Renormalisation of Chiral Gauge Theories with Non-Anticommuting γ_5 at the Multi-Loop Level

Matthias Weißwange

In Collaboration with

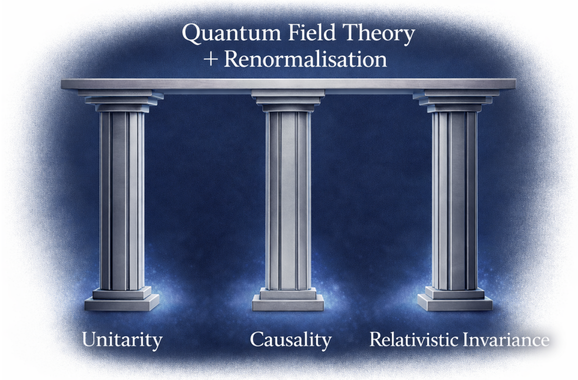
Dominik Stöckinger, Andreas von Manteuffel, Paul Kühler

Renormalisation and Symmetries

Renormalisation

Quantum Corrections $\Longrightarrow$ UV Divergences $\Longrightarrow$ Renormalisation

- Mathematically: extending products of distributions to the full spacetime
- Physically: reparametrisation of the theory
- Conditions:



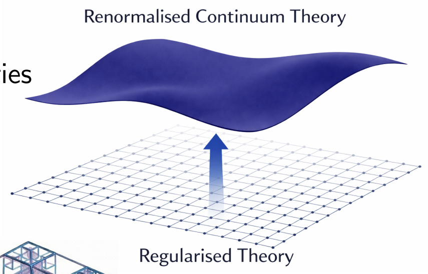
- Physically meaningful: Relations between Observables

Regularisation and Renormalisation

$$\text{QFT} = \lim_{\epsilon \to 0} \left(\text{Family of Regularised Theories } \mathcal{L}^{(\epsilon)} \right)$$

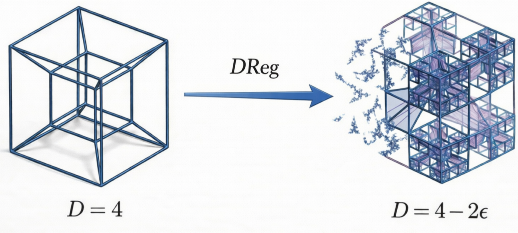
Regularisation:

- Regularisation $\implies$ Isolate Singularities
- Regularisation can and may in general break Symmetries
- Various Options: UV-cutoff, Pauli-Villars, DReg, ...



Dimensional Regularisation (DReg):

- Satisfies Causality, Lorentz Invariance and Unitarity
- Most commonly used

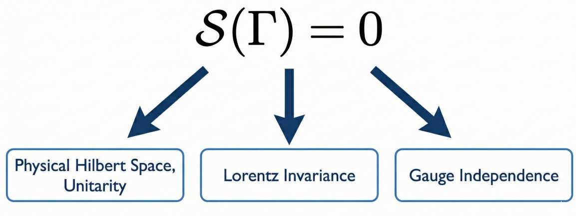


Symmetries — Gauge and BRST Invariance

- The Slavnov-Taylor identity reflects symmetries in the full quantum theory

$$\mathcal{S}(\Gamma) = \int \mathrm{d}^4x \, \langle s\phi_i \rangle \frac{\delta\Gamma}{\delta\phi_i(x)} = \int \mathrm{d}^4x \, \frac{\delta\Gamma}{\delta K_i(x)} \frac{\delta\Gamma}{\delta\phi_i(x)} = 0$$

- Gauge theories at the quantum level: BRST invariance $\Longrightarrow$ Slavnov-Taylor identity

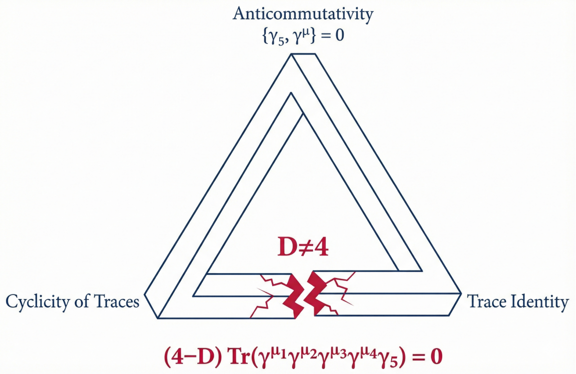


- The Slavnov-Taylor identity needs to be obeyed after renormalisation for consistency
  - physical Hilbert space with positive norm states
  - unitary and gauge independent physical S-matrix

The γ_5-Problem

The γ_5-Problem

- Electroweak interactions act on chiral fermions
- γ_5 is manifestly 4-dimensional



- Give up at least one property
  - Breitenlohner-Maison/'t Hooft-Veltman (BMHV) scheme [HV'72, BM'77]
  - Alternative γ_5-schemes (Naive anticommuting? Reading point? ...)

Breitenlohner-Maison/'t Hooft-Veltman (BMHV) Scheme

Abandoning anticommutativity of $\gamma_5 \longrightarrow$ BMHV algebra

- Decomposing the D dimensional space

$$\mathbb{M} = \mathbb{M}_4 \oplus \mathbb{M}_{-2\epsilon}, \qquad \eta_{\mu\nu} = \overline{\eta}_{\mu\nu} + \hat{\eta}_{\mu\nu}, \qquad X^\mu = \overline{X}^\mu + \hat{X}^\mu$$

- BMHV algebra: $\{\gamma^\mu, \gamma_5\} \neq 0$

$$\{\gamma^\mu, \gamma_5\} = \{\hat{\gamma}^\mu, \gamma_5\} = 2\,\hat{\gamma}^\mu\,\gamma_5, \qquad \{\overline{\gamma}^\mu, \gamma_5\} = 0, \qquad [\hat{\gamma}^\mu, \gamma_5] = 0$$

- Regularisation induced symmetry breaking: $\mathcal{S}_D(\Gamma_{\mathrm{DReg}}) \neq 0$
- Broken symmetry has to be restored $\Longrightarrow$ more complicated renormalisation

$$S_{\mathrm{ct}} = S_{\mathrm{sct}} + S_{\mathrm{fct}} = S_{\mathrm{sct,inv}} + S_{\mathrm{sct,non-inv}} + S_{\mathrm{fct}}$$

- But: unitary and proven to be consistent to all orders!

