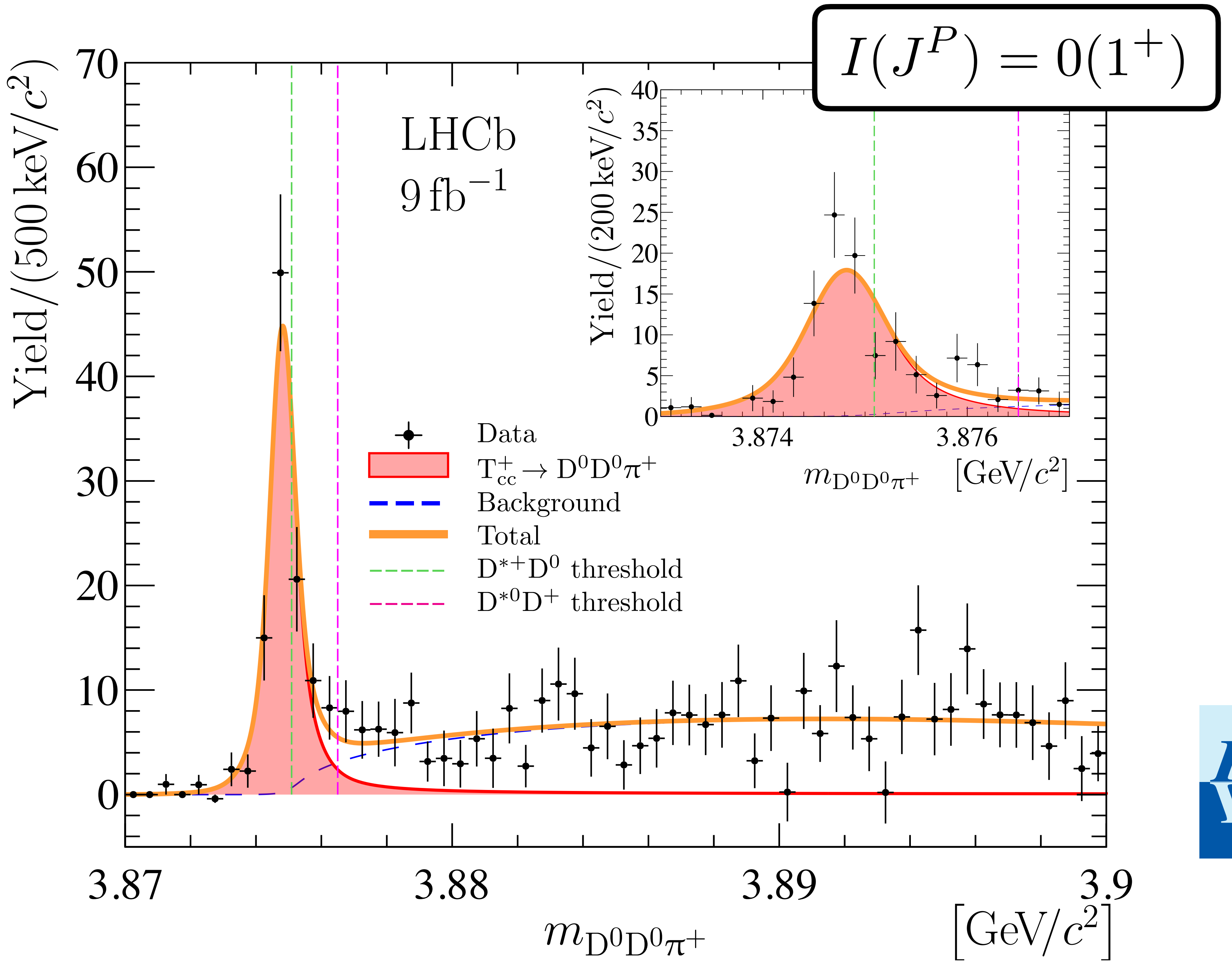
Phys.Rev.D, **87** (2013) 034505

# Three-Hadron Spectroscopy

Continue pushing these ideas to three-particle systems

- Many resonances decay to three (or more) particles — *Can we access these states?*
- Most exotics discovered have strong coupling to three-particles — *What is significance?*
- BSM Physics involves studying multi-body flavor decays — *Can we constrain dynamics?*

**Example:** $T_{cc}$ tetraquark candidate




R. Aaij et al.,
Nature Physics **18**, 751–754 (2022)

**Other Three-Body Talks**

**Maxwell Hansen** (Tuesday)
**Haobo Yan** (Wednesday)
**Maxim Mai** (Wednesday)
**Stephen Sharpe** (Thursday)

# Three-Pion Scattering

Focus on three-pion systems

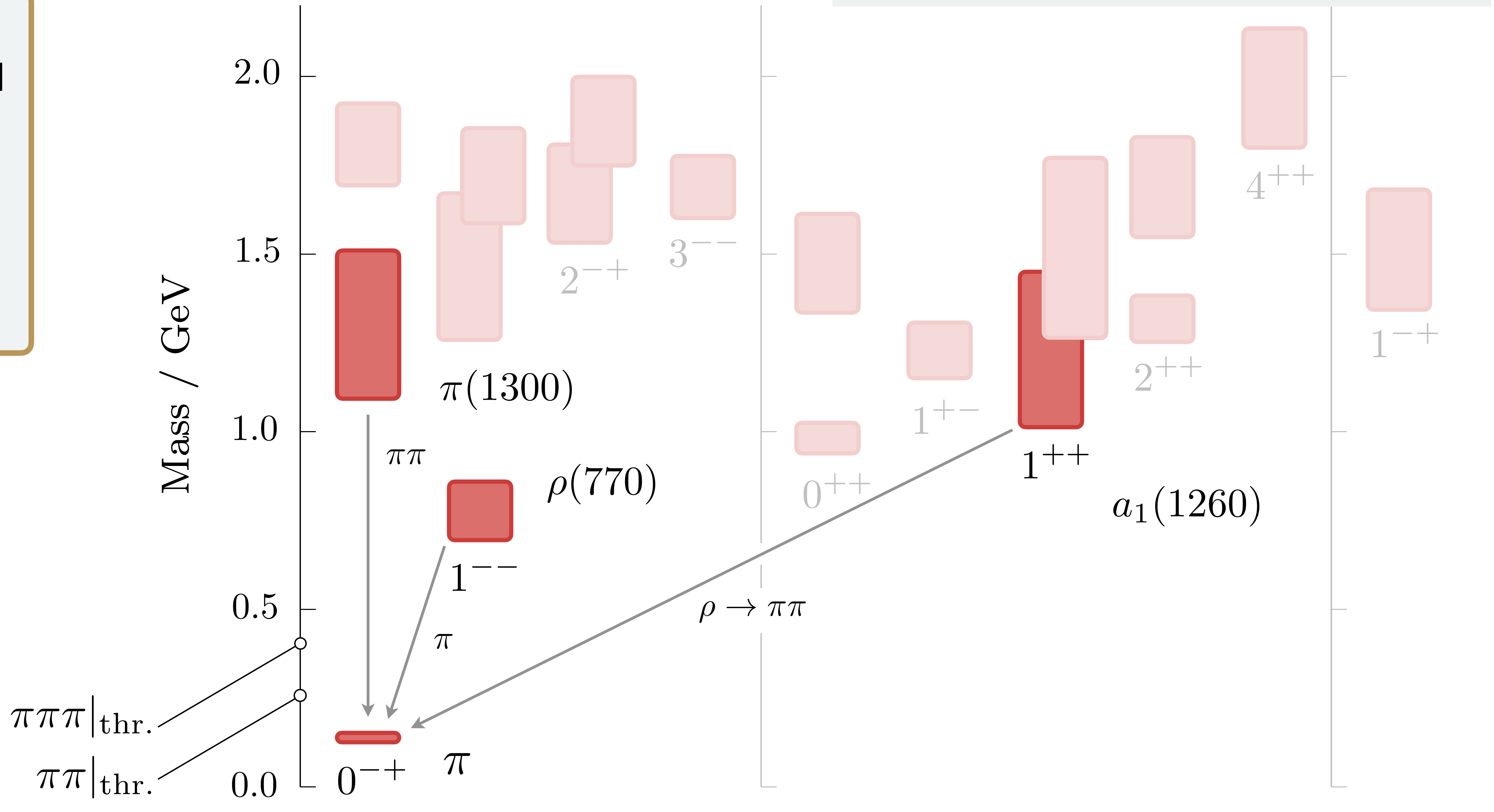
- Many states couple to three pions — decades of phenomenological knowledge
- Three-pion amplitudes essential to other hadronic phenomena
- ‘Simple’ playground to test new formalisms, construct observable frameworks

**Three-Hadron Systems**

Studying three-hadron states is considerably more involved

- No simple notion of *phase shift*
- Kinematic rescattering effects ubiquitous
- Relation between K matrices to amplitudes involves integral equations — more analysis required

