Setting the physical scale for lattice simulations using the gluon propagator and the understanding of finite size effects

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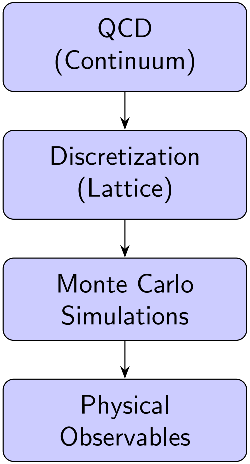
Outline

1. Introduction and Motivation
2. Lattice Setup
3. Lattice Artefacts & Finite Size Effects
4. Summary & Outlook

Lattice QCD and Scale Setting



- LQCD is a **first-principles, nonperturbative** approach to QCD
- Simulations yield **dimensionless quantities**
- Comparison with experiment need a **physical scale**, i.e. **lattice spacing** a in physical units
- **Traditional Methods:**
  - Heavy quark potential / Sommer scale (r_0, r_1).
  - Wilson flow scales (t_0, w_0).
  - Hadronic observables (m_Ω, f_π).
  - **The Ambiguity:** Different methods typically show systematic variations at the few-percent level.

- Issues not related:
  - **breaking of rotational symmetry**

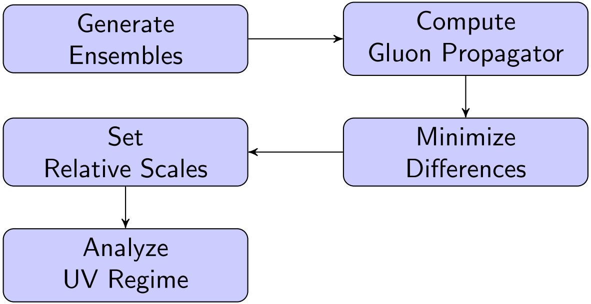
    The symmetry group is $H4$ and not $O(4)$
  - **finite volume effects**

The Landau Gauge Gluon Propagator and the Physical Scale

Advantages

- Relies **solely** on the Landau gauge gluon propagator.
- **Computational Efficiency:** Avoids ghost propagator computation, which is very time-consuming.
- **Accessibility:** Modern LQCD codes include a gluon propagator module (gauge fixing, and $D(p^2)$).
- **Large statistical ensembles** ($\sim 10^4$ configurations per ensemble) produce smooth propagator.
- **Efficient** way of accessing $a(\beta)$

Methodology Overview



1. Generate large ensembles of gauge configurations for different β values.
2. Compute the Landau gauge gluon propagator for each β value.
3. Minimize the difference between propagators from different setups to get the relative scale.
4. Study the UV regime to identify and handle finite size effects.

Important: the function $a(\beta)$ depends on discretization (action) used and on the momentum definitions.

Definitions & Tensorial Structure

The gluon propagator in the Landau gauge is the two-point vacuum correlation function:

$$\langle\, A^a_\mu(p')\; A^b_\nu(p)\,\rangle \;=\; V\;\delta(p'+p)D^{ab}_{\mu\nu}(p)$$

In the Landau gauge, the propagator is strictly orthogonal to the gluon momentum

$$D^{ab}_{\mu\nu}(p) \;=\; \delta^{ab}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) D(p^2) \;=\; \delta^{ab}\left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\frac{Z(p^2)}{p^2}$$

$\longrightarrow$ $Z(p^2)$ is the gluon dressing function

Lattice Setup

Lattice Setup

- Pure Yang-Mills SU(3) theory with Wilson action
- Two different β values and Two different physical volumes
- 10^4 configurations per ensemble
- Only the Landau gauge will be considered (Gribov noise will be disregarded)

β	Volume	# Configs	a^{-1}(GeV)	L (fm)
6.0	32^4	10^4	1.943	3.25
6.0	48^4	9034	1.943	4.88
6.2	48^4	10^4	2.718	3.49
6.2	64^4	10^4	2.718	4.64

- Lattice spacing measured from the string tension.
- $a^{-1}(\beta = 6.0)$ will be used to benchmark the relative value of $a^{-1}(\beta = 6.2)$.