Chiral Gauge Theories in D Dimensions and the BMHV Scheme

Chiral gauge theory

$$\mathcal{L} = i\overline{\psi}_i\gamma^\mu\partial_\mu\psi_i - g\mathcal{Y}_{Rij}B_\mu\overline{\psi}_i\mathbb{P}_\mathrm{L}\gamma^\mu\mathbb{P}_\mathrm{R}\psi_j + \dots$$

Challenges in D dimensions:

1. Kinetic term, i.e. γ^μ, must be D-dimensional

$$\overline{\psi}\partial\!\!\!/\psi = \overline{\psi_L}\overline{\partial\!\!\!/}\psi_L + \overline{\psi_R}\overline{\partial\!\!\!/}\psi_R + \overline{\psi_L}\widehat{\partial\!\!\!/}\psi_R + \overline{\psi_R}\widehat{\partial\!\!\!/}\psi_L$$

   $\Longrightarrow$ kinetic term necessarily mixes chiralities

2. Analytic continuation to D dimensions not unique
   - $\mathbb{P}_\mathrm{L}\gamma^\mu\mathbb{P}_\mathrm{R}$ admits many inequivalent but equally correct extensions
   - use the most symmetric option $\Longrightarrow$ most natural and symmetric choice

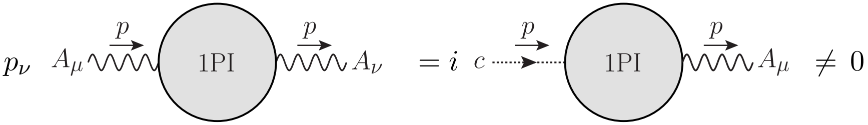
$$\overline{\psi}\mathbb{P}_\mathrm{L}\gamma^\mu\mathbb{P}_\mathrm{R}\psi = \overline{\psi}\mathbb{P}_\mathrm{L}\overline{\gamma}^\mu\mathbb{P}_\mathrm{R}\psi = \overline{\psi_R}\overline{\gamma}^\mu\psi_R$$

Always: Mismatch D versus 4 $\Longrightarrow$ breaks gauge invariance

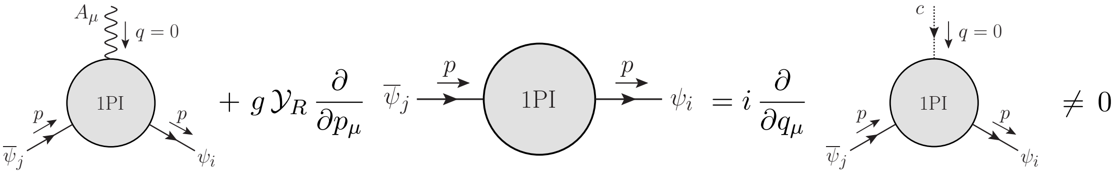
Example: Broken Ward Identities in Abelian Chiral Gauge Theories

Spurious symmetry breakings induced by the BMHV algebra, e.g.:

- Violation of the transversality of the gauge boson self energy



- Violation of the fermion self energy to fermion gauge boson interaction current relation



Symmetry Breaking

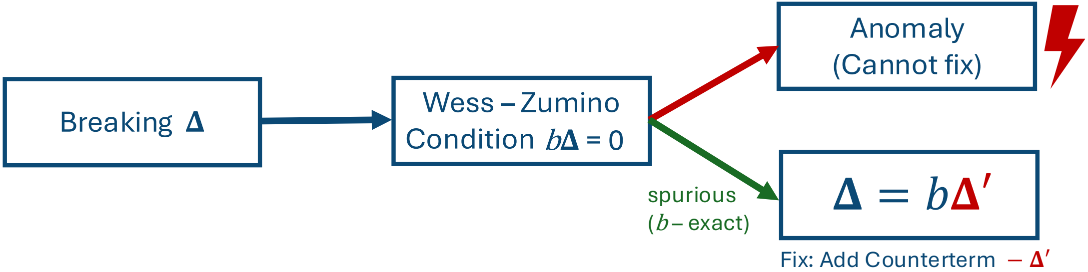
Distinguish two cases

(1) $\mathcal{S}(\Gamma_{\text{reg}}) = 0$: symmetry is manifestly preserved at all orders

(2) $\mathcal{S}(\Gamma_{\text{reg}}) \neq 0$: classical symmetry is broken in either of two ways:

  (a) spurious symmetry breaking: the symmetry can be restored, such that $\mathcal{S}(\Gamma_{\text{ren}}) = 0$;

  (b) anomalous symmetry breaking: the symmetry cannot be restored, i.e. $\mathcal{S}(\Gamma_{\text{ren}}) \neq 0$;



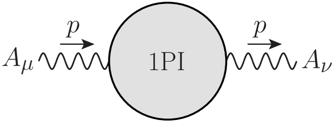
Symmetry Restoration

Symmetry Restoration — Explicit Evaluation

Explicit evaluation of all Green functions in $\mathcal{S}_D(\Gamma^{(n)}_{\text{subren}} + S^n_{\text{sct}})$ [HB et al. '21,'23]

- Consider gauge boson self energy at 1-loop

$$i\,\widetilde{\Gamma}^{\nu\mu}_{AA}(p)\Big|^1 =$$



$$\Big|^1$$

$$= \frac{ie^2}{16\pi^2}\frac{\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\Bigg\{\frac{2}{\epsilon}\left[\left(\overline{p}^\mu\overline{p}^\nu - \overline{p}^2\overline{\eta}^{\mu\nu}\right) - \frac{1}{2}\widehat{p}^2\overline{\eta}^{\mu\nu}\right]$$

$$+\left[\left(\frac{10}{3} - 2\ln(-p^2)\right)\left(\overline{p}^\mu\overline{p}^\nu - \overline{p}^2\overline{\eta}^{\mu\nu}\right) - \left(\overline{p}^2 + \widehat{p}^2\left(\frac{8}{3} - \ln(-p^2)\right)\right)\overline{\eta}^{\mu\nu}\right]\Bigg\}$$

- Drop finite evanescent contribution, which vanishes in $\mathrm{LIM}_{D\to 4}$
- Regularisation breaks transversality

Symmetry Restoration — Explicit Evaluation

Exhibit the breaking explicitly — probe the gauge boson Ward identity [HB et al. '21,'23]

$$[\mathcal{S}(\Gamma)]^1_{A^\mu c} = i\, p_\nu \widetilde{\Gamma}^{\nu\mu}_{AA}(p)\Big|^1 = \frac{ie^2}{16\pi^2}\frac{\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\left[-\frac{1}{\epsilon}\widehat{p}^2\overline{p}^\mu - \overline{p}^2\overline{p}^\mu\right] \neq 0$$

Need 3 types of counterterms

- singular and symmetric

$$S^1_{\text{sct,inv}} = -\frac{e^2}{16\pi^2\epsilon}\frac{2\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\int d^Dx\left(-\frac{1}{4}\overline{F}^{\mu\nu}\,\overline{F}_{\mu\nu}\right) + \ldots$$

- singular and non-symmetric

$$S^1_{\text{sct,non-inv}} = -\frac{e^2}{16\pi^2\epsilon}\frac{\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\int d^Dx\,\frac{1}{2}\overline{A}_\mu\widehat{\partial}^2\overline{A}^\mu + \ldots$$

- finite symmetry-restoring

$$S^1_{\text{fct}} = -\frac{e^2}{16\pi^2}\frac{\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\int d^4x\,\frac{1}{2}\overline{A}_\mu\overline{\partial}^2\overline{A}^\mu + \ldots$$