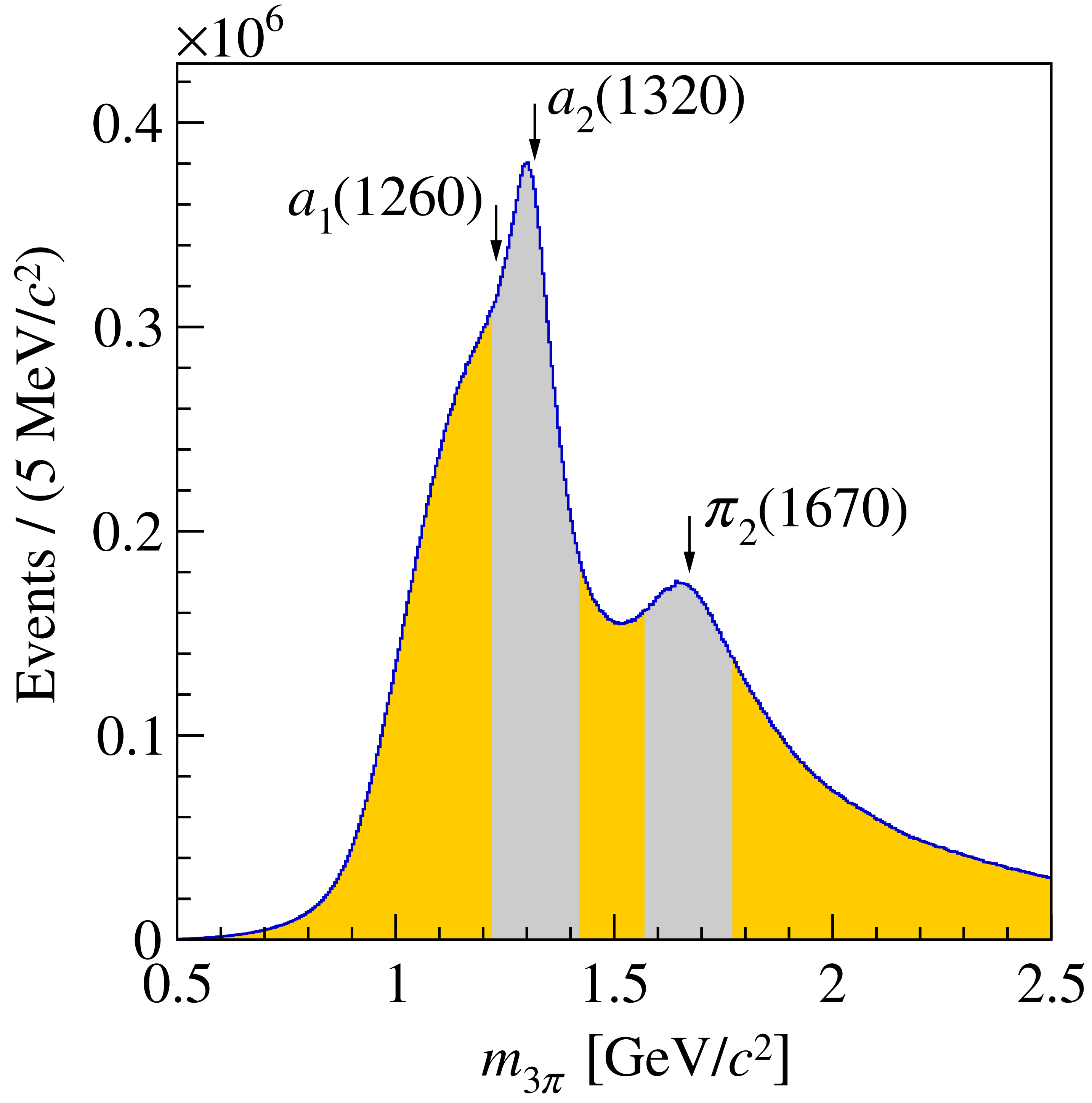
observed light isovector mesons

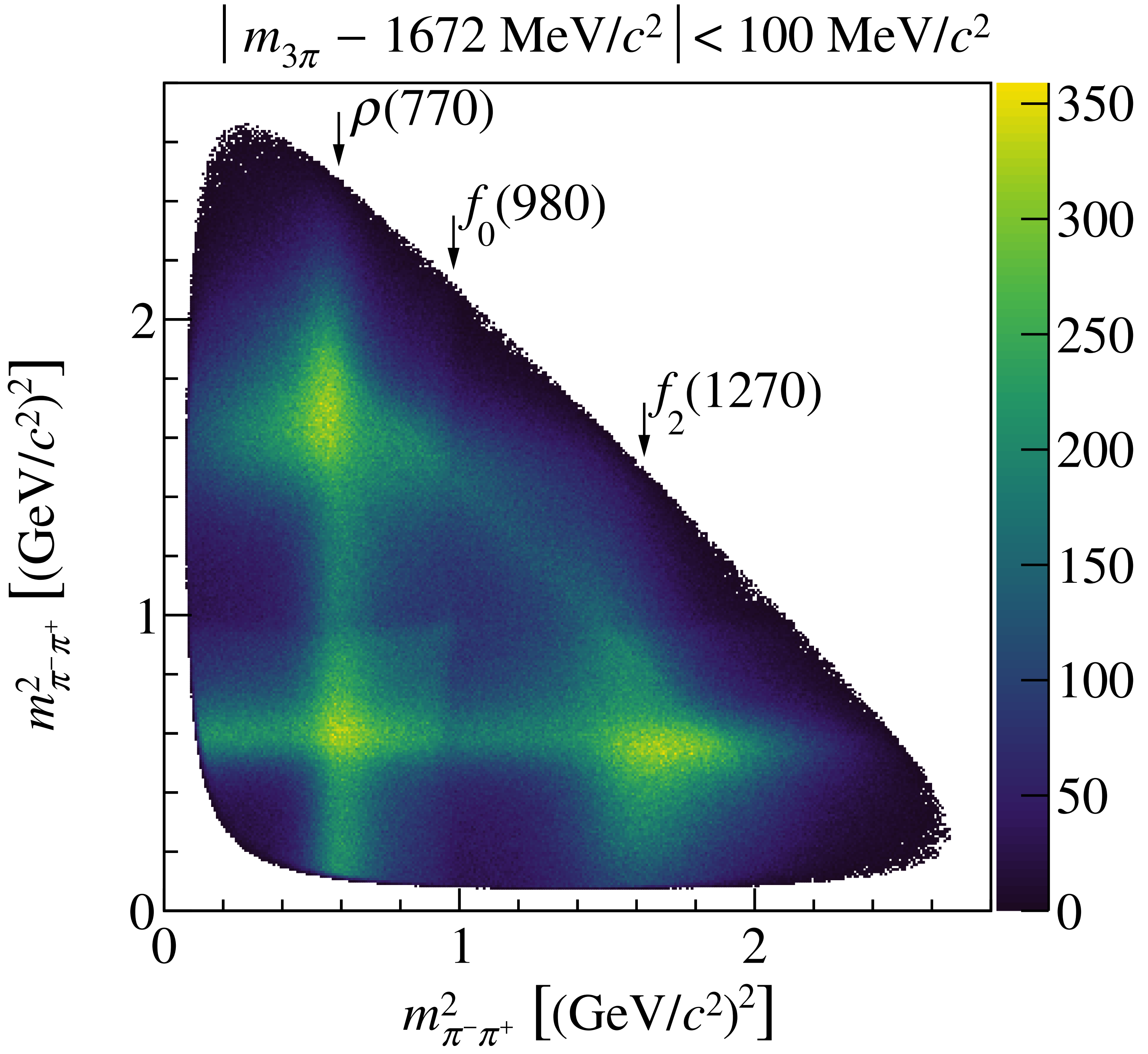


# Three-Pion Scattering

Focus on three-pion systems

- Many states couple to three pions — decades of phenomenological knowledge
- Three-pion amplitudes essential to other hadronic phenomena
- ‘Simple’ playground to test new formalisms, construct observable frameworks



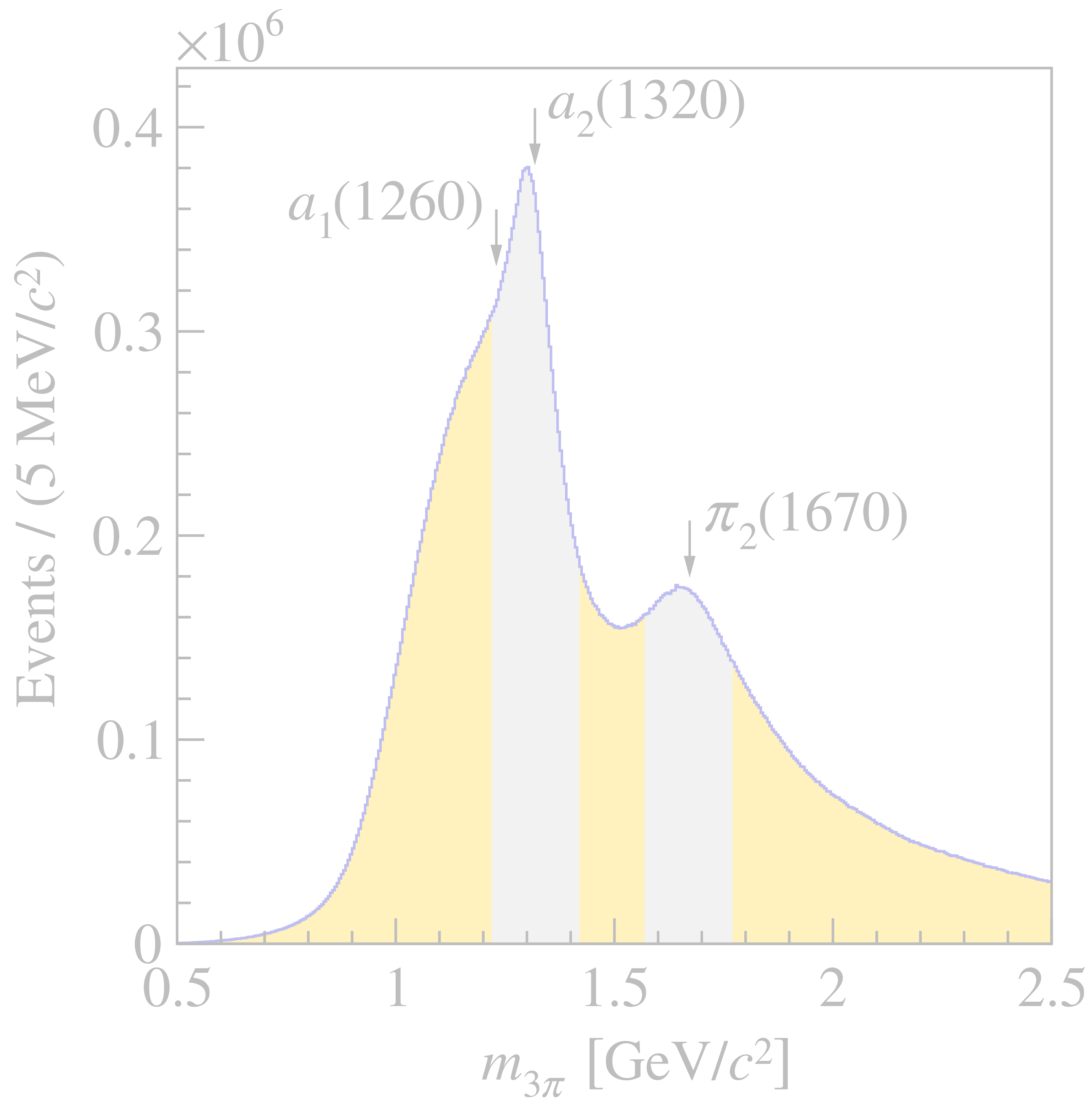



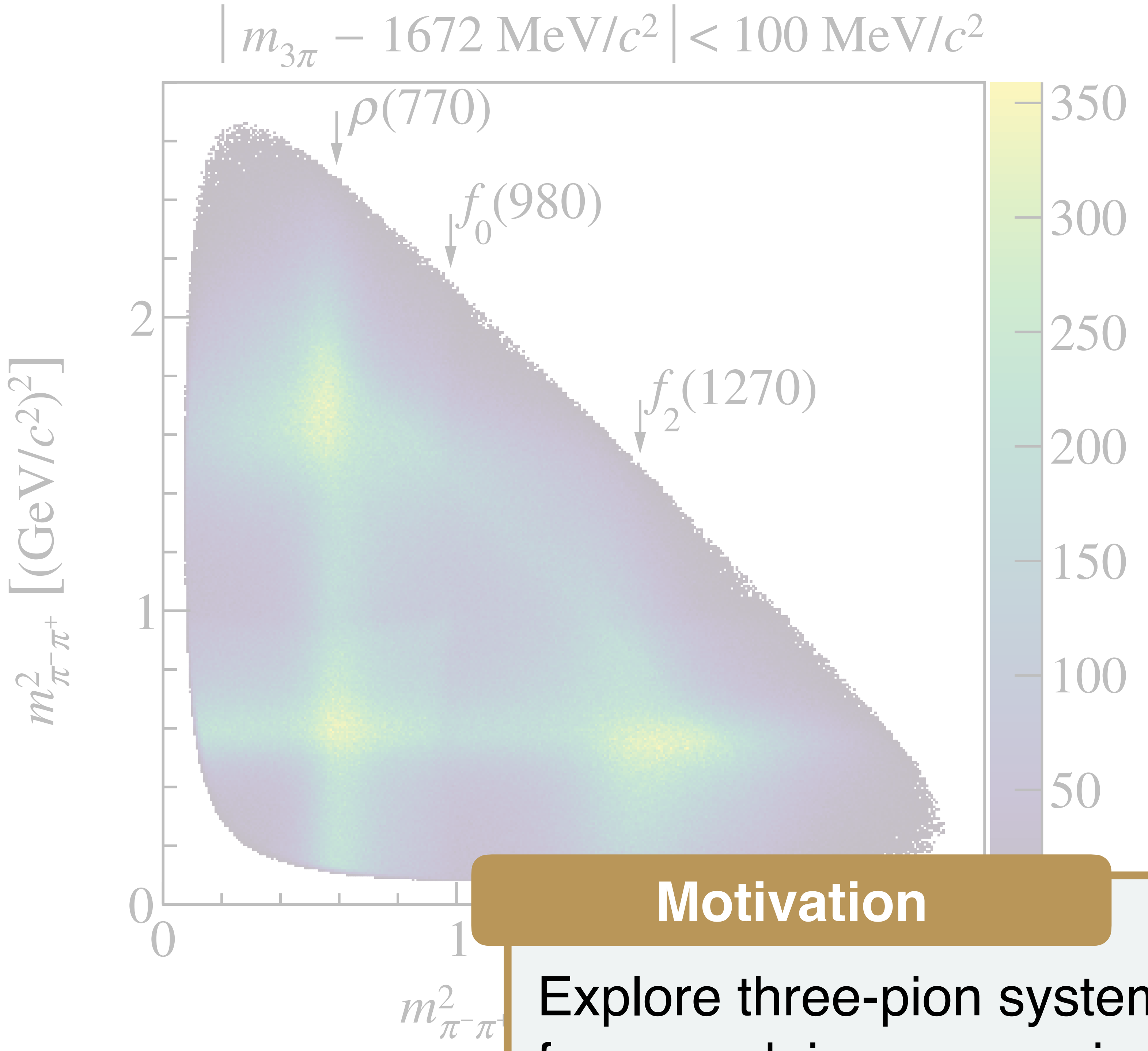

C.Adolph, et al
Phys.Rev.D, **95** (2017) 032004

# Three-Pion Scattering

Focus on three-pion systems

- Many states couple to three pions — decades of phenomenological knowledge
- Three-pion amplitudes essential to other hadronic phenomena
- ‘Simple’ playground to test new formalisms, construct observable frameworks




C.Adolph, et al
Phys.Rev.D, **95** (2017) 032004

**Motivation**

Explore three-pion systems with lattice QCD — establish analysis framework in manner similar to experiment

# Three-Pion Scattering from Lattice QCD

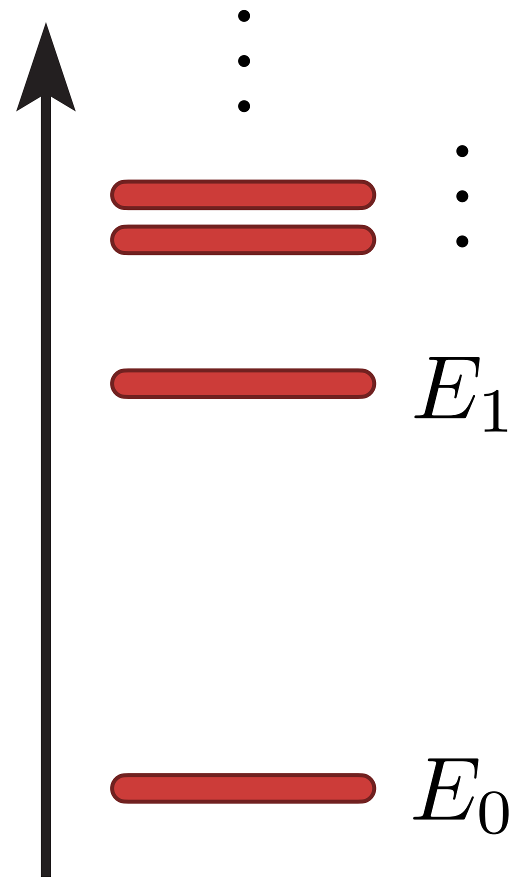
Follow usual *hadron spectrum collaboration* (**hadspec**) methodology

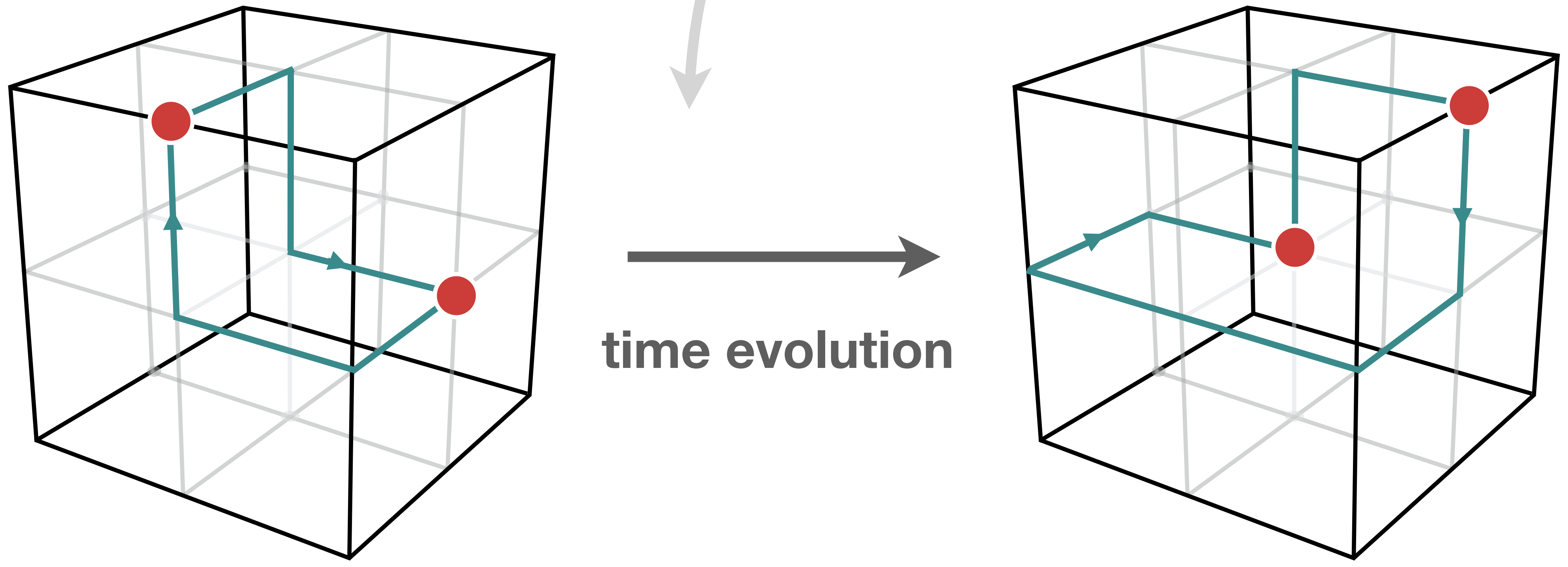
- $N_f = 2 + 1$ Anisotropic Wilson-Clover fermion action, Syzmanzik improved gauge action
- Large multi-hadron operator basis with distillation — $\pi\pi\pi$ operators & $\rho\pi$ operators
- Solve GEVP for finite-volume spectrum