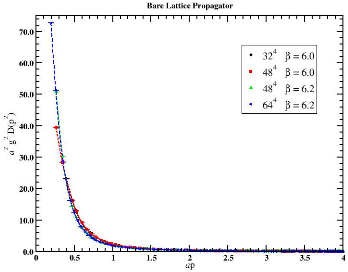
Further definitions

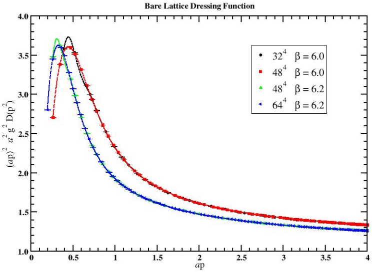
- Use as momentum the improved momentum

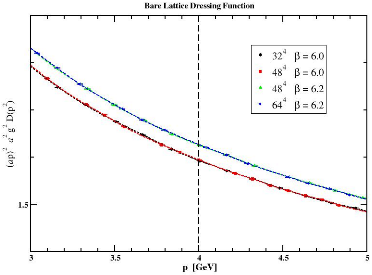
$$p_\mu = \frac{2}{a} \sin\left(\frac{\pi}{L}\, n_\mu\right) \qquad \text{with} \qquad n_\mu = -L/2, \, \ldots, \, -1, \, 0, \, 1, \, \ldots, \, L/2 - 1 \; ;$$

- $D(p^2)$ renormalized in the MOM scheme, i.e. $D(\mu^2) = 1/\mu^2$ at $\mu = 4$ GeV.
- Only data surviving **cylindrical and conical momentum cuts for** $p > 0.7$ **GeV** will be considered (reducing the effects of breaking of rotational symmetry). For **smaller momenta all data** will be included.
- The large statistical ensembles allows the use of **cubic splines to smoothly interpolate** data. In the interpolation the zero momentum points will be excluded (they introduce a bias for the lower momentum).

Bare Lattice Data







The Relative Scale Setting Minimization

Minimization Target

On each volume, rescale the momentum for $\beta = 6.2$ and minimize the renormalized difference

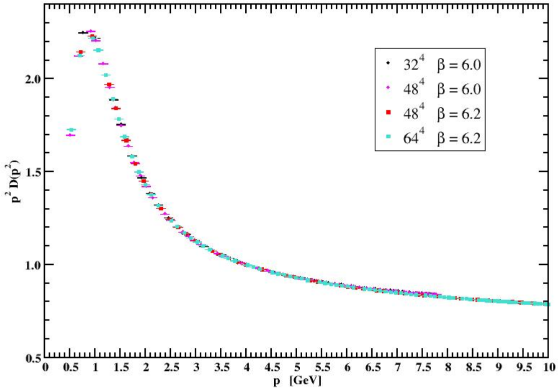
$$\chi^2_{\text{rel}} = \sum_i \int_{p_{min}}^{p_{max}} dp \left[D_{\beta=6.0}(p^2) - D_{\beta=6.2}(p^2) \right]^2$$

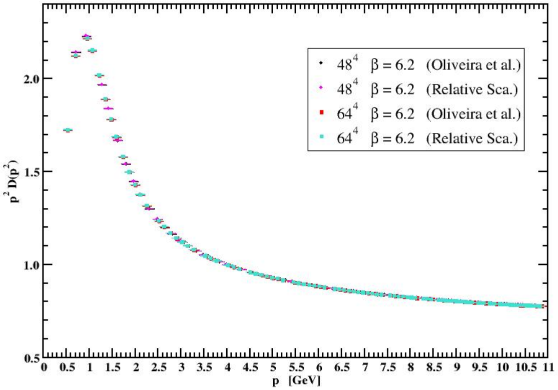
- Set $p_{max} = 7$ GeV (no difference if used instead $p_{max} = 5$ GeV).
- Set $p_{min} = 1.5$ GeV.
- By scaling the momentum space values properly, the data seamlessly collapses onto a unique, well-defined smooth curve.
- The minimization gives

$$a^{-1}(\beta = 6.0) = 1.943 \text{ GeV} \qquad \text{and} \qquad a^{-1}(\beta = 6.2) = 2.7052 \text{ GeV}.$$

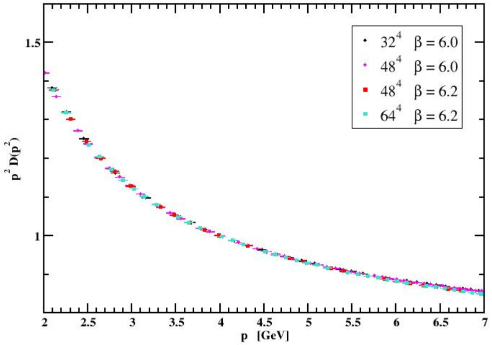
The latter should be compared with the value on the first table $a^{-1}(\beta = 6.2) = 2.718$ GeV.