Symmetry Restoration — Quantum Action Principle

- The ultimate symmetry requirement is the Slavnov-Taylor identity

$$\underset{D\to 4}{\mathrm{LIM}}\left(\mathcal{S}_D(\Gamma_{\mathrm{DRen}})\right) = 0$$

- Regularised Quantum Action Principle of Dimensional Regularisation

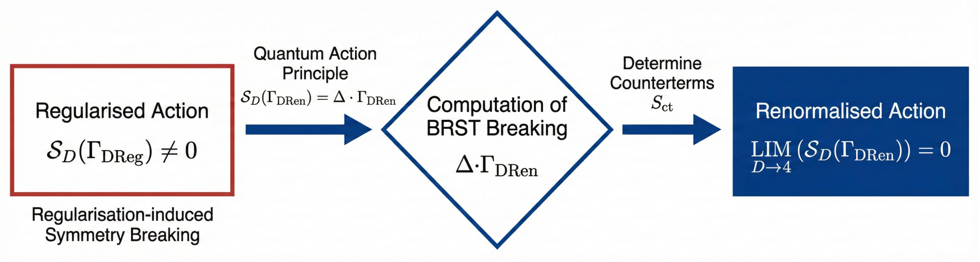
DReg:[BM'77], DRed:[DS'05]
Rev:[OP,SS'95, HB et al.'23]

$$\mathcal{S}_D(\Gamma_{\mathrm{DRen}}) = (\widehat{\Delta} + \Delta_{\mathrm{ct}}) \cdot \Gamma_{\mathrm{DRen}}$$

- Possible symmetry breaking can be rewritten as a composite operator insertion

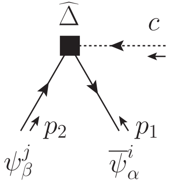
$$\widehat{\Delta} = \mathcal{S}_D(S_0), \qquad \widehat{\Delta} + \Delta_{\mathrm{ct}} = \mathcal{S}_D(S_0 + S_{\mathrm{ct}})$$

Symmetry Restoration — Practical Application



- Perturbative requirement: $\underset{D\to 4}{\text{LIM}}\left(\widehat{\Delta}\cdot\Gamma^{(n)}_{\text{DRen}} + \sum_{k=1}^{n-1}\Delta^{(k)}_{\text{ct}}\cdot\Gamma^{(n-k)}_{\text{DRen}} + \Delta^{(n)}_{\text{ct}}\right) = 0$

- Tree-level breaking:

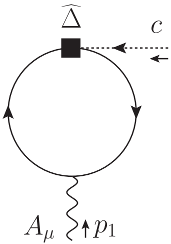


$$= -g\,\mathcal{Y}_{Rij}\left(\widehat{\not{p}}_1\mathbb{P}_{\text{R}} + \widehat{\not{p}}_2\mathbb{P}_{\text{L}}\right)_{\alpha\beta}$$

Symmetry Restoration — 1-Loop Example

Breaking of the gauge boson self energy at 1-loop [HB et al. '21,'23]

$$i\big(\widehat{\Delta}\cdot\Gamma^{(1)}\big)_{A_\mu c} =$$



$$= \frac{e^2}{16\pi^2}\frac{\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\Big[-\frac{1}{\epsilon}\widehat{p}^2\overline{p}^\mu - \overline{p}^2\overline{p}^\mu\Big]$$

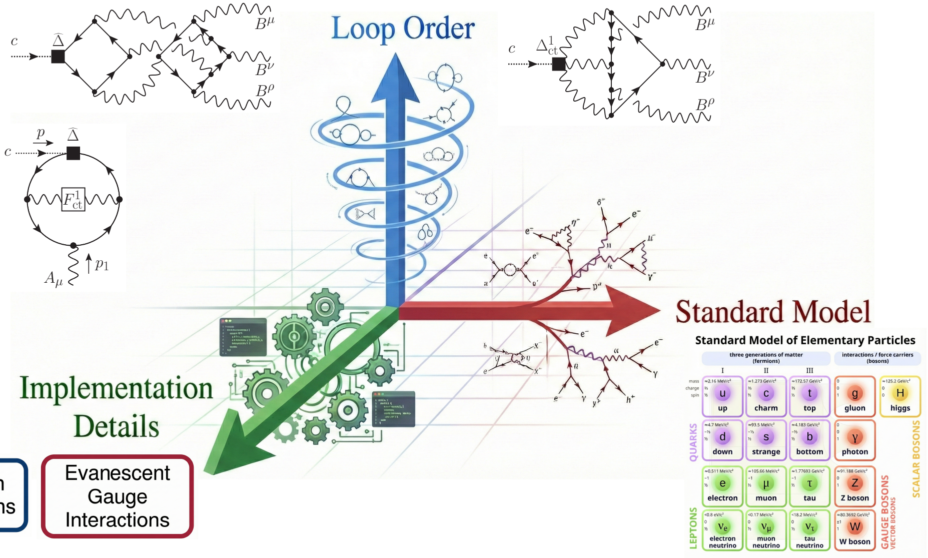
$$= -i\,[\mathcal{S}(\Gamma)]^{(1)}_{A^\mu c} = p_\nu \Gamma^{\nu\mu}_{AA}(p)\Big|^{(1)}$$

1-loop breaking of the Slavnov-Taylor identity

$$\big(\widehat{\Delta}\cdot\Gamma\big)^{(1)} = -\frac{1}{16\pi^2}\int d^Dx\,\frac{e^2\,\mathrm{Tr}\big(\mathcal{Y}_R^2\big)}{3}\Big[\frac{1}{\epsilon}\,c\,\overline{\partial}_\mu\,\widehat{\partial}^2\,\overline{A}^\mu + c\,\overline{\partial}_\mu\,\overline{\partial}^2\,\overline{A}^\mu\Big] + \ldots$$