$$C_{ij}(t) = \langle \mathcal{O}_i(t) \mathcal{O}_j(0) \rangle$$

$$C_{ij}(t) v^j_{\mathfrak{n}} = \lambda_{\mathfrak{n}} C_{ij}(t_0) v^j_{\mathfrak{n}}$$

$$\lambda_{\mathfrak{n}} \sim e^{E_{\mathfrak{n}}(t-t_0)}$$

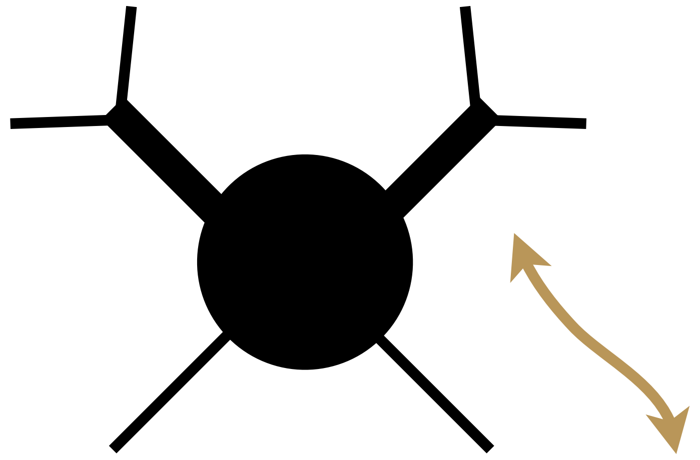


# Three-Pion Scattering from Lattice QCD

Here, focus on new results for three pions in $I = 2$

- Two-pion sub-channel resonances, no three-pion resonances
- Strongly coupled partial waves



“$\rho$”, $[\pi\pi]_{I=2}$

$m_\pi \approx 400$ MeV

| $(L/a_s)^3 \times (T/a_t)$ | $N_{\text{cfgs}}$ | $N_{\text{tsrc}}$ | $N_v$ |
|---|---|---|---|
| $16^3 \times 128$ | 479 | 4 | 64 |
| $20^3 \times 256$ | 286 | 4 | 128 |
| $24^3 \times 128$ | 553 | 1 | 160 |

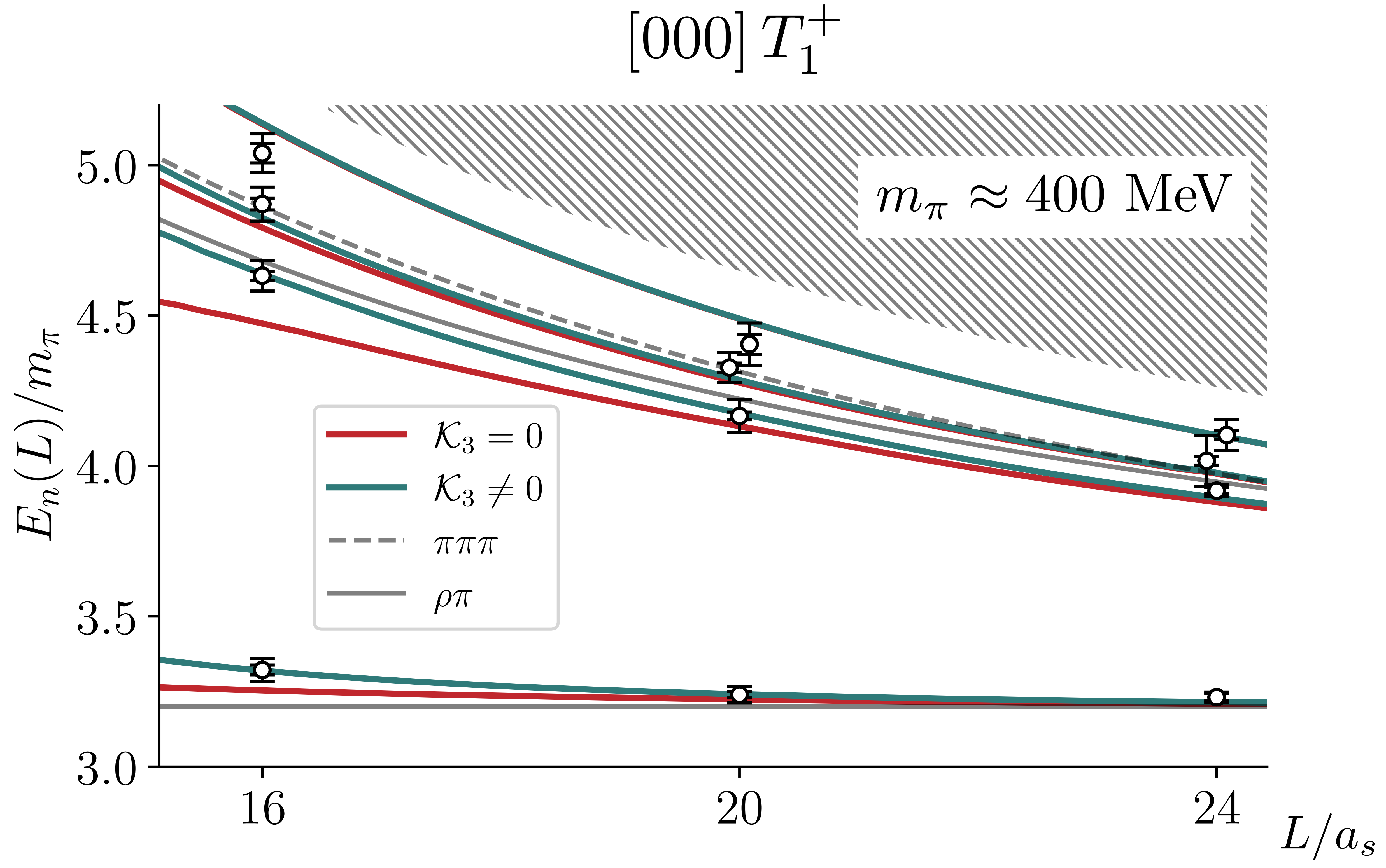
| $I^G_{3\pi}$ | $J^{PC}$ | $([\pi\pi]^I_\ell \pi)_L$ |
|---|---|---|
| $3^-$ | $0^{-+}$ | $([\pi\pi]^2_S \pi)_S$ |
| | $1^{-+}$ | none |
| | $1^{++}$ | $([\pi\pi]^2_S \pi)_P$ |
| $2^-$ | $0^{--}$ | $([\pi\pi]^2_S \pi)_S$, $([\pi\pi]^1_P \pi)_P$ |
| | $1^{--}$ | $([\pi\pi]^1_P \pi)_P$ |
| | $1^{+-}$ | $([\pi\pi]^2_S \pi)_P$, $([\pi\pi]^1_P \pi)_S$, $([\pi\pi]^1_P \pi)_D$ |
| $1^-$ | $0^{-+}$ | $([\pi\pi]^{0,2}_S \pi)_S$, $([\pi\pi]^1_P \pi)_P$ |
| | $1^{-+}$ | $([\pi\pi]^1_P \pi)_P$ |
| | $1^{++}$ | $([\pi\pi]^{0,2}_S \pi)_P$, $([\pi\pi]^1_P \pi)_S$, $([\pi\pi]^1_P \pi)_D$ |
| $0^-$ | $0^{--}$ | $([\pi\pi]^1_P \pi)_P$ |
| | $1^{--}$ | $([\pi\pi]^1_P \pi)_P$ |
| | $1^{+-}$ | $([\pi\pi]^1_P \pi)_S$, $([\pi\pi]^1_P \pi)_D$ |

R. Briceño, M. Hansen, AJ, R. Edwards, and C. Thomas
arXiv:2510.24894, *accepted* to PRL

# Three-Pion Scattering from Lattice QCD

Here, focus on new results for three pions in $I = 2$

- Two-pion sub-channel resonances, no three-pion resonances
- Strongly coupled partial waves




| **Operators** |
| --- |
| $[(\pi_{[001]})_{A_2}(\pi_{[00-1]})_{A_2}]_{T_1^-}(\pi_{[000]})_{A_1^-}$ |
| $[(\pi_{[011]})_{A_2}(\pi_{[0-1-1]})_{A_2}]_{T_1^-}(\pi_{[000]})_{A_1^-}$ |
| $[(\pi_{[-101]})_{A_2}(\pi_{[100]})_{A_2}]_{A_1}(\pi_{[00-1]})_{A_2}$ |
| $[(\pi_{[-101]})_{A_2}(\pi_{[100]})_{A_2}]_{E_2}(\pi_{[00-1]})_{A_2}$ |
| $[(\pi_{[111]})_{A_2}(\pi_{[000]})_{A_1^-}]_{A_1}(\pi_{[-1-1-1]})_{A_2}$ |
| $[(\pi_{[002]})_{A_2}(\pi_{[00-1]})_{A_2}]_{A_1}(\pi_{[00-1]})_{A_2}$ |
| $(\rho^{\mathfrak{n}=0}_{[000]})_{T_1^-}(\pi_{[000]})_{A_1^-}$ |
| $(\rho^{\mathfrak{n}=0}_{[001]})_{A_1}(\pi_{[00-1]})_{A_2}$ |
| $(\rho^{\mathfrak{n}=0}_{[001]})_{E2}(\pi_{[00-1]})_{A_2}$ |
| $(\rho^{\mathfrak{n}=1}_{[001]})_{A_1}(\pi_{[00-1]})_{A_2}$ |
| $(\rho^{\mathfrak{n}=0}_{[011]})_{A_1}(\pi_{[0-1-1]})_{A_2}$ |
| $(\rho^{\mathfrak{n}=0}_{[011]})_{B_1}(\pi_{[0-1-1]})_{A_2}$ |
| $(\rho^{\mathfrak{n}=0}_{[011]})_{B_2}(\pi_{[0-1-1]})_{A_2}$ |

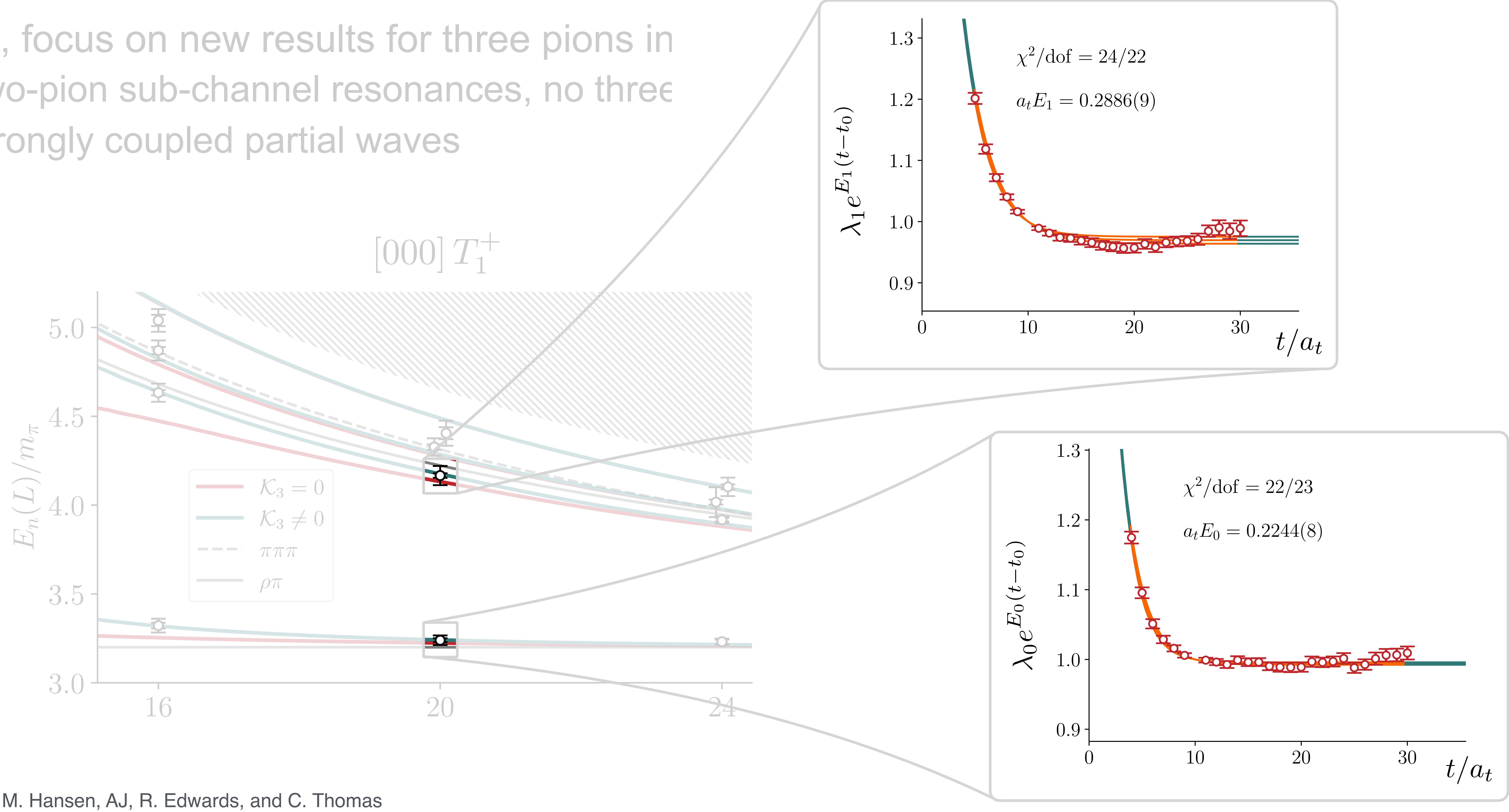
R. Briceño, M. Hansen, AJ, R. Edwards, and C. Thomas
arXiv:2510.24894, *accepted* to PRL