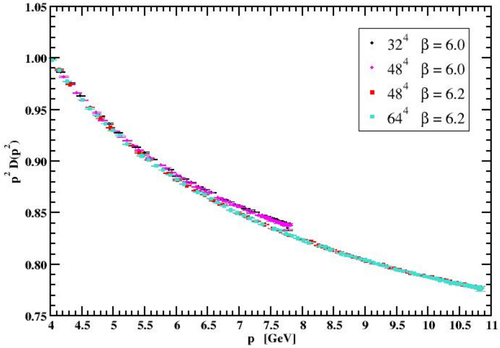
Lattice Data I





Lattice Data II





Comments on the UV renormalized data

- **Observation I:** good agreement for each lattice spacing and both volumes (small volume effects?)
- **Observation II:** scaling violations above 5 GeV (lattice spacing effects?)
- **Observation III:** all data sets compatible with the functional form for $D(p^2)$ predicted by RG-improved perturbation theory.

- **Origin:** interplay of finite lattice spacing and finite volume effects

- **Our approach:** Assume that the UV differences are due to lattice spacing effects and rely on perturbation theory to try to understand them.

An Effective Description of Finite Size Effects

Working hypothesis

$$D_L(p^2) \;=\; D_C(p^2)\left(1+\sum_j c_j(a,V)\;\alpha_s^j(\mu)\;\ln\frac{p^2}{\zeta^2}\right)^j$$

where μ is the **renormalization scale**, ζ is a mass scale and, at large momentum,

$$D_C(p^2) \;=\; \frac{Z(\mu)}{p^2}\left(1+\frac{33\,\alpha_s(\mu)}{12\,\pi}\;\ln\frac{p^2}{\zeta^2}\right)^{-\gamma} \qquad \text{(one-loop RG improved)}$$

with $\gamma = 13/22$ being the gluon anomalous dimension.

Lattice Log-Log Plots

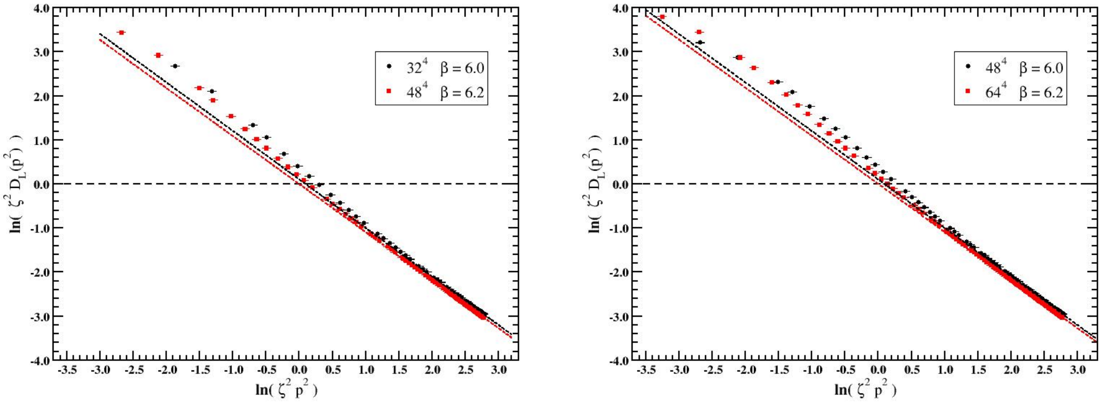


Figure 1: Left: Log-Log plot for the smallest volume. Right: Log-Log plot for the smallest volume. Dashed Lines: outcome of linear fit

Understanding the data in the UV

Stochastic Lattice Perturbation Theory (extrapolated to the continuum)

$$D_{sPt}(p^2) \; = \; Z \; \frac{2.29368 \; - \; 0.24697 \; \ln\left(a^2 p^2\right)}{p^2}$$