Research Programme

Renormalisation Programme — 3 Dimensions of Complexity



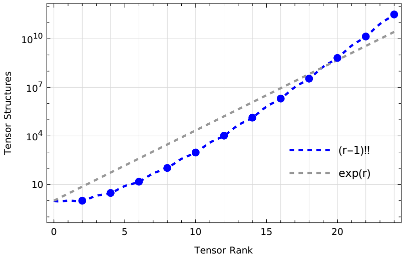
Fermions in *D*-Dimensions

Evanescent Gauge Interactions

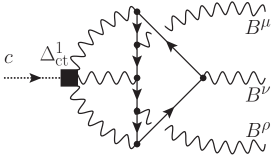
Loop Order

Multi-Loop Calculations — Challenges

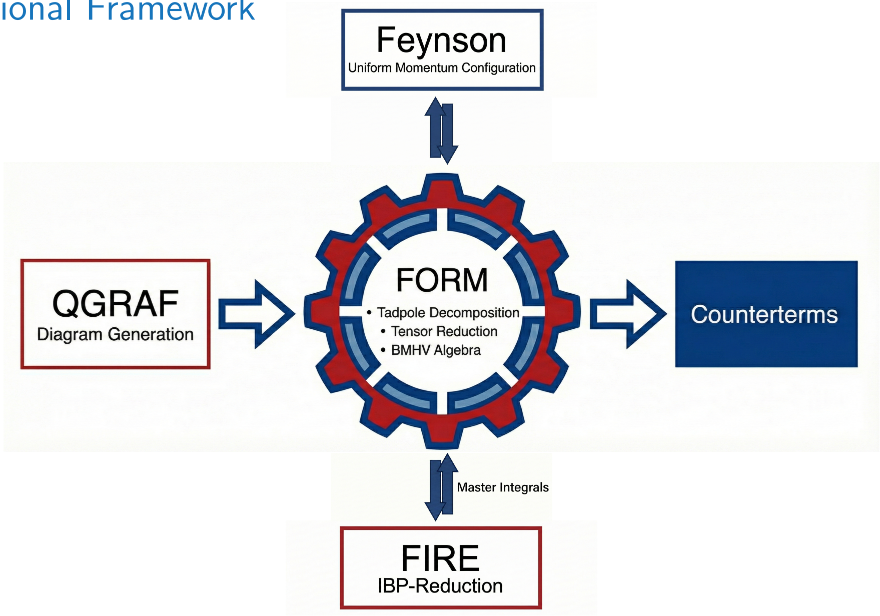
- Number of Feynman diagrams grows rapidly
- Complexity per Feynman diagrams
  - Dirac algebra
  - Loop integration (IBP-reduction, master integrals)
  - Tensor structures



- BMHV-specific challenges
  - Symmetry breaking in intermediate steps
  - Proliferation of Lorentz covariants $(\overline{\eta}^{\mu\nu}, \widehat{\eta}^{\mu\nu}, \overline{\gamma}^{\mu}, \widehat{\gamma}^{\mu}, \overline{p}_i^{\mu}, \ldots)$

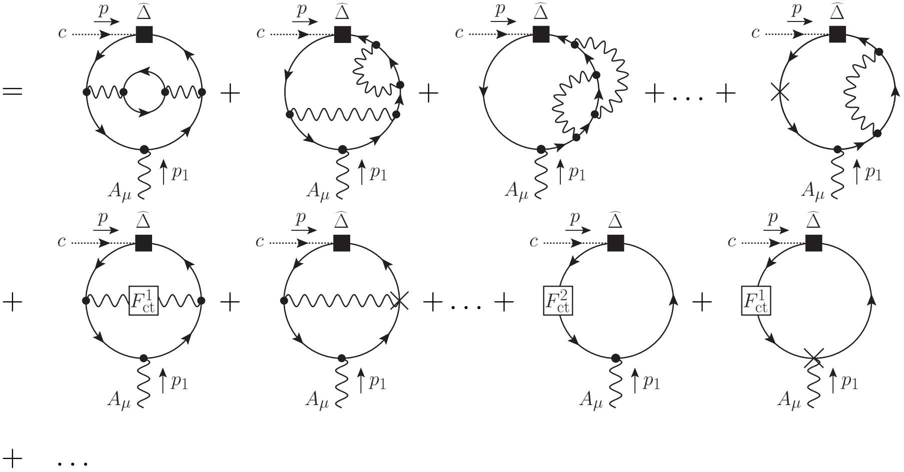


Computational Framework



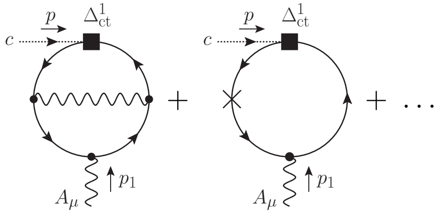
Symmetry Restoration — 3-Loop Application

$$i\left(\widehat{\Delta}\cdot\Gamma^3\right)_{A_\mu c} =$$

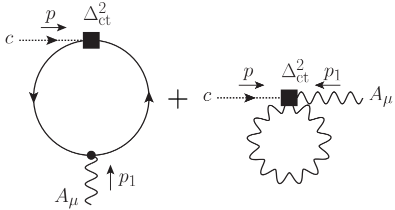


Symmetry Restoration — 3-Loop Application

$$i\left(\Delta^{1}_{\mathrm{ct}} \cdot \Gamma^{2}\right)_{A_\mu c} =$$

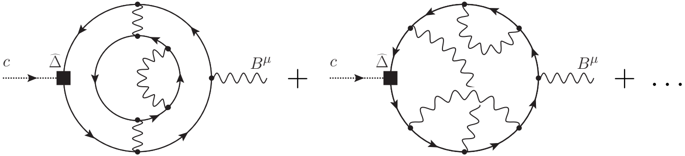


$$i\left(\Delta^{2}_{\mathrm{ct}} \cdot \Gamma^{1}\right)_{A_\mu c} =$$

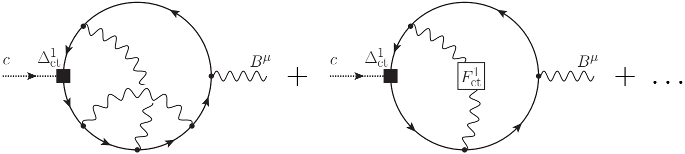


Symmetry Restoration — 4-Loop Application

$$i\big(\widehat{\Delta}\cdot\Gamma^4\big)_{B_\mu c} =$$



$$i\big(\Delta^1_{\rm ct}\cdot\Gamma^3\big)_{B_\mu c} =$$



- Analogously for $i\big(\Delta^2_{\rm ct}\cdot\Gamma^2\big)_{B_\mu c}$ and $i\big(\Delta^3_{\rm ct}\cdot\Gamma^1\big)_{B_\mu c}$

Gauge Boson Self Energy in an Abelian Chiral Gauge Theory

4-loop result [AM,DS,MW'25]

$$i\Gamma^{\nu\mu}_{BB}(p)\Big|^{4}_{\text{div}} = -\frac{ig^8}{(16\pi^2)^4}\,\delta Z^{(4)}_B\left(\overline{p}^\mu\overline{p}^\nu - \overline{p}^2\overline{\eta}^{\mu\nu}\right) + \frac{ig^8}{(16\pi^2)^4}\left(\delta\widehat{X}^{(4)}_{BB}\,\widehat{p}^2\,\overline{\eta}^{\mu\nu} + \delta\overline{X}^{(4)}_{BB}\,\overline{p}^2\,\overline{\eta}^{\mu\nu}\right),$$

with

$$\delta Z^{(4)}_B = \mathcal{A}^{4,\text{inv}}_{BB}\frac{1}{\epsilon} + \mathcal{B}^{4,\text{inv}}_{BB}\frac{1}{\epsilon^2} + \mathcal{C}^{4,\text{inv}}_{BB}\frac{1}{\epsilon^3},$$

$$\delta\widehat{X}^{(4)}_{BB} = \widehat{\mathcal{A}}^{4,\text{break}}_{BB}\frac{1}{\epsilon} + \widehat{\mathcal{B}}^{4,\text{break}}_{BB}\frac{1}{\epsilon^2} + \widehat{\mathcal{C}}^{4,\text{break}}_{BB}\frac{1}{\epsilon^3} + \widehat{\mathcal{D}}^{4,\text{break}}_{BB}\frac{1}{\epsilon^4},$$

$$\delta\overline{X}^{(4)}_{BB} = \overline{\mathcal{A}}^{4,\text{break}}_{BB}\frac{1}{\epsilon} + \overline{\mathcal{B}}^{4,\text{break}}_{BB}\frac{1}{\epsilon^2},$$