# Three-Pion Scattering from Lattice QCD

Here, focus on new results for three pions in

- Two-pion sub-channel resonances, no three

- Strongly coupled partial waves




R. Briceño, M. Hansen, AJ, R. Edwards, and C. Thomas
arXiv:2510.24894, *accepted* to PRL

# Three-Pion Scattering from Lattice QCD

## Map finite-volume spectrum to K matrices

**Quantization Condition**

$$\det_{I,\boldsymbol{k},S,m}\left[\mathbf{F}_3^{-1}(E_n, L|\boldsymbol{\eta}_2) + \mathbf{K}_3(E_n, |\boldsymbol{\eta}_3)\right] = 0$$

$$\mathbf{F}_3(E, L|\boldsymbol{\eta}_2) = \frac{1}{L^3}\left[\Big(\mathbf{F}_2(E, L) + \mathbf{G}(E, L)\Big)^{-1} + \mathbf{K}_2(E|\boldsymbol{\eta}_2)\right]^{-1}$$

*Equivalence proofs*


M. Hansen and S. Sharpe
Phys. Rev. D **90**, (2014) 116003

M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak
Eur. Phys. J. A **53**, 177 (2017)

AJ et al. [JPAC]
Eur. Phys. J. C **79**, no. 1, 56 (2019)

*and many more….*

AJ et al. [JPAC]
Phys. Rev. D **100**, 034508 (2019)

T. Blanton and S. Sharpe
Phys. Rev. D **102**, 054515 (2020)

AJ
Phys. Rev. D **108**, 034505 (2023)

# Three-Pion Scattering from Lattice QCD

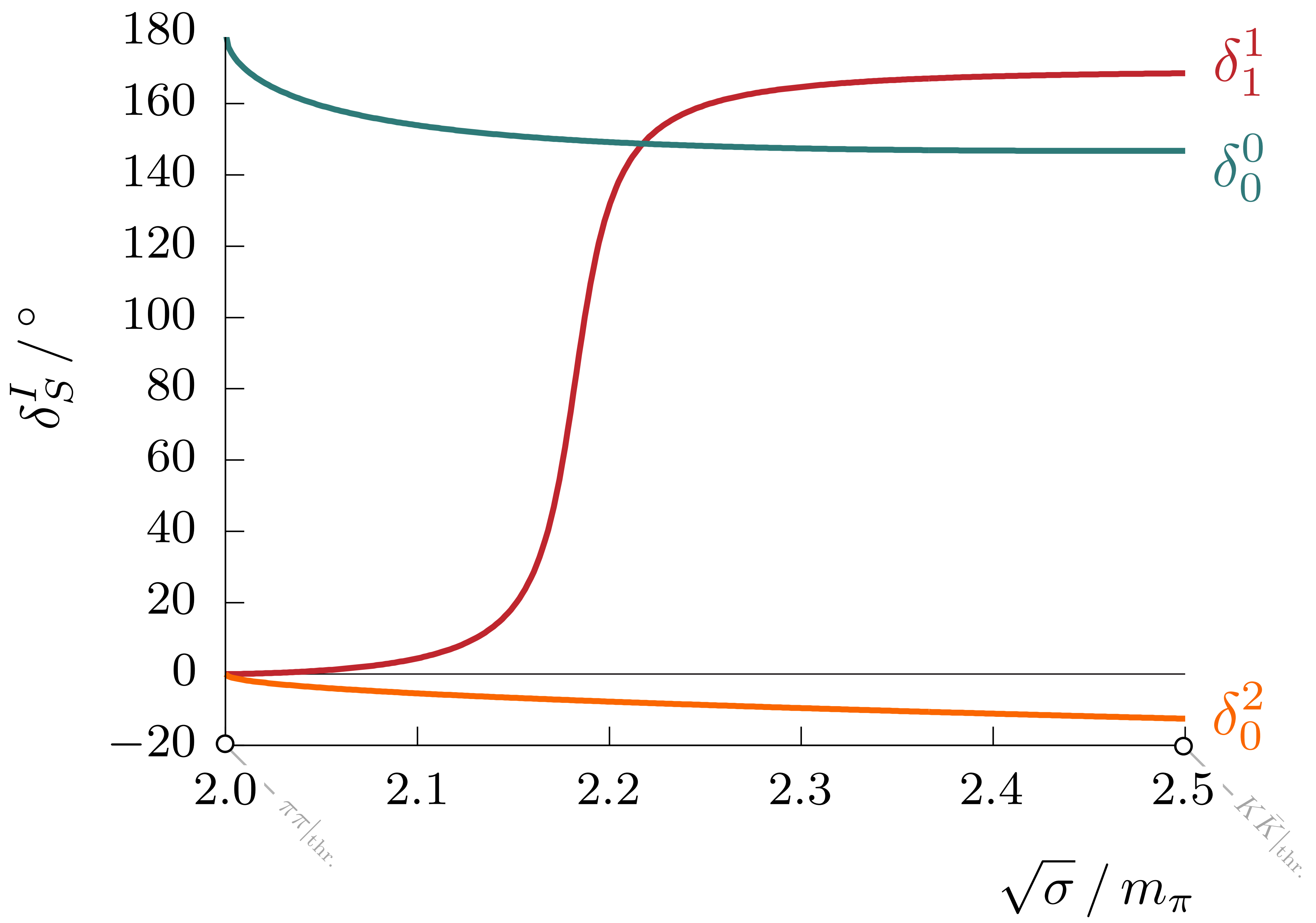
Map finite-volume spectrum to K matrices

- Use previously lattice QCD determined hadspec $\pi\pi$ phase shifts (input)

**Quantization Condition**

$$\det_{I,\boldsymbol{k},S,m}\left[\mathbf{F}_3^{-1}(E_n,L|\boldsymbol{\eta}_2)+\mathbf{K}_3(E_n,|\boldsymbol{\eta}_3)\right]=0$$

Two-pion phase shifts @ $m_\pi \approx 400\,\mathrm{MeV}$





J.Dudek, R.Edwards, & C.Thomas
Phys.Rev.D **86** (2012) 034031

J.Dudek, R.Edwards, & C.Thomas
Phys.Rev.D, **87** (2013) 034505

R. Briceño, J.Dudek, R.Edwards, & D.Wilson
Phys.Rev.Lett. **118** (2017) 2, 022002

# Three-Pion Scattering from Lattice QCD

## Map finite-volume spectrum to K matrices

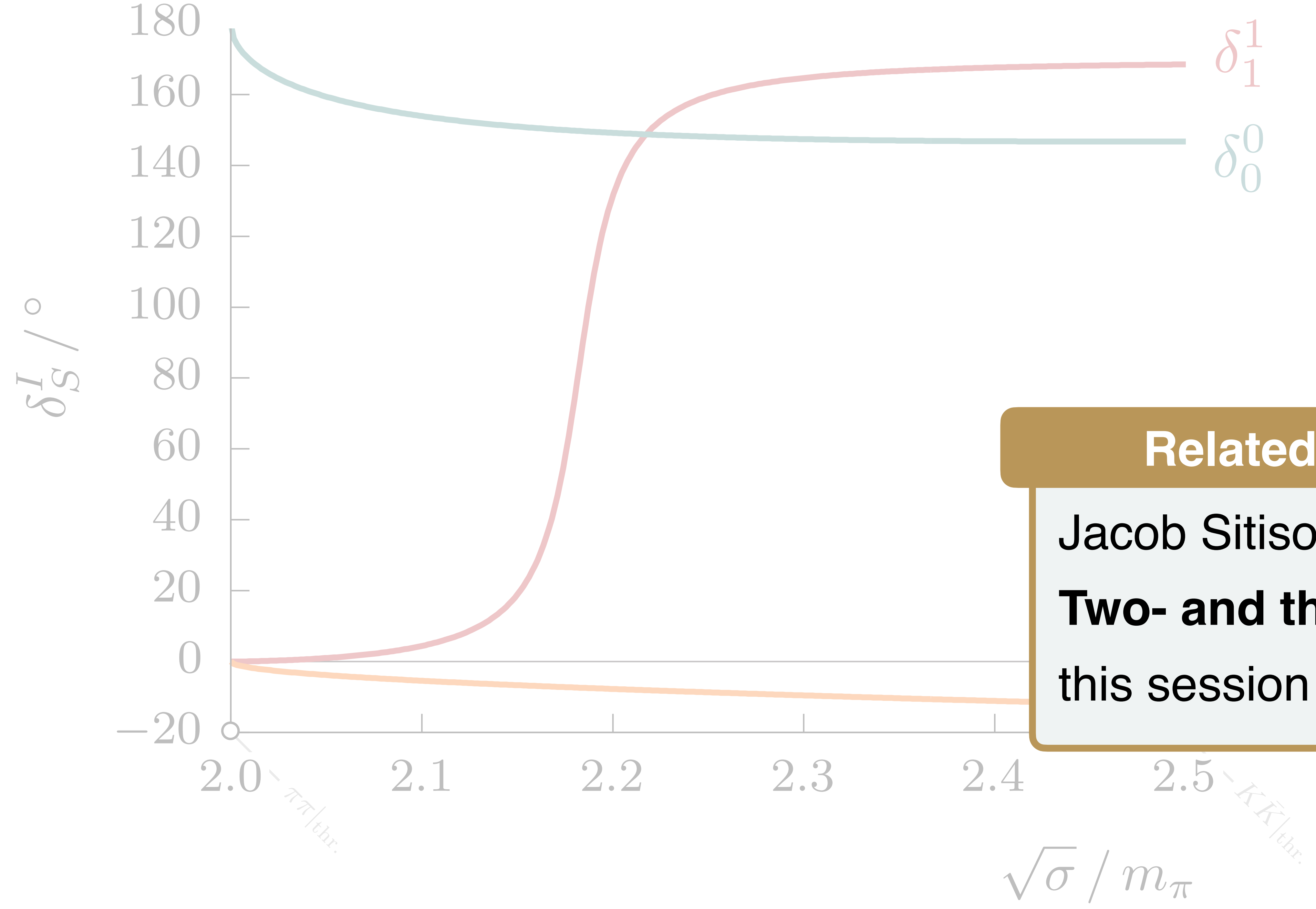
- Use previously lattice QCD determined hadspec $\pi\pi$ phase shifts (input)

**Quantization Condition**

$$\det_{I,\boldsymbol{k},S,m}\left[\mathbf{F}_3^{-1}(E_n,L|\boldsymbol{\eta}_2)+\mathbf{K}_3(E_n,|\boldsymbol{\eta}_3)\right]=0$$

$$\mathbf{F}_3(E,L|\eta_2)=\frac{1}{\ldots}\left[\left(\mathbf{F}_2(E,L)+\mathbf{G}(E,L)\right)\ldots\right]^{-1}$$

Two-pion phase shifts @ $m_\pi \approx 400\,\mathrm{MeV}$




Phys.Rev.D, 87 (2013) 034505

R. Briceño, J.Dudek, R.Edwards, & D.Wilson
Phys.Rev.Lett. **118** (2017) 2, 022002

**Related Talk**

Jacob Sitison

**Two- and three-pion scattering in multiple three-body isospin channels**

this session

**Jacob Sitison**
U Colorado Boulder

# Three-Pion Scattering from Lattice QCD

Map finite-volume spectrum to K matrices

- Use previously lattice QCD determined hadspec $\pi\pi$ phase shifts (input)
- Parameterize $\mathbf{K}_3$ with many flexible parameterizations

$$\mathbf{K}_3 = \mathbf{h}\,\widetilde{\mathcal{K}}_3\,\mathbf{h}$$

*Known barrier factors, subduction coefficients,…*

$$m_\pi^2\,\widetilde{\mathcal{K}}({}^{2S'+1}\ell_1'|{}^{2S+1}\ell_1) = \alpha({}^{2S'+1}\ell_1'|{}^{2S+1}\ell_1) + \left(\frac{s-(3m_\pi)^2}{m_\pi^2}\right)\beta({}^{2S'+1}\ell_1'|{}^{2S+1}\ell_1)$$

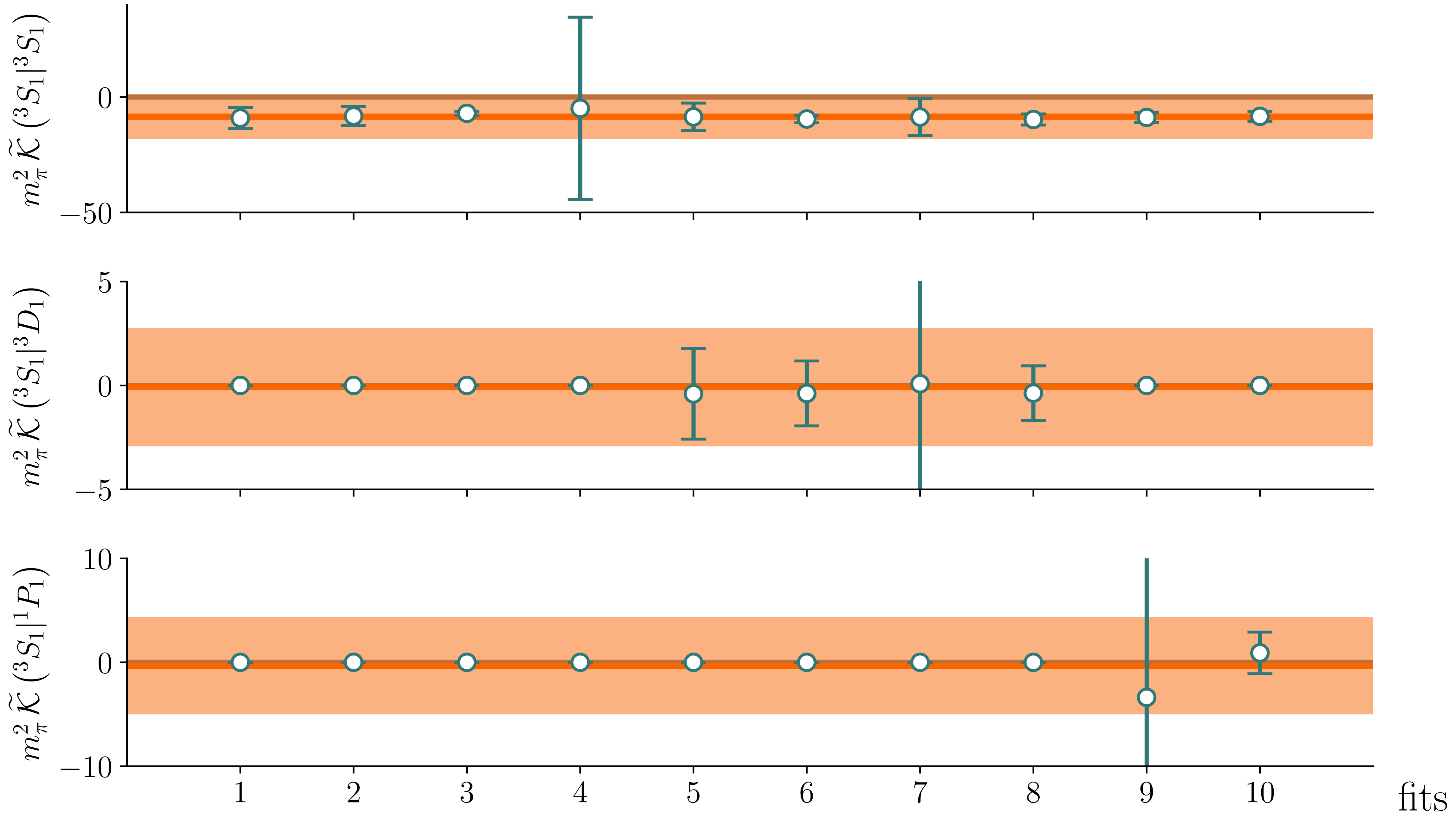
$$h_{\ell S}(k) = \frac{k^\ell q_k^{\star\,S}}{m_\pi^{\ell+S}}\left(1+\gamma({}^{2S+1}\ell_1)\,\frac{q_k^{\star\,2}}{m_\pi^2}\right)$$