Effective Parameterization

$$D_L(p^2) \; = \; D_{sPt}(p^2) \; + \; \sum_{j=0} c_j \; p^{2j}$$

Understanding the data in the UV

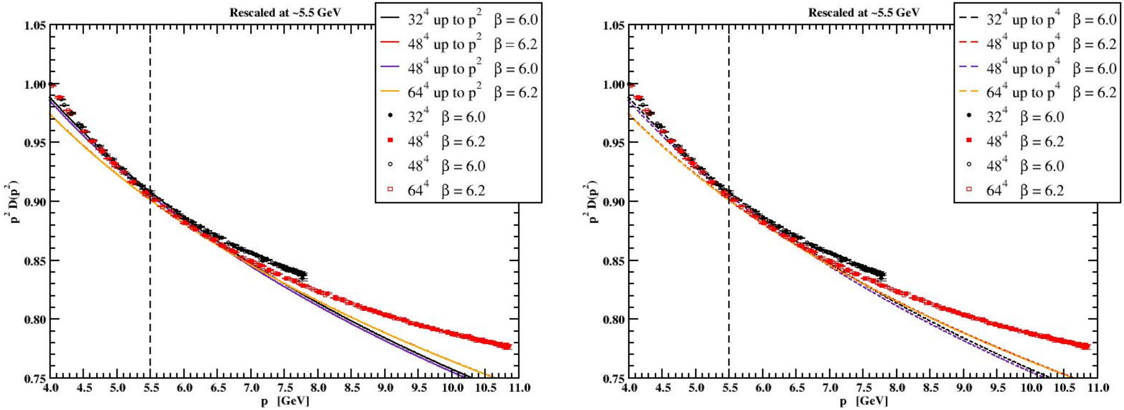


Figure 2: **UV gluon dressing function corrected for the lattice artefacts**. The lines in the figures refer to the outcome of the fits when only terms up to p^2 are included to model the lattice artefacts. The dashed lines in the figures were obtained including terms up to p^4 to model the lattice artefacts.

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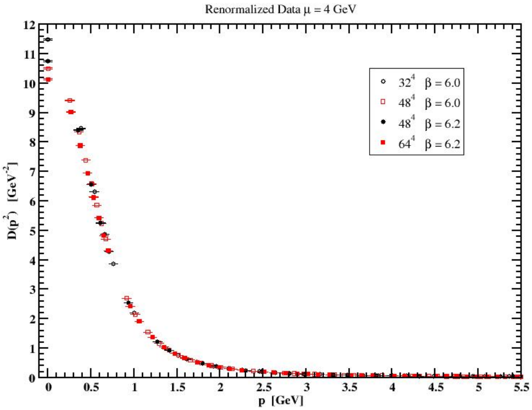
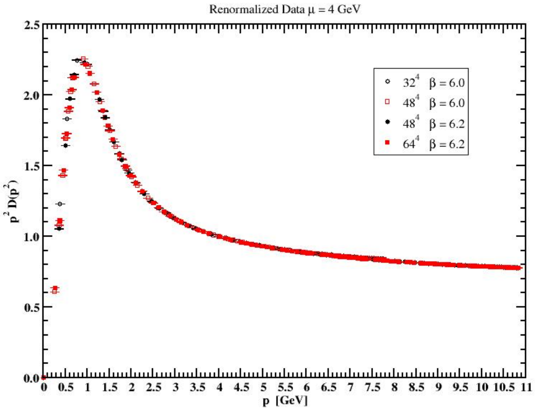
All Lattice Data Corrected in UV





Summary and Conclusions

- **Accmplishment:** large statistical ensembles + cuts produce smooth gluon propagator data
- **Simple & Robust Method:** The Landau gauge gluon propagator offers a reliable scale-setting technique
- **UV Control:** Effective description and correction of finite size lattice artefacts that dominate the UV regime (> 5 GeV).

- **Extension to other N-point Green functions:** The minimization and correction procedure discussed can be adapted to other Green functions but need large statistical ensembles
- **Further developments:** help with the sampling and noise reduction techniques will be welcome

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Thank you!

Questions & Comments?