- Transversality of the gauge boson self energy is violated
- Violation is local
- Symmetry restoration necessary and possible

Violation of the Gauge Boson Transversality

4-loop result [AM,DS,MW'25]

$$i\Delta \cdot \Gamma\Big|^{4}_{B_\mu c} = i\widehat{\Delta} \cdot \Gamma^{4}_{B_\mu c} + i\Delta^{1}_{\text{ct}} \cdot \Gamma^{3}_{B_\mu c} + i\Delta^{2}_{\text{ct}} \cdot \Gamma^{2}_{B_\mu c} + i\Delta^{3}_{\text{ct}} \cdot \Gamma^{1}_{B_\mu c}$$
$$= \frac{g^8}{(16\pi^2)^4}\left[\delta\widehat{X}^{(4)}_{BB}\,\widehat{p}^2\,\overline{p}^\mu + \left(\delta\overline{X}^{(4)}_{BB} + \mathcal{F}^{4,\text{break}}_{BB}\right)\overline{p}^2\,\overline{p}^\mu\right],$$

with $\delta\widehat{X}^{(4)}_{BB}$ and $\delta\overline{X}^{(4)}_{BB}$ as before, as well as

$$\mathcal{F}^{4,\text{break}}_{BB} = \left(\frac{403759}{25920} + \frac{\pi^4}{3375} + \frac{30113\zeta(3)}{4500} - \frac{403\zeta(5)}{45}\right)\mathrm{Tr}\left(\mathcal{Y}^8_R\right)$$
$$+ \left(\frac{4525151}{11664000} - \frac{17\,\pi^4}{20250} - \frac{15569\zeta(3)}{40500}\right)\mathrm{Tr}\left(\mathcal{Y}^6_R\right)\mathrm{Tr}\left(\mathcal{Y}^2_R\right)$$
$$+ \left(\frac{38768057}{7776000} + \frac{12061\zeta(3)}{900} - \frac{713\zeta(5)}{45}\right)\mathrm{Tr}\left(\mathcal{Y}^4_R\right)^2 - \frac{4240349}{69984000}\mathrm{Tr}\left(\mathcal{Y}^4_R\right)\mathrm{Tr}\left(\mathcal{Y}^2_R\right)^2$$

Counterterm Action

At the L-loop level (with $L \in \{1,2,3,4\}$): [AM,DS,MW'25]

$$S^{(L)}_{\text{sct,inv}} = \frac{g^{2L}}{(16\pi^2)^L} \int d^D x \left\{ \delta Z^{(L)}_B \left(-\frac{1}{4} \overline{F}^{\mu\nu} \overline{F}_{\mu\nu} \right) + \delta Z^{(L)}_{\psi,kj} \overline{\psi}_i i \overline{\not{D}}_{\mathrm{R},ik} \psi_j \right\},$$

$$S^{(L)}_{\text{sct,break}} = \frac{g^{2L}}{(16\pi^2)^L} \int d^D x \left\{ \delta \widehat{X}^{(L)}_{BB} \frac{1}{2} \overline{B}_\mu \widehat{\partial}^2 \overline{B}^\mu + \delta \overline{X}^{(L)}_{BB} \frac{1}{2} \overline{B}_\mu \overline{\partial}^2 \overline{B}^\mu \right.$$
$$\left. + g^2 \delta \overline{X}^{(L)}_{BBBB} \frac{1}{8} \overline{B}_\mu \overline{B}^\mu \overline{B}_\nu \overline{B}^\nu + \delta \overline{X}^{(L)}_{\overline{\psi}\psi,ij} \overline{\psi}_i i \overline{\not{\partial}} \mathbb{P}_\mathrm{R} \psi_j \right\},$$

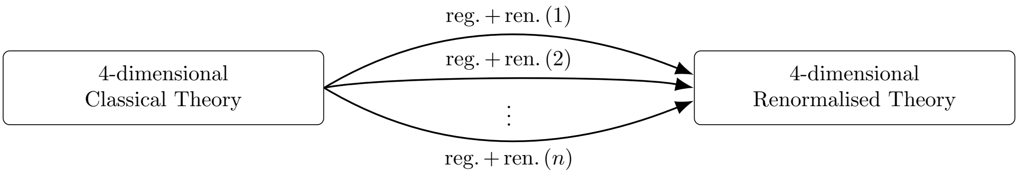
and

$$S^{(L)}_{\text{fct}} = \frac{g^{2L}}{(16\pi^2)^L} \int d^4 x \left\{ \mathcal{F}^{L,\text{break}}_{BB} \frac{1}{2} \overline{B}_\mu \overline{\partial}^2 \overline{B}^\mu + g^2 \mathcal{F}^{L,\text{break}}_{BBBB} \frac{1}{8} \overline{B}_\mu \overline{B}^\mu \overline{B}_\nu \overline{B}^\nu + \mathcal{F}^{L,\text{break}}_{\overline{\psi}\psi,ij} \overline{\psi}_i i \overline{\not{\partial}} \mathbb{P}_\mathrm{R} \psi_j \right\}.$$

Implementation Details

Freedom in Regularisation and Renormalisation

- Regularisation $\Longrightarrow$ family of theories labelled by regulator (e.g. $\Lambda, \epsilon, \ldots$)
- Renormalisation $=$ Reparametrisation $\Longrightarrow$ family of theories labelled by free parameters
- Imaginary parts fixed by unitarity, non-local real parts fixed by causality
- Remaining (local) freedom fixed by
  - choice of renormalisation scheme
  - relating free parameters to observables
- Any consistent regularisation that satisfies the fundamental requirements is admissible



- Dimensional Regularisation:
  - require proper regularisation $\Longrightarrow$ fully D-dimensional propagator denominators
  - BUT: non-uniqueness of D-dimensional extension $\Longrightarrow$ freedom regarding evanescent details

Fermions in D Dimensions

In principal 2 options [PE,PK,DS,MW'24]

1. Option 1: $(\psi_L + \psi_R \longrightarrow \psi)$
   - electron: $e = e_L + e_R$

$$\mathcal{L}_{\mathrm{kin},e} = \overline{e} i \not{\partial} e = \overline{e_L} i \overline{\not{\partial}} e_L + \overline{e_R} i \overline{\not{\partial}} e_R + \overline{e_L} i \widehat{\not{\partial}} e_R + \overline{e_R} i \widehat{\not{\partial}} e_L$$

   - $\widehat{\not{\partial}}$-terms mix physical fields with different gauge quantum numbers