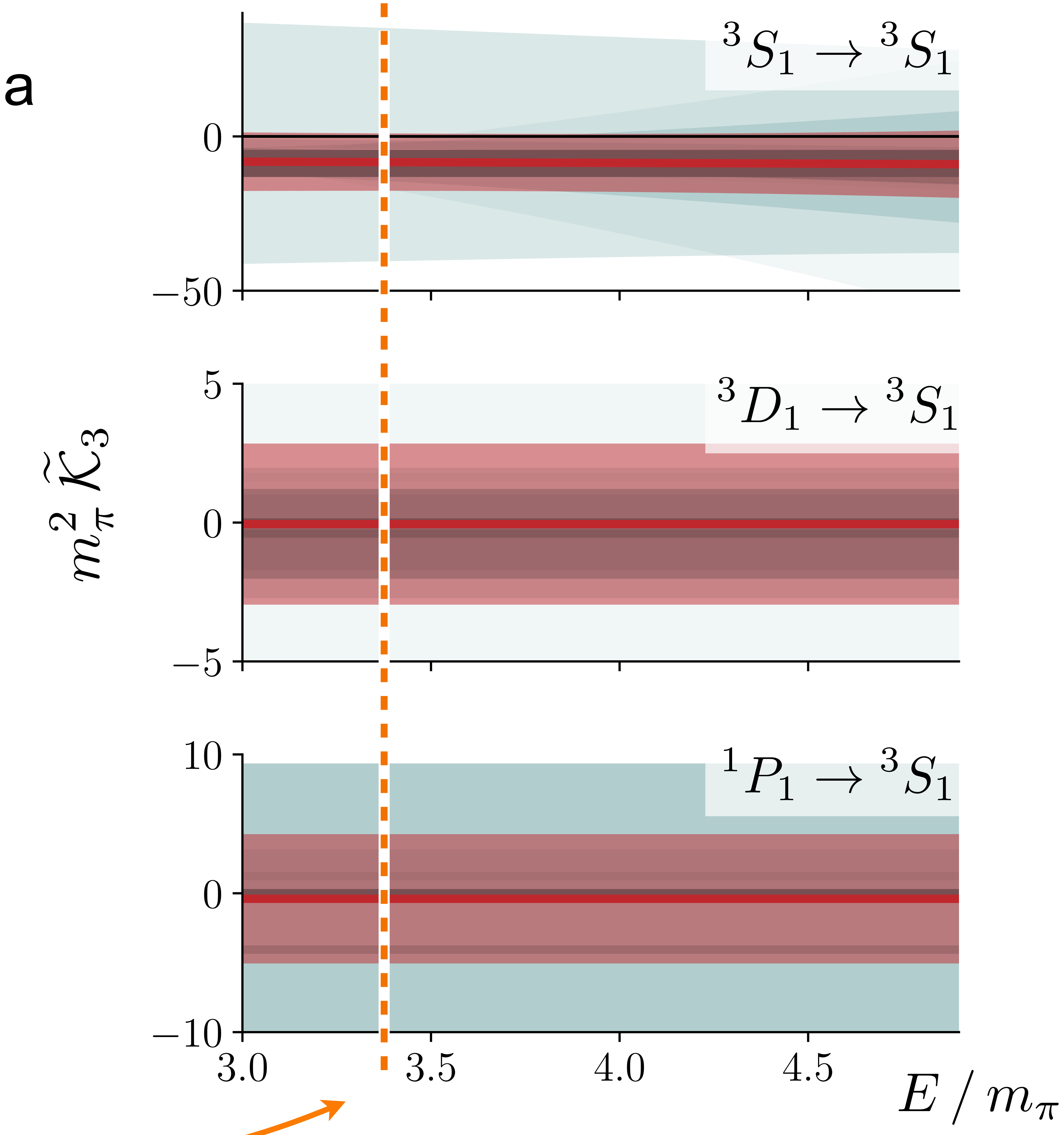
# Three-Pion Scattering from Lattice QCD

Map finite-volume spectrum to K matrices

- Use previously lattice QCD determined hadspec $\pi\pi$ pha
- Parameterize $\mathbf{K}_3$ with many flexible parameterizations

$$\mathbf{K}_3 = \mathbf{h}\,\widetilde{\mathcal{K}}_3\,\mathbf{h}$$




$E = 3.4 m_\pi$

R. Briceño, M. Hansen, AJ, R. Edwards, and C. Thomas
arXiv:2510.24894, *accepted* to PRL

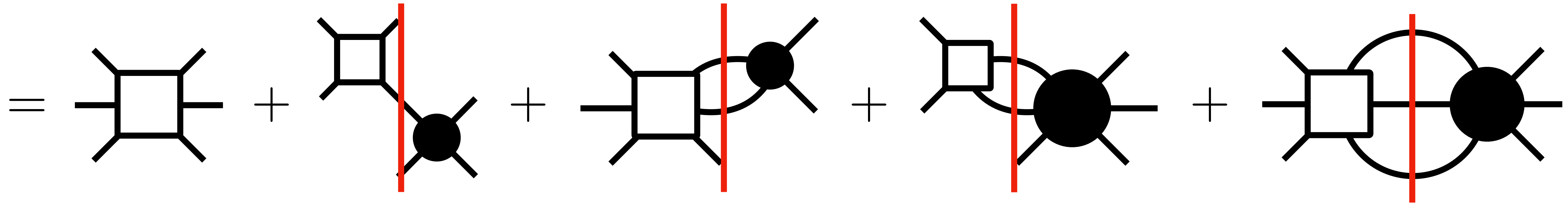
# Three-Pion Scattering from Lattice QCD

Reconstruct amplitudes via set of integral equations

- Subchannel resonances, strongly coupled partial waves

$$\mathcal{M}_3 = \sum_{J^P} \mathcal{M}_3^{J^P} \, \mathcal{R}_{J^P}$$

K matrix representation is set of integral equations



*Integral equations are solved numerically via Nyström method*
*— Approximate integral by quadrature, solve linear system*
*— care must be taken since kernels contain singularities*


AJ
Phys.Rev.D **108** (2023) 3, 034505

AJ and R. Briceño
Phys.Rev.D **109** (2024) 9, 096030

R. Briceño, C. Costa, AJ
Phys.Rev.D **111** (2025) 3, 036029

S. Dawid, M. Islam, R. Briceño
Phys. Rev. A **108**, (2023) 034016

S. Dawid, M. Islam, R. Briceño, AJ
Phys. Rev. A **109**, (2024) 043325

AJ, N. Chambers, R. Briceño
Phys.Rev.D **113** (2026) 1, 014019

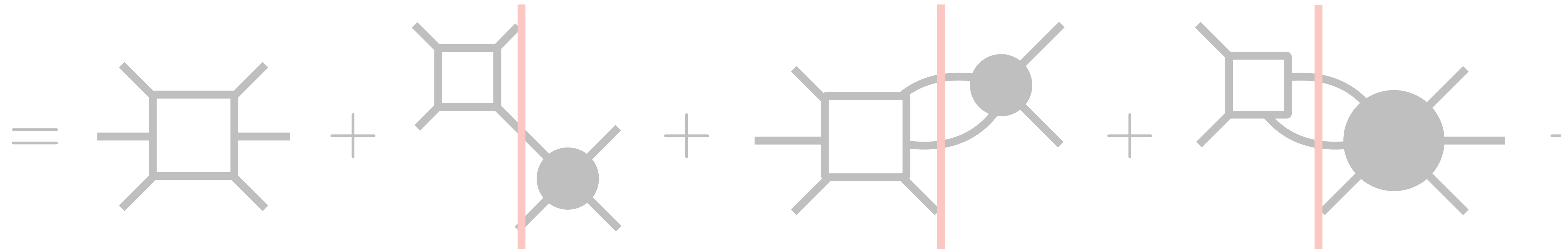
# Three-Pion Scattering from Lattice QCD

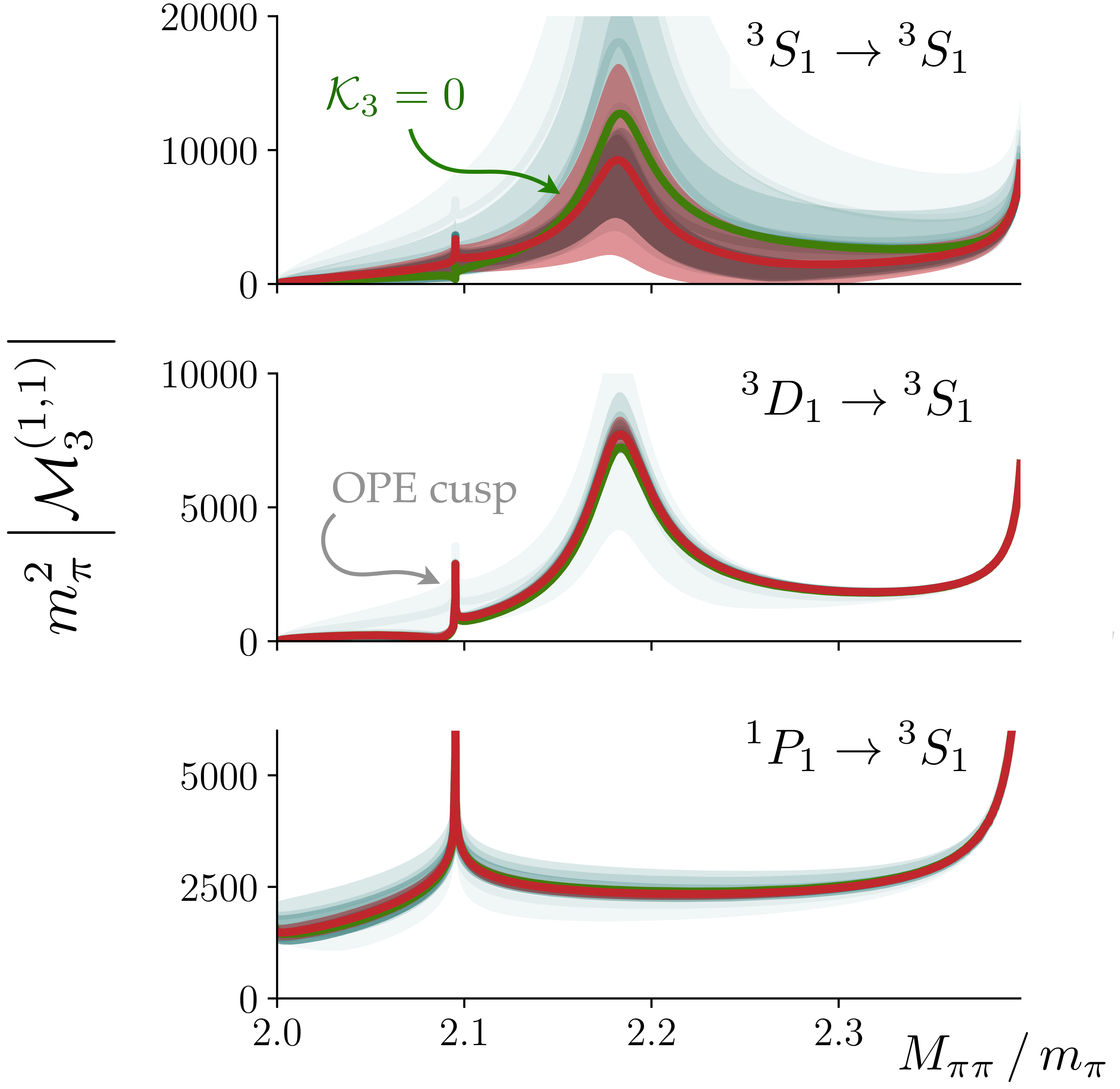
Reconstruct amplitudes via set of integral equations