2. Option 2: $(\psi_L + \psi_R^{\mathrm{st}} \longrightarrow \psi_1)$ and $(\psi_L^{\mathrm{st}} + \psi_R \longrightarrow \psi_2)$
   - electron: $e_1 = e_L + e_R^{\mathrm{st}}$ and $e_2 = e_L^{\mathrm{st}} + e_R$

$$\begin{aligned}\mathcal{L}_{\mathrm{kin},e} = \overline{e}_1 i \not{\partial} e_1 + \overline{e}_2 i \not{\partial} e_2 = {} & \overline{e_L} i \overline{\not{\partial}} e_L + \overline{e_R^{\mathrm{st}}} i \overline{\not{\partial}} e_R^{\mathrm{st}} + \overline{e_L} i \widehat{\not{\partial}} e_R^{\mathrm{st}} + \overline{e_R^{\mathrm{st}}} i \widehat{\not{\partial}} e_L \\ & + \overline{e_L^{\mathrm{st}}} i \overline{\not{\partial}} e_L^{\mathrm{st}} + \overline{e_R} i \overline{\not{\partial}} e_R + \overline{e_L^{\mathrm{st}}} i \widehat{\not{\partial}} e_R + \overline{e_R} i \widehat{\not{\partial}} e_L^{\mathrm{st}}\end{aligned}$$

   - preserves global symmetry
   - proliferation of terms
   - complicated propagator in the massive case

Evanescent Interactions in the BMHV Scheme

Example: W-electron-neutrino interaction [PE,PK,DS,MW'24]

- 4 dimensions: $\overline{e}\,\overline{\gamma}^\mu \mathbb{P}_\mathrm{L} \nu W^-_\mu + \mathrm{h.c.}$
- D dimensions: $\overline{e}\,\mathbb{P}_\mathrm{R}\overline{\gamma}^\mu \mathbb{P}_\mathrm{L} \nu \overline{W}^-_\mu + \overline{e}\,\mathbb{P}_\mathrm{L}\widehat{\gamma}^\mu \mathbb{P}_\mathrm{L} \nu \widehat{W}^-_\mu + \mathrm{h.c.}$

  or: $\overline{e}\,\mathbb{P}_\mathrm{R}\overline{\gamma}^\mu \mathbb{P}_\mathrm{L} \nu \overline{W}^-_\mu + \mathrm{h.c.}$

In general:

$$\mathcal{L} \supset \overline{\psi}_i i \partial\!\!\!/ \psi_i - g\mathcal{Y}_{Rij} B_\mu \overline{\psi}_i \mathbb{P}_\mathrm{L}\gamma^\mu \mathbb{P}_\mathrm{R}\psi_j - g\mathcal{Y}_{Lij} B_\mu \overline{\psi}_i \mathbb{P}_\mathrm{R}\gamma^\mu \mathbb{P}_\mathrm{L}\psi_j$$
$$- g\mathcal{Y}_{LRij} B_\mu \overline{\psi}_i \mathbb{P}_\mathrm{R}\gamma^\mu \mathbb{P}_\mathrm{R}\psi_j - g\mathcal{Y}_{RLij} B_\mu \overline{\psi}_i \mathbb{P}_\mathrm{L}\gamma^\mu \mathbb{P}_\mathrm{L}\psi_j$$

Evanescent interaction currents

- restricted by hermiticity, CPT-invariance and non-broken symmetries
- additional tree-level breaking

$$\mathcal{S}_D(S_0) = \widehat{\Delta} = \widehat{\Delta}_{c\overline{\psi}\psi} + \widehat{\Delta}_{cB\overline{\psi}\psi}$$

Standard Model Application

The Standard Model: Fields and Generators

- Fermion doublets

$$l_I^a = \begin{pmatrix} \nu_I \\ e_I \end{pmatrix}^a, \qquad q_I^{i,a} = \begin{pmatrix} u_I^i \\ d_I^i \end{pmatrix}^a$$

- Fermion generators

$$T^A_{\overline{\alpha}\beta,ab} = \begin{cases} g_Y \mathcal{Y}^l_{\overline{\alpha}\beta,ab}, & A = 1 \\ g_W t^A_{W,\overline{\alpha}\beta,ab}, & A \in \{2,3,4\} \\ 0, & A \in \{5,\ldots,12\} \end{cases}, \quad \mathcal{T}^A_{\overline{\alpha}\beta,ab,ij} = \begin{cases} g_Y \mathcal{Y}^q_{\overline{\alpha}\beta,ab}\delta_{ij}, & A = 1 \\ g_W t^A_{W,\overline{\alpha}\beta,ab}\delta_{ij}, & A \in \{2,3,4\} \\ g_s t^A_{S,\overline{\alpha}\beta,ij}\delta_{ab}, & A \in \{5,\ldots,12\} \end{cases}$$

- Explicit form of the generators in "chirality space"

$$\left(\mathcal{Y}^f_{\alpha\beta,ab}\right) = \begin{pmatrix} \mathcal{Y}^f_{L,ab} & \widehat{\mathcal{Y}}^f_{ab} \\ \left(\widehat{\mathcal{Y}}^{f\dagger}\right)_{ab} & \mathcal{Y}^f_{R,ab} \end{pmatrix}, \ \left(t^A_{W,\alpha\beta,ab}\right) = \begin{pmatrix} t^A_{L,ab} & \widehat{t}^A_{ab} \\ \left(\widehat{t}^{A\dagger}\right)_{ab} & 0 \end{pmatrix}, \ \left(t^A_{S,\alpha\beta,ij}\right) = \begin{pmatrix} t^A_{s,ij} & \widehat{t}^A_{s,ij} \\ \left(\widehat{t}^{A\dagger}_s\right)_{ij} & t^A_{s,ij} \end{pmatrix}$$

BMHV Regularisation in the Standard Model

- Generic fermionic Lagrangian [MW'25]

$$\mathcal{L}_{\text{fermion}} = \bar{l}^a_I i\partial\!\!\!/ l^a_I - T^A_{\overline{\alpha}\beta,ab}\bar{l}^a_I \,\mathbb{P}_\alpha \mathcal{V}\!\!\!/^A \mathbb{P}_\beta \, l^b_I + \overline{q}^{i,a}_I i\partial\!\!\!/ q^{i,a}_I - \mathcal{T}^A_{\overline{\alpha}\beta,ab,ij}\overline{q}^{i,a}_I \,\mathbb{P}_\alpha \mathcal{V}\!\!\!/^A \mathbb{P}_\beta \, q^{j,b}_I,$$

- QED and QCD are vector-like gauge theories
- Fully D-dimensional QCD $\Longrightarrow \widehat{t}^A_s = t^A_s$
- What about QED? Consider electric charge generator: $\mathcal{Q}^f_{\overline{\alpha}\beta,ab} = t^3_{W,\overline{\alpha}\beta,ab} + \mathcal{Y}^f_{\overline{\alpha}\beta,ab}$

$$\left(\mathcal{Q}^f_{\overline{\alpha}\beta,ab}\right) = \begin{pmatrix} \mathcal{Y}^f_{L,ab} + t^3_{L,ab} & \widehat{\mathcal{Y}}^f_{ab} + \widehat{t}^3_{ab} \\ \left(\widehat{\mathcal{Y}}^{f\dagger}\right)_{ab} + \left(\widehat{t}^{3\dagger}\right)_{ab} & \mathcal{Y}^f_{R,ab} \end{pmatrix} \overset{!}{=} \begin{pmatrix} Q^f_{ab} & Q^f_{ab} \\ Q^f_{ab} & Q^f_{ab} \end{pmatrix}, \qquad f \in \{l,q\}.$$

- Choice: BMHV Implementation
  - Fermions in D Dimensions $\Longrightarrow$ Option 1: $(\psi_L + \psi_R \longrightarrow \psi)$
  - Evanescent gauge interactions:

$$\widehat{\mathcal{Y}}^f_{ab} = c_{\text{QED}}\, \mathcal{Y}^f_{R,ab}, \qquad f \in \{l,q\},$$
$$\widehat{t}^A_{ab} \equiv 0,$$
$$\widehat{t}^A_{s,ij} = c_{\text{QCD}}\, t^A_{s,ij},$$

Gauge Interactions in the Standard Model

Standard Model quarks in the BMHV scheme [MW'25]

- photon interaction current

$$eQ_{ab}\overline{q}_I^{i,a}\not{A}q_I^{i,b} - (1 - c_{\mathrm{QED}})eQ_{ab}\overline{q}_I^{i,a}\widehat{\not{A}}q_I^{i,b}$$

- Z_μ interaction current

$$-\frac{s_W}{c_W}eQ_{ab}\overline{q}_I^{i,a}\not{Z}q_I^{i,b} + \frac{g_W}{c_W}t^3_{L,ab}\overline{q}_I^{i,a}\overline{\not{Z}}\mathbb{P}_{\mathrm{L}}q_I^{i,b} + (1 - c_{\mathrm{QED}})\frac{s_W}{c_W}eQ_{ab}\overline{q}_I^{i,a}\widehat{\not{Z}}q_I^{i,b}$$

- $W_\mu^\pm$ interaction current

$$\frac{g_W}{\sqrt{2}}\tau^+_{ab}\,\overline{q}_I^{i,a}\overline{\not{W}^+}\mathbb{P}_{\mathrm{L}}q_I^{i,b} + \frac{g_W}{\sqrt{2}}\tau^-_{ab}\,\overline{q}_I^{i,a}\overline{\not{W}^-}\mathbb{P}_{\mathrm{L}}q_I^{i,b}$$

- gluon interaction current

$$g_s t^A_{s,ij}\overline{q}_I^{i,a}\not{G}^A q_I^{j,a} - (1 - c_{\mathrm{QCD}})g_s t^A_{s,ij}\overline{q}_I^{i,a}\widehat{\not{G}^A}q_I^{j,a}$$

Regularisation Induced Symmetry Breaking in the Standard Model

Tree-level breaking: [MW'25]

$$\widehat{\Delta} = \mathcal{S}_D(S_0) = -\int d^Dx\, c^A \Big\{ \overline{q}_I^{i,a} \Big[\mathbb{P}_\mathrm{R} \Big(\mathcal{T}^A_{R,ab,ij} \overleftarrow{\widehat{\partial\!\!\!/}} + \mathcal{T}^A_{L,ab,ij} \overrightarrow{\widehat{\partial\!\!\!/}} - \mathcal{T}^A_{LR,ab,ij} \big(\overleftarrow{\widehat{\partial\!\!\!/}} + \overrightarrow{\widehat{\partial\!\!\!/}} \big) \Big) \mathbb{P}_\mathrm{R}$$

$$+ \mathbb{P}_\mathrm{L} \Big(\mathcal{T}^A_{R,ab,ij} \overrightarrow{\widehat{\partial\!\!\!/}} + \mathcal{T}^A_{L,ab,ij} \overleftarrow{\widehat{\partial\!\!\!/}} - \mathcal{T}^A_{RL,ab,ij} \big(\overleftarrow{\widehat{\partial\!\!\!/}} + \overrightarrow{\widehat{\partial\!\!\!/}} \big) \Big) \mathbb{P}_\mathrm{L} \Big] q_I^{j,b} \Big\}$$

$$- i \int d^Dx\, c^A \Big\{ \overline{q}_I^{i,a} \Big[\Big(\mathcal{T}^A_R \mathcal{T}^B_{RL} - \mathcal{T}^B_{RL} \mathcal{T}^A_L - i \mathscr{C}^{ABC} \mathcal{T}^C_{RL} \Big)_{ab,ij} \mathbb{P}_\mathrm{L} \widehat{\mathcal{V}\!\!\!/}^B \mathbb{P}_\mathrm{L}$$

$$+ \Big(\mathcal{T}^A_L \mathcal{T}^B_{LR} - \mathcal{T}^B_{LR} \mathcal{T}^A_R - i \mathscr{C}^{ABC} \mathcal{T}^C_{LR} \Big)_{ab,ij} \mathbb{P}_\mathrm{R} \widehat{\mathcal{V}\!\!\!/}^B \mathbb{P}_\mathrm{R} \Big] q_I^{j,b} \Big\}$$

$+$ lepton-contributions

$$=: \widehat{\Delta}_1[c, \overline{l}, l] + \widehat{\Delta}_1[c, \overline{q}, q] + \widehat{\Delta}_2[c, \mathcal{V}, \overline{l}, l] + \widehat{\Delta}_2[c, \mathcal{V}, \overline{q}, q]$$

QCD Contributions to the Symmetry Breaking

- Question: is it sufficient to consider symmetry restoration of the EWSM? [MW'25]
- Complete tree-level breaking of the full SM:

$$\widehat{\Delta} = \int d^D x \Big[\widehat{\Delta}_Y(x) + \widehat{\Delta}_W(x) + \widehat{\Delta}_S(x) + \widehat{\Delta}_{YW}(x) + \widehat{\Delta}_{YS}(x) + \widehat{\Delta}_{WS}(x)\Big]$$

- Setting evanescent gauge interactions to zero $\Longrightarrow$ mixed contributions vanish
- QCD contributions to the tree-level breaking:

$$\begin{aligned}
\widehat{\Delta}_S(x) &= -(1 - c_{\mathrm{QCD}}) g_s t^A_{s,ij} c^A \widehat{\partial}_\mu \big(\overline{q}^{i,a}_I \widehat{\gamma}^\mu q^{j,a}_I\big), \\
\widehat{\Delta}_{YS}(x) &= i c_{\mathrm{QCD}} g_Y g_s \big(\mathcal{Y}^q_R - \mathcal{Y}^q_L\big)_{ab} t^A_{s,ij} c_B \overline{q}^{i,a}_I \Big[\mathbb{P}_{\mathrm{R}} \widehat{\not{G}}^A \mathbb{P}_{\mathrm{R}} - \mathbb{P}_{\mathrm{L}} \widehat{\not{G}}^A \mathbb{P}_{\mathrm{L}}\Big] q^{j,b}_I, \\
\widehat{\Delta}_{WS}(x) &= -i c_{\mathrm{QCD}} g_W g_s t^A_{L,ab} t^B_{s,ij} c^A \overline{q}^{i,a}_I \Big[\mathbb{P}_{\mathrm{R}} \widehat{\not{G}}^A \mathbb{P}_{\mathrm{R}} - \mathbb{P}_{\mathrm{L}} \widehat{\not{G}}^A \mathbb{P}_{\mathrm{L}}\Big] q^{j,b}_I.
\end{aligned}$$

- There is no choice of c_{QED} and c_{QCD} that eliminates all strong contributions $\propto g_s$
- Answer: NO, need to include QCD for a proper renormalisation!