- Subchannel resonances, strongly coupled partial waves

$$\mathcal{M}_3 = \sum_{J^P} \mathcal{M}_3^{J^P} \mathcal{R}_{J^P}$$

K matrix repres






AJ
Phys.Rev.D **108** (2023) 3, 034505

AJ and R. Briceño
Phys.Rev.D **109** (2024) 9, 096030

R. Briceño, C. Costa, AJ
Phys.Rev.D **111** (2025) 3, 036029

S. Dawid, M. Islam, R. Briceño
Phys. Rev. A **108**, (2023) 034016

S. Dawid, M. Islam, R. Briceño, AJ
Phys. Rev. A **109**, (2024) 043325

AJ, N. Chambers, R. Briceño
Phys.Rev.D **113** (2026) 1, 014019

R. Briceño, M. Hansen, AJ, R. Edwards, and C. Thomas
arXiv:2510.24894, *accepted* to PRL

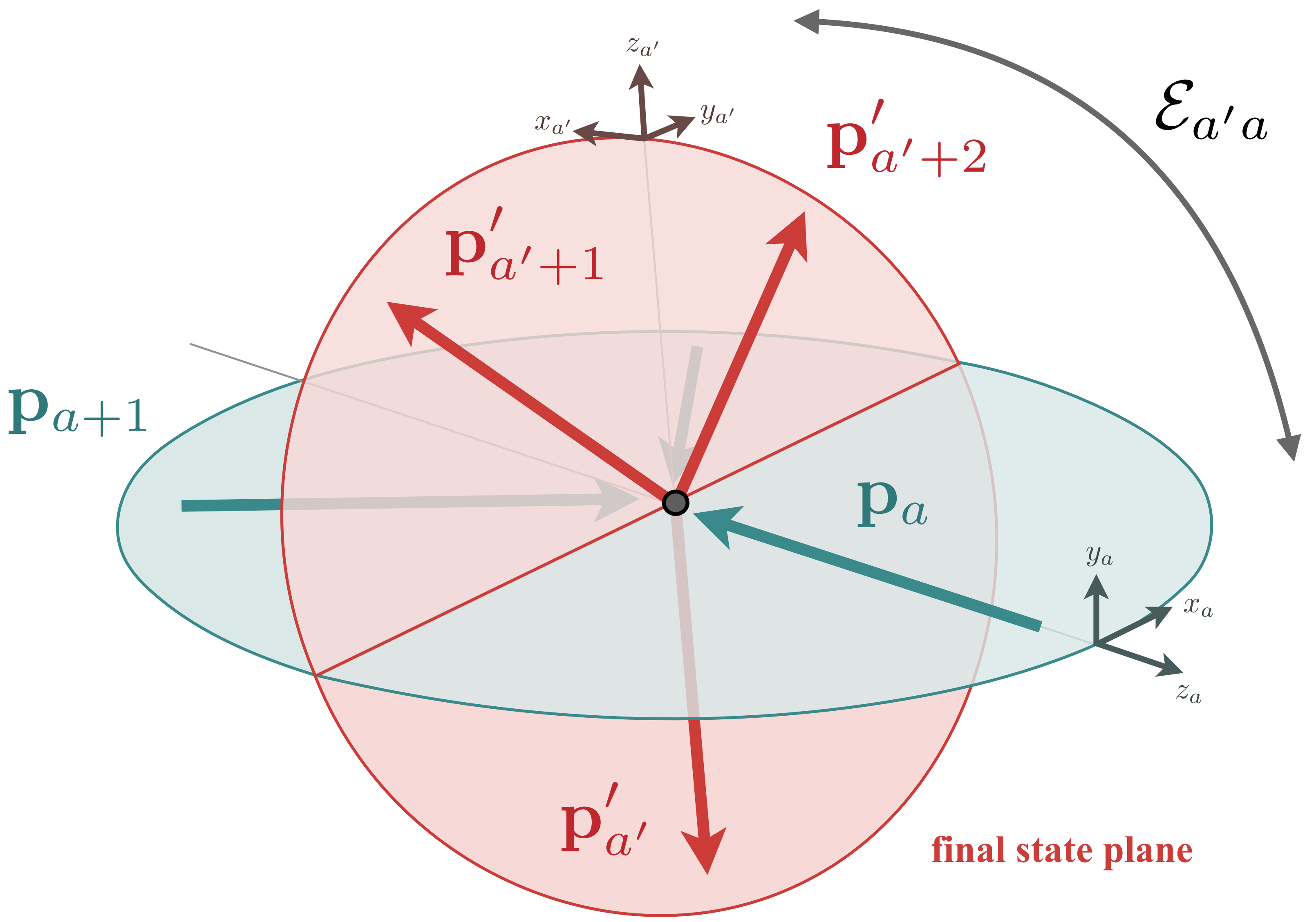
# Dalitz observables

Can study amplitude features through **intensity distributions**

- Full amplitude requires recoupling between spectator choices (*symmetrizing*)

$$\mathcal{I} = \sum_{I_2', I_2} \sum_{\lambda, \lambda'} \left| \mathcal{M}_{3, I_2'\lambda', I_2\lambda} \right|^2$$

$$\mathcal{M}_{3, I_2'\lambda', I_2\lambda} = \sum_{a,b} \mathcal{R}^{(a)}_{I_2'\lambda'} \cdot \mathcal{M}_3^{(a,b)} \cdot \mathcal{R}^{(b)\,\top}_{I_2\lambda}$$




AJ, N.Chambers, & R. Briceño
Phys.Rev.D **113** (2026) 014019

**Related Talk**

Nick Chambers

**Symmetrizing Three-body Partial-Wave Amplitudes**

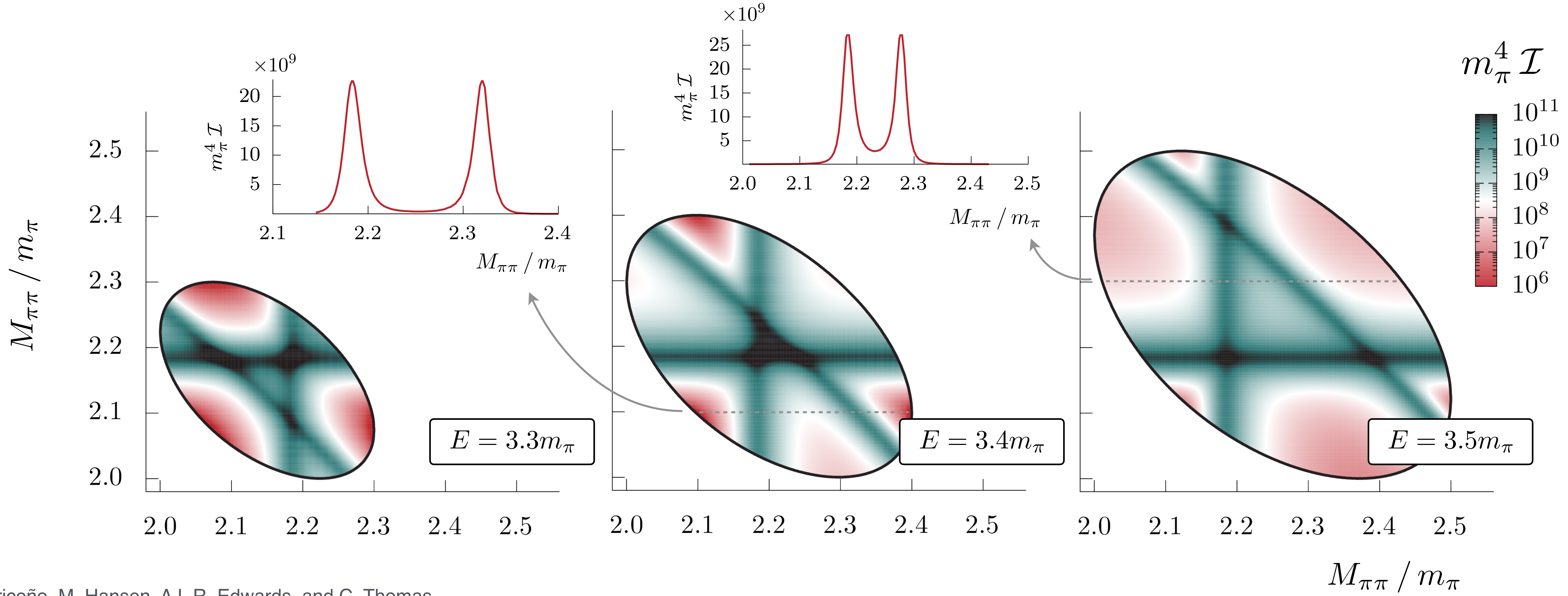
*this session*

**Nick Chambers**
William & Mary

# Dalitz observables

Can study amplitude features through **intensity distributions**

- Full amplitude requires recoupling between spectator choices (*symmetrizing*)
- e.g., view sub-channel resonances via Dalitz-plots




R. Briceño, M. Hansen, AJ, R. Edwards, and C. Thomas
arXiv:2510.24894, *accepted* to PRL

# Summary

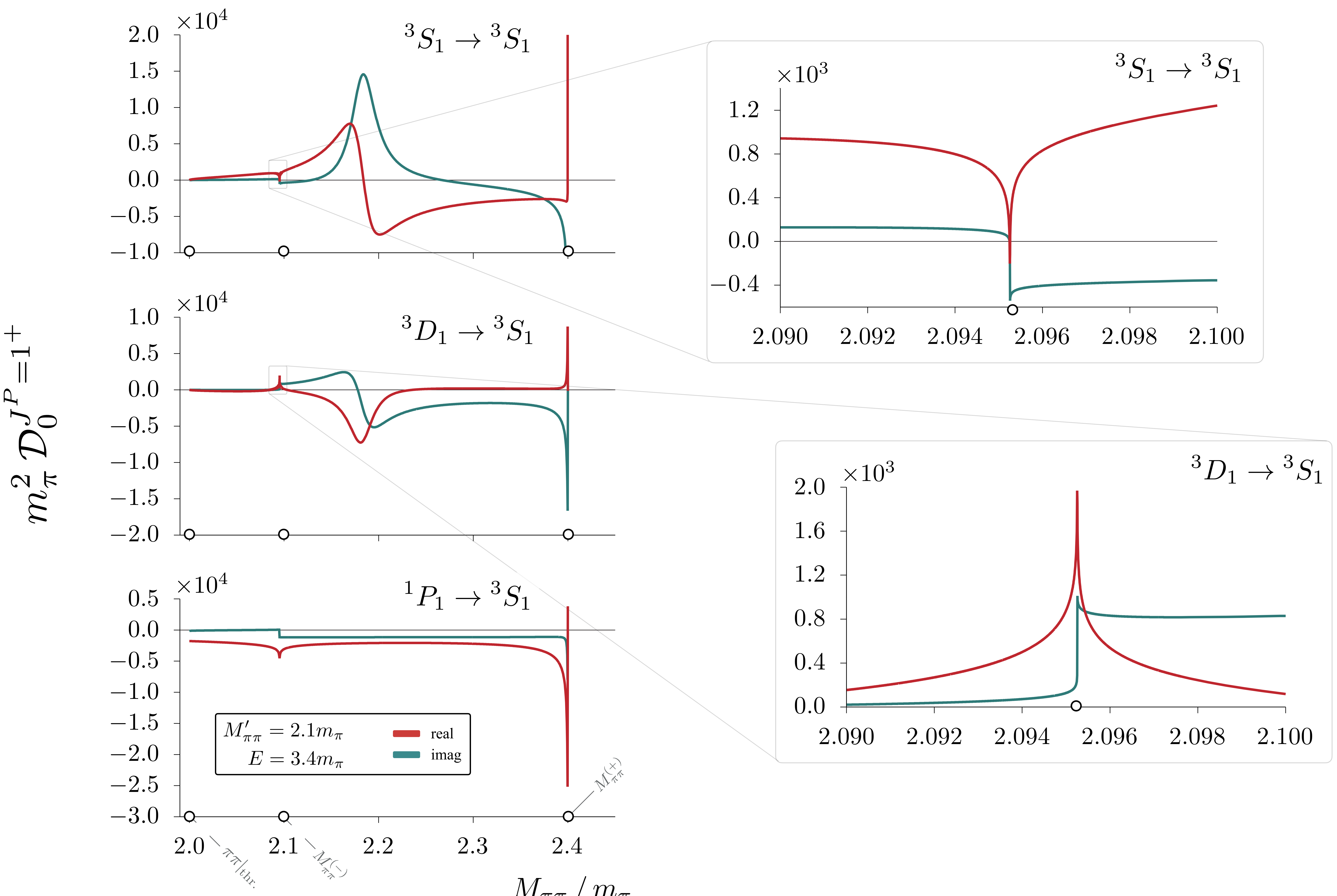
Lattice QCD spectroscopy is very active, making immense progress in few-body dynamics

- First calculation of isotensor $3\pi$ scattering amplitude
- Three-hadron framework was years in development, now have first sets of calculations
- Future applications will aim to extract more interesting reactions and push new observables

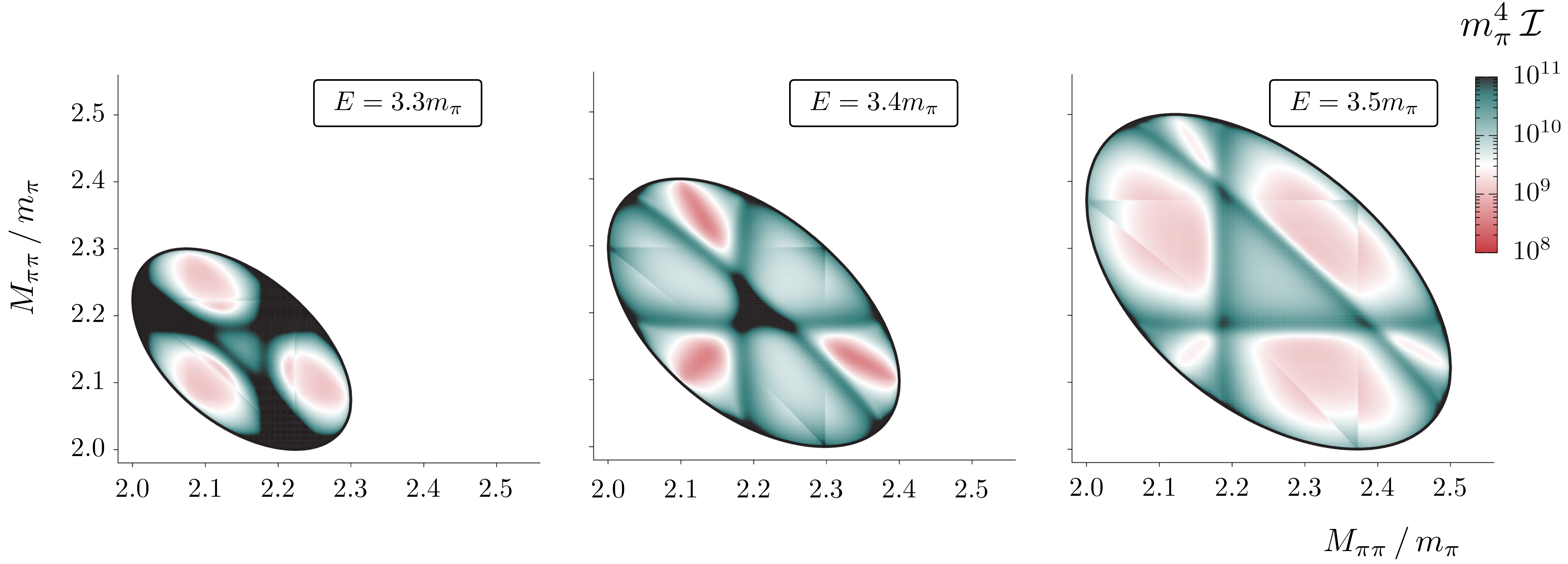
# K Matrix Parameters

| fit | $\alpha(^3S_1\|^3S_1)$ | $\beta(^3S_1\|^3S_1)$ | $\gamma(^3S_1)$ | $\alpha(^3S_1\|^3D_1)$ | $\alpha(^3D_1\|^3D_1)$ | $\alpha(^1P_1\|^1P_1)$ | dof | Corr | $\chi^2$ | $e^{-(2n_p+\chi^2)/2}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $-9(5)$ | | | | | | 11 | 1 | 5.1 | 0.029 |
| 2 | $-8(5)$ | $-0.2(0.4)$ | | | | | 10 | $\begin{pmatrix} 1.00 & -0.56 \\ & 1.00 \end{pmatrix}$ | 5.0 | 0.011 |
| 3 | $-7.1(0.8)$ | | $-4(2)$ | | | | 10 | $\begin{pmatrix} 1.00 & -0.29 \\ & 1.00 \end{pmatrix}$ | 4.7 | 0.013 |
| 4 | $-6(40)$ | $-0.1(0.9)$ | $-4(40)$ | | | | 9 | $\begin{pmatrix} 1.00 & -0.62 & -0.97 \\ & 1.00 & 0.42 \\ & & 1.00 \end{pmatrix}$ | 4.6 | 0.005 |
| 5 | $-8(4)$ | $0(1)$ | | $0(2)$ | | | 9 | $\begin{pmatrix} 1.00 & 0.52 & 0.70 \\ & 1.00 & 0.94 \\ & & 1.00 \end{pmatrix}$ | 4.6 | 0.005 |
| 6 | $-10(2)$ | | | $0(2)$ | | | 10 | $\begin{pmatrix} 1.00 & 0.63 \\ & 1.00 \end{pmatrix}$ | 4.7 | 0.013 |
| 7 | $-8(2)$ | $0(3)$ | | $0(20)$ | $0(5)$ | | 8 | $\begin{pmatrix} 1.00 & 0.82 & -0.83 & -0.83 \\ & 1.00 & -1.00 & -0.99 \\ & & 1.00 & 1.00 \\ & & & 1.00 \end{pmatrix}$ | 4.4 | 0.002 |
| 8 | $-10(2)$ | | | $0(1)$ | $0(1)$ | | 9 | $\begin{pmatrix} 1.00 & 0.02 & -0.53 \\ & 1.00 & -0.29 \\ & & 1.00 \end{pmatrix}$ | 4.7 | 0.0047 |
| 9 | $-9(2)$ | | | | | $-3(10)$ | 10 | $\begin{pmatrix} 1.00 & -0.21 \\ & 1.00 \end{pmatrix}$ | 5.0 | 0.011 |
| 10 | $-8(2)$ | $-0.2(0.3)$ | | | | $1(2)$ | 9 | $\begin{pmatrix} 1.00 & -0.01 & 0.04 \\ & 1.00 & 0.03 \\ & & 1.00 \end{pmatrix}$ | 5.0 | 0.0042 |

# OPE Singularities



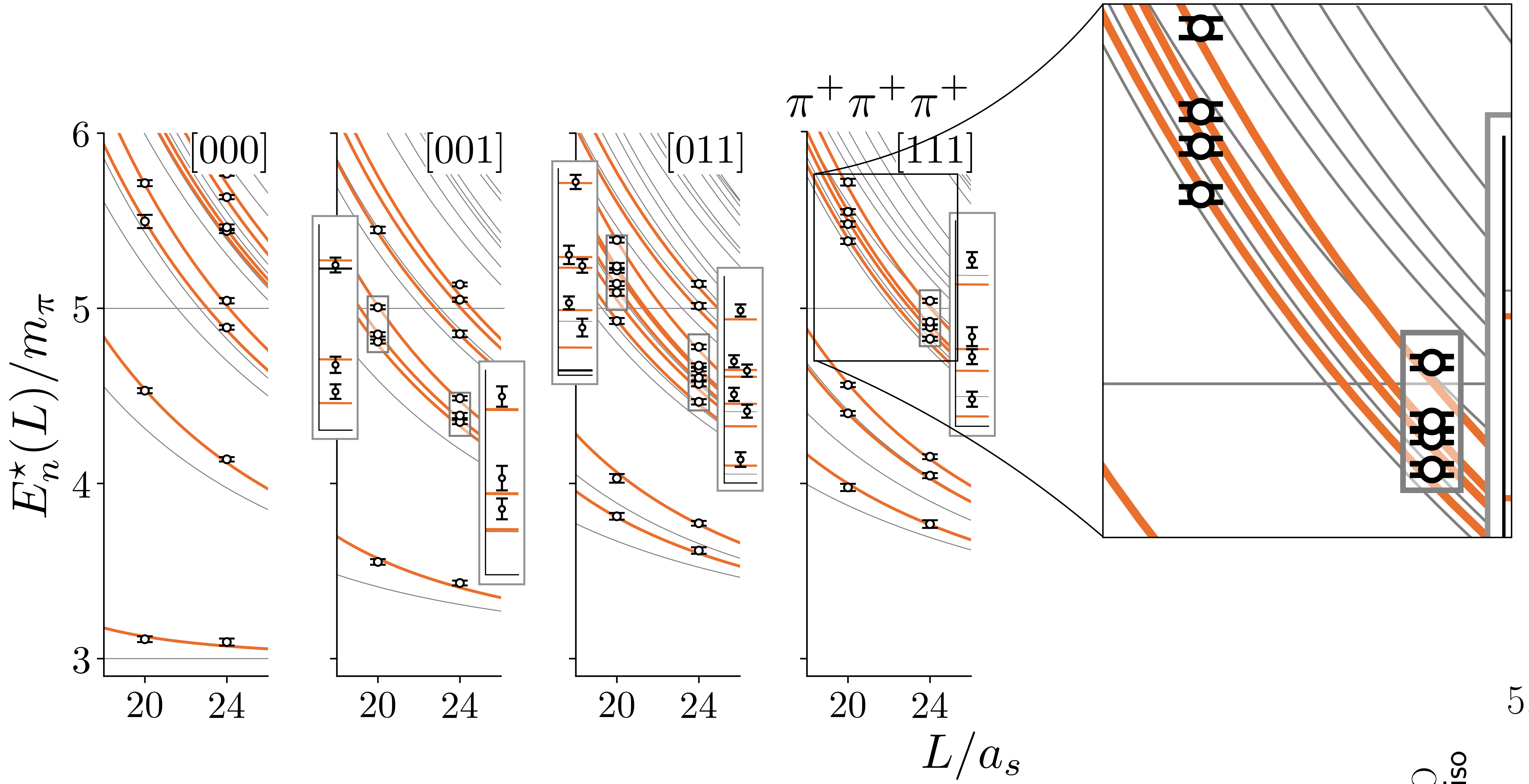
# OPE Dalitz Plots



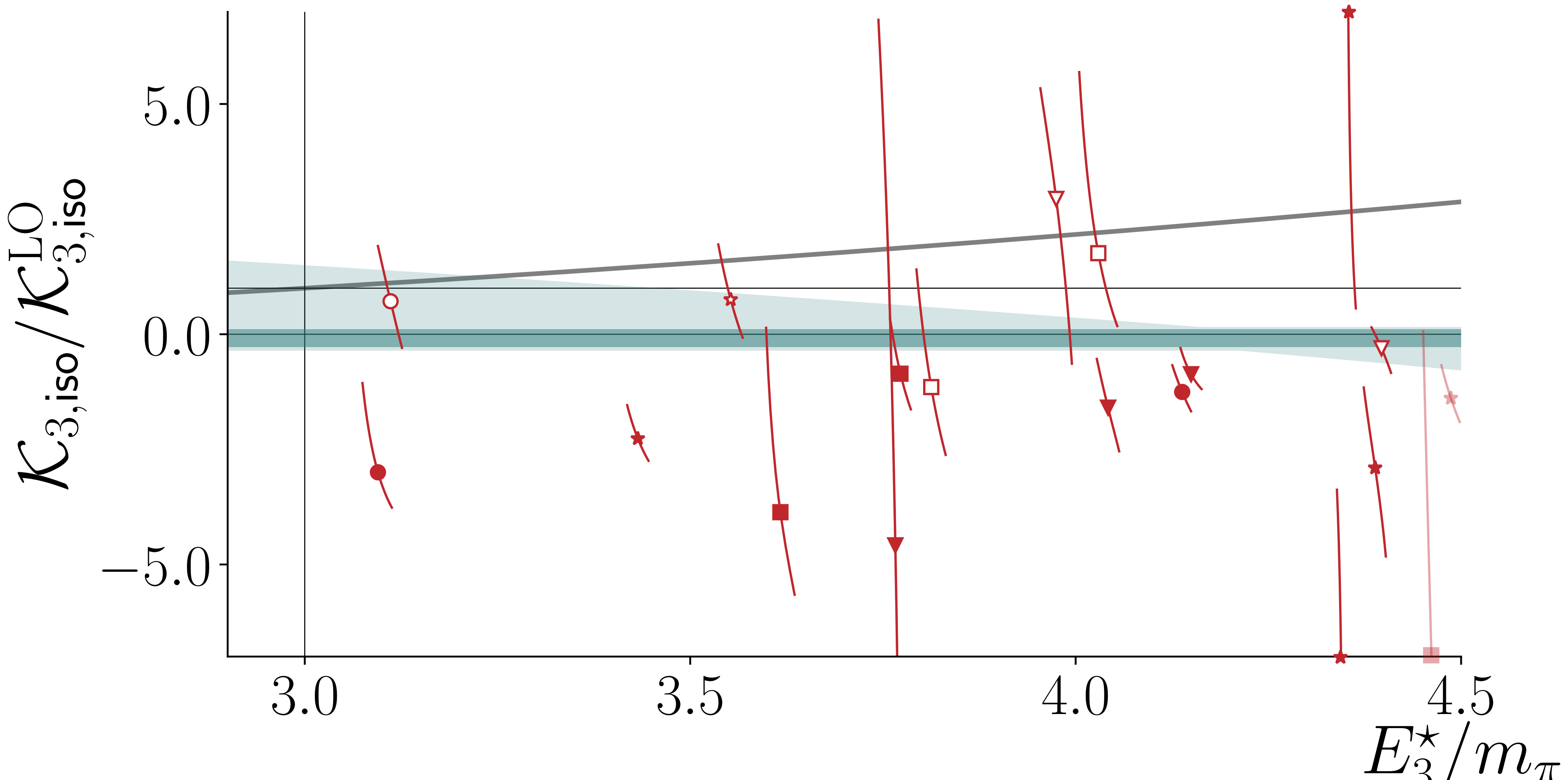
# Three Pions

First applications were in $3\pi^+$ in $I(J^P) = 3(0^-)$

- Simple application without contamination from higher partial waves




*103 energy levels described by three numbers $m_\pi$, $a_{\pi\pi}^{I=2}$, $\mathscr{K}_{3,\mathrm{iso}}^{I=3}$*



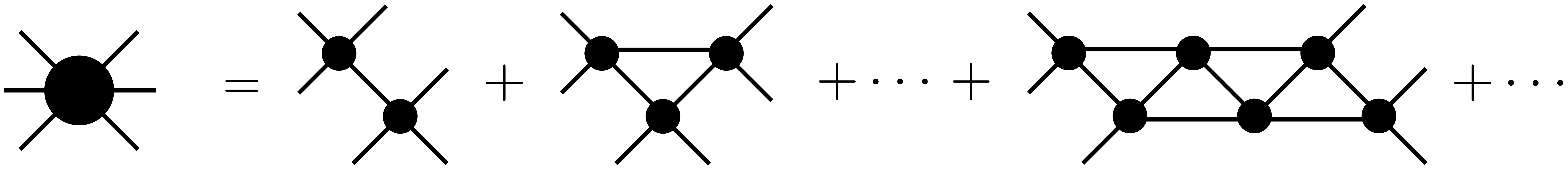

hadspec

M. Hansen et al. [HadSpec]
Phys. Rev. Lett. **126**, (2021) 012001

# Three Pions

First applications were in $3\pi^+$ in $I(J^P) = 3(0^-)$

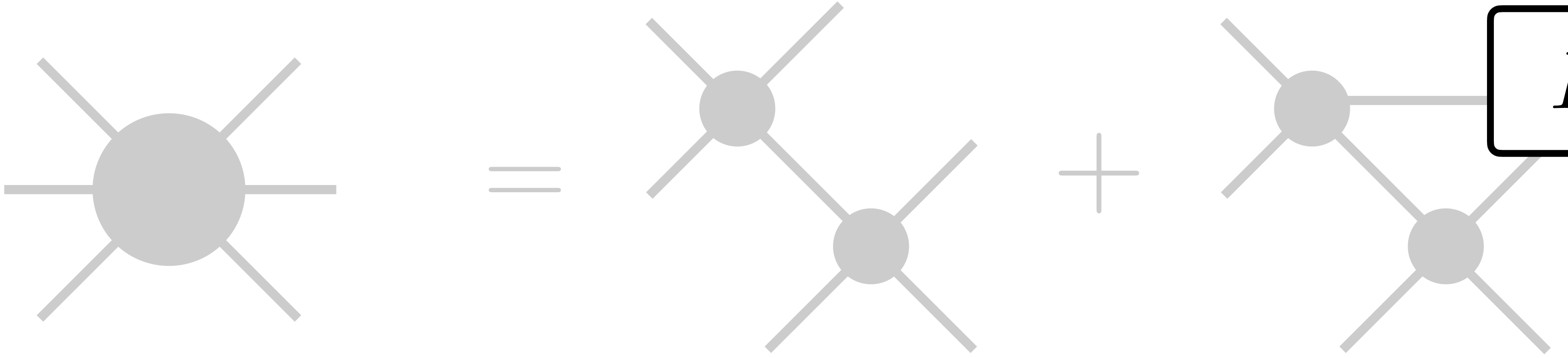
- Simple application without contamination from higher partial waves
- System dominated by ladder of exchanges

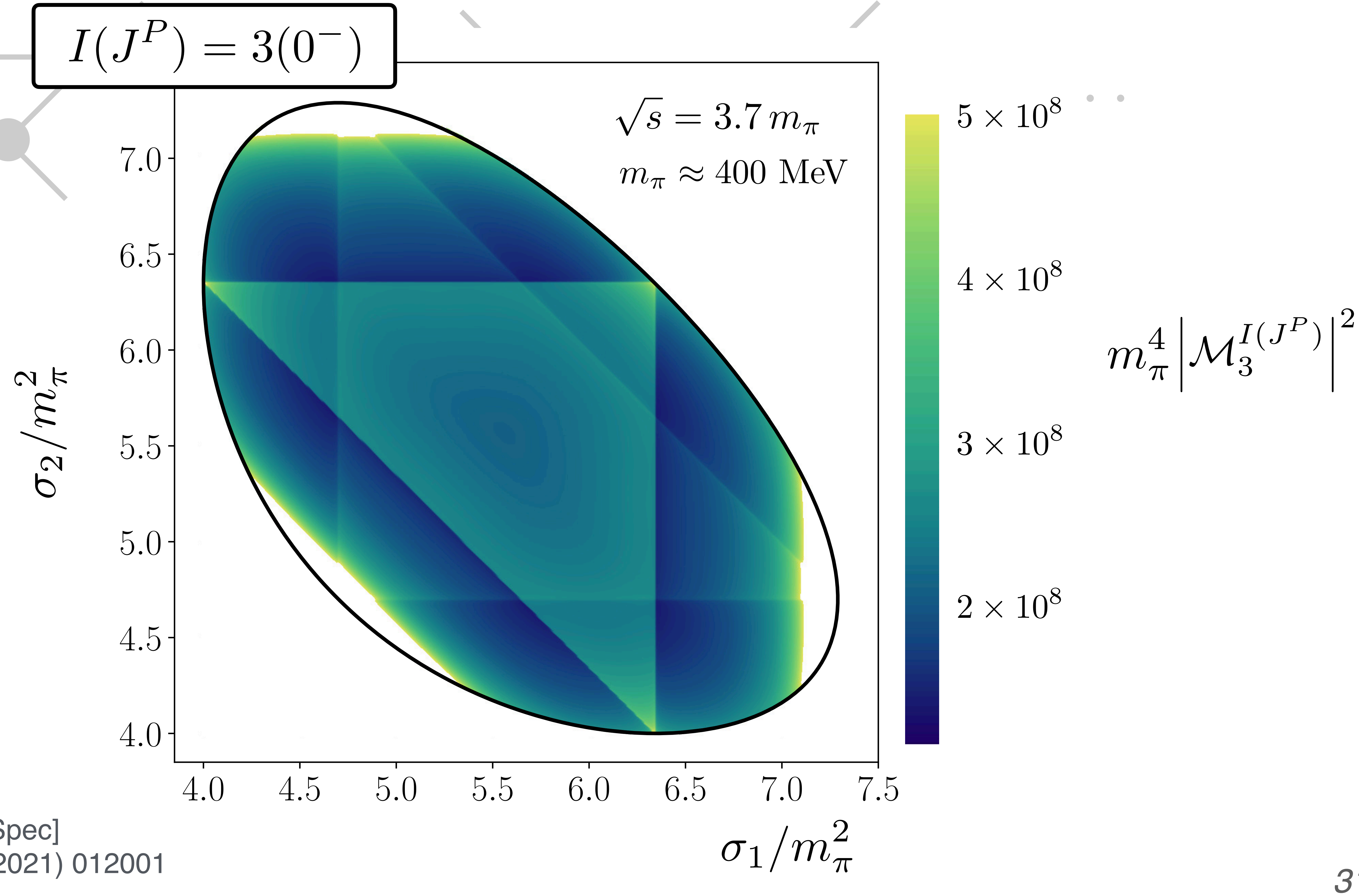


# Three Pions

First applications were in $3\pi^+$ in $I(J^P) = 3(0^-)$

- Simple application without contamination from higher partial waves
- System dominated by ladder of exchanges






$\sigma_j = (\sqrt{s} - E_j)^2 - p_j^2$

M. Hansen et al. [HadSpec]
Phys. Rev. Lett. **126**, (2021) 012001

# Higher Partial Waves

## Can access higher waves — coupled integral equations

$$\mathcal{M}_3 = \sum_{J^P} \mathcal{M}_3^{J^P}\, \mathcal{R}_{J^P}$$

***Rotational Dependence***
*— Clebsch-Gordan coefficients*
*— Wigner D matrix elements*
*— associated factors*

***Partial Wave Amplitudes***
*— depends on 'energy' variables only*
*— Matrix in $LS$-space for given $J^P$*

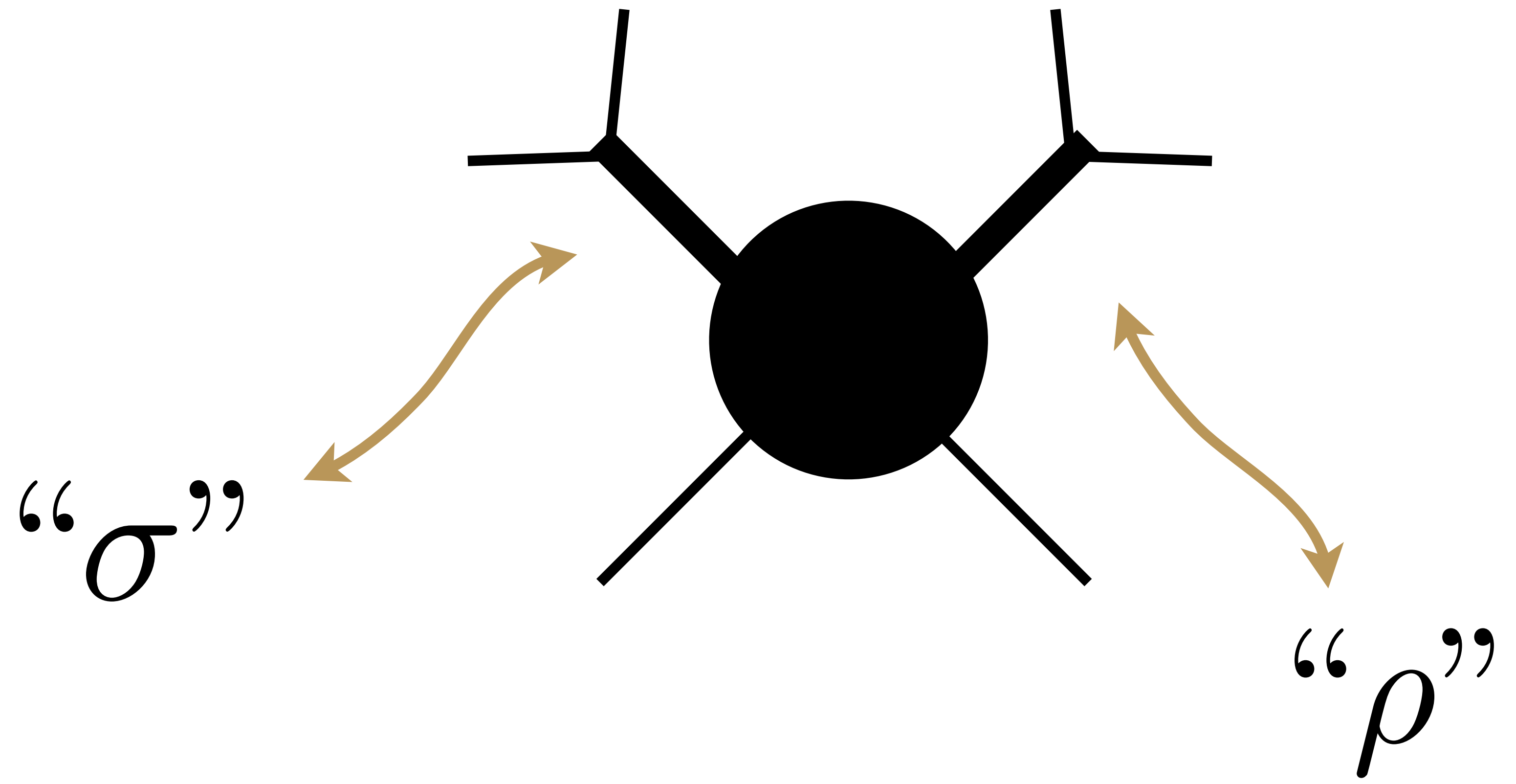
$$\mathcal{R}_{J^P} \sim \sum_{m_J} Z_{L'S'}^{Jm_J\,*}(\hat{\mathbf{p}}) Z_{LS}^{Jm_J}(\hat{\mathbf{k}})$$

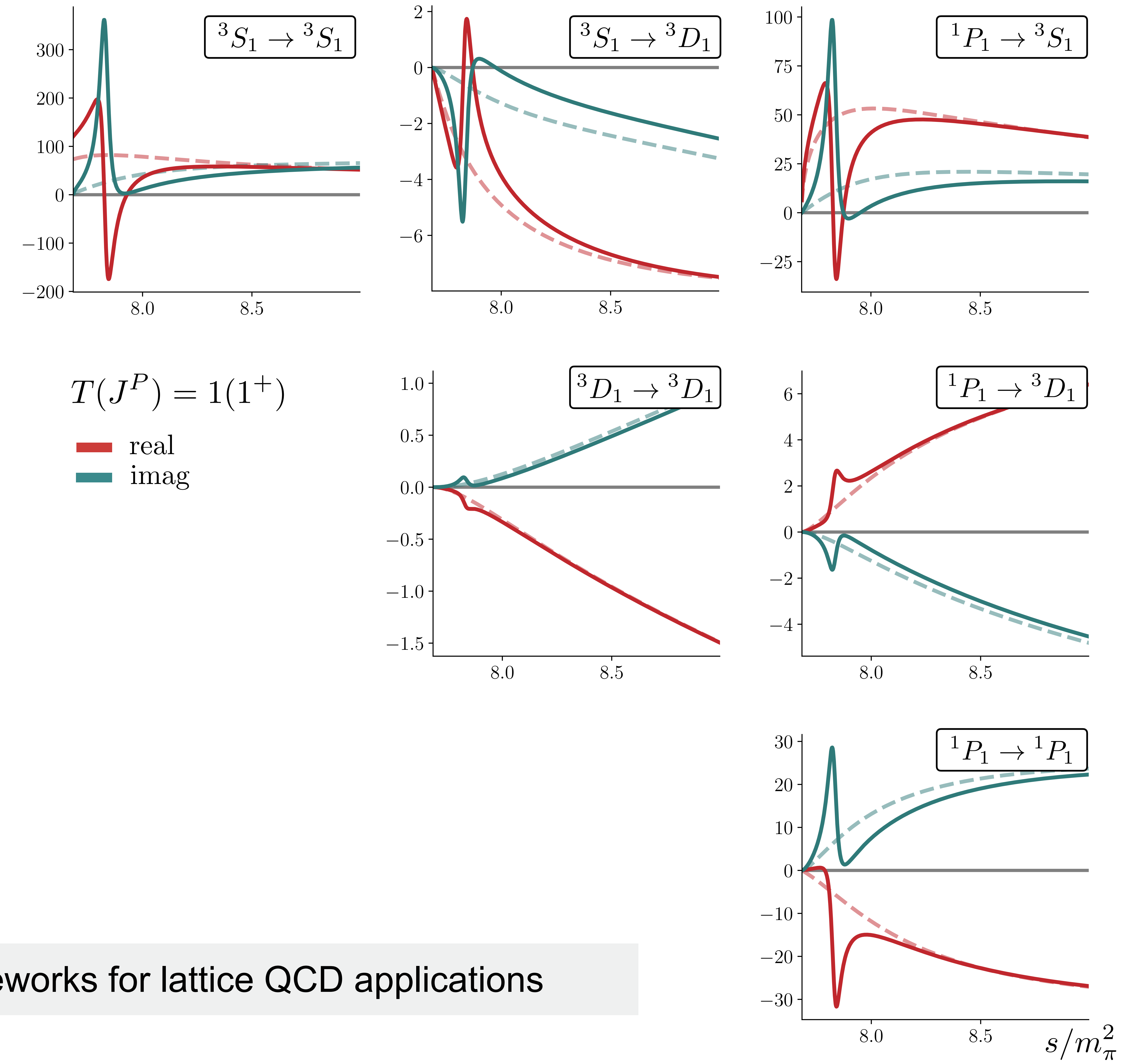
$$Z_{LS}^{Jm_J}(\hat{\mathbf{k}}) = \sqrt{4\pi(2L+1)} \sum_{\lambda} \langle J\lambda | L0, S\lambda \rangle\, D_{m_J\lambda}^{(J)}(\hat{\mathbf{k}})\, Y_{S\lambda}^{*}(\hat{\mathbf{a}})$$

| $I_{3\pi}^G$ | $J^{PC}$ | $([\pi\pi]_\ell^I \pi)_L$ |
|---|---|---|
| $3^-$ | $0^{-+}$ | $([\pi\pi]_S^2\pi)_S$ |
| | $1^{-+}$ | none |
| | $1^{++}$ | $([\pi\pi]_S^2\pi)_P$ |
| $2^-$ | $0^{--}$ | $([\pi\pi]_S^2\pi)_S$, $([\pi\pi]_P^1\pi)_P$ |
| | $1^{--}$ | $([\pi\pi]_P^1\pi)_P$ |
| | $1^{+-}$ | $([\pi\pi]_S^2\pi)_P$, $([\pi\pi]_P^1\pi)_S$, $([\pi\pi]_P^1\pi)_D$ |
| $1^-$ | $0^{-+}$ | $([\pi\pi]_S^{0,2}\pi)_S$, $([\pi\pi]_P^1\pi)_P$ |
| | $1^{-+}$ | $([\pi\pi]_P^1\pi)_P$ |
| | $1^{++}$ | $([\pi\pi]_S^{0,2}\pi)_P$, $([\pi\pi]_P^1\pi)_S$, $([\pi\pi]_P^1\pi)_D$ |
| $0^-$ | $0^{--}$ | $([\pi\pi]_P^1\pi)_P$ |
| | $1^{--}$ | $([\pi\pi]_P^1\pi)_P$ |
| | $1^{+-}$ | $([\pi\pi]_P^1\pi)_S$, $([\pi\pi]_P^1\pi)_D$ |

# Higher Partial Waves

Can access higher waves — coupled integral equations




AJ and R. Briceño
Phys.Rev.D **109** (2024) 9, 096030

R. Briceño, C. Costa, AJ
Phys.Rev.D **111** (2025) 3, 036029

Testing frameworks for lattice QCD applications