Excerpt: 1-Loop Results in the Standard Model

- Finite symmetry-restoring counterterms of the fermionic sector [MW'25]

$$S^{(1)}_{\text{fct,fermion}} = \frac{1}{16\pi^2}\int d^4x \Big\{ \delta F^{(1)}_{l,ab} \bar{l}^a_I i\overline{\partial\!\!\!/} l^b_I - \delta F^{(1),A}_{\bar{l}\mathcal{V}l,R,ab} \bar{l}^a_I \overline{\mathcal{V}\!\!\!/}^A \mathbb{P}_\mathrm{R} l^b_I - \delta F^{(1),A}_{\bar{l}\mathcal{V}l,L,ab,IJ} \bar{l}^a_I \overline{\mathcal{V}\!\!\!/}^A \mathbb{P}_\mathrm{L} l^b_J$$
$$+ \delta F^{(1)}_{q,ab} \bar{q}^{i,a}_I i\overline{\partial\!\!\!/} q^{i,b}_I - \delta F^{(1),A}_{\bar{q}\mathcal{V}q,R,ab,ij} \bar{q}^{i,a}_I \overline{\mathcal{V}\!\!\!/}^A \mathbb{P}_\mathrm{R} q^{j,b}_I - \delta F^{(1),A}_{\bar{q}\mathcal{V}q,L,ab,ij,IJ} \bar{q}^{i,a}_I \overline{\mathcal{V}\!\!\!/}^A \mathbb{P}_\mathrm{L} q^{j,b}_J \Big\}$$

- Finite symmetry-restoring corrections to the BRST transformations

$$S^{(1)}_{\text{fct,ext}} = -\frac{i\delta F^{(1)}_{R_f}}{16\pi^2}\int d^4x \Big\{ c^A \overline{R}^a_{l,I} \left[T^A_{R,ab}\mathbb{P}_\mathrm{R} + T^A_{L,ab}\mathbb{P}_\mathrm{L}\right] l^b_I$$
$$+ c^A \overline{R}^{i,a}_{q,I} \left[\mathcal{T}^A_{R,ab,ij}\mathbb{P}_\mathrm{R} + \mathcal{T}^A_{L,ab,ij}\mathbb{P}_\mathrm{L}\right] q^{j,b}_I - c^A \overline{R}^{i,a}_{q,I} \widehat{\mathcal{T}}^A_{ab,ij} q^{j,b}_I + \text{h.c.} \Big\}$$

- Sample counterterm coefficients

$$\delta F^{(1)}_{l,ab} = \frac{3-\xi_F}{3}\mathcal{C}^{ab}_2(F^l_{LR}) - \frac{\xi_F}{3}\mathcal{C}^{ab}_2(F^l_L, F^l_{LR}), \qquad \delta F^{(1)}_{R_f} = -\frac{1-\xi_F}{4}\mathcal{C}_2(G^{A'})$$

- Simplest and most economical expressions obtained for $(c_{\mathrm{QED}} = 0, c_{\mathrm{QCD}} = 0)$

Towards higher Orders in the Standard Model

Determination of symmetry-restoring counterterms

- Computation of the symmetry-breaking at the n-loop level

$$(\Delta \cdot \Gamma)^{(n)} = \widehat{\Delta} \cdot \Gamma^{(n)} + \sum_{k=1}^{n-1} \Delta_{\mathrm{ct}}^{(k)} \cdot \Gamma^{(n-k)} = -\Delta_{\mathrm{ct}}^{(n)} + \mathcal{O}(\hat{.})$$

- Determination of the non-invariant counterterms from the n-loop breaking operator

$$\Delta_{\mathrm{ct}}^{(n)} = \left[\mathcal{S}_D\left(S_0 + \sum_{k=1}^{n} S_{\mathrm{ct}}^{(k)}\right)\right]^{(n)} = \left[\int d^D x \, \frac{\delta\left(S_0 + \sum_{k=1}^{n} S_{\mathrm{ct}}^{(k)}\right)}{\delta \phi_i(x)} \frac{\delta\left(S_0 + \sum_{k=1}^{n} S_{\mathrm{ct}}^{(k)}\right)}{\delta K_i(x)}\right]^{(n)}$$

Abelian

- BRST transformations do not renormalise
- $\delta S_{\mathrm{ct}}^{(n)} / \delta K_i(x) \equiv 0$
- $s_D S_{\mathrm{ct}}^{(n)} = \Delta_{\mathrm{ct}}^{(n)} = -(\Delta \cdot \Gamma)^{(n)} + \mathcal{O}(\hat{.})$

Non-Abelian

- BRST transformations renormalise
- $\delta S_{\mathrm{ct}}^{(n)} / \delta K_i(x) \neq 0$
- Non-trivial evaluation of $\left[\mathcal{S}_D\left(S_0 + \sum_{k=1}^{n} S_{\mathrm{ct}}^{(k)}\right)\right]^{(n)}$

Summary and Outlook

Summary of Contributions

Conceptual

Analysis of BMHV Implementation [2411.02543].

Renormalisation and Symmetry Restoration [2303.09120].

Computational

Automated FORM framework.

Handling billions of terms.

Application

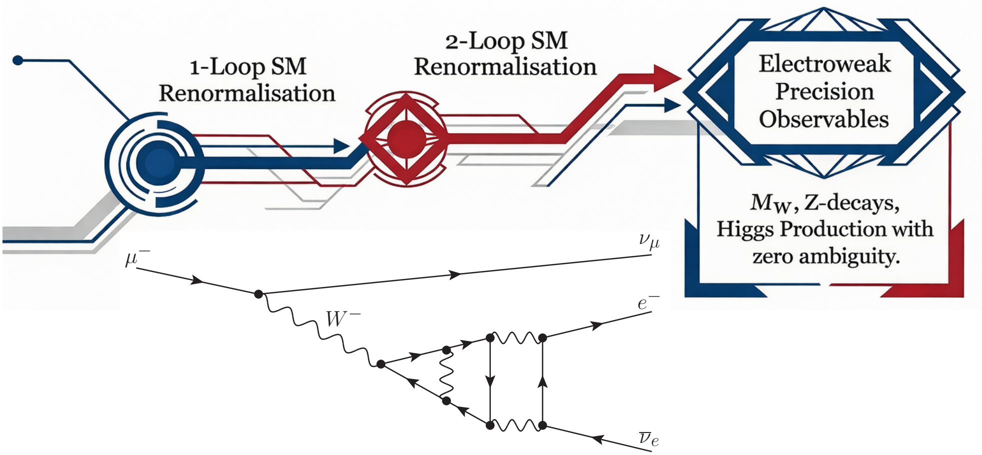
3-Loop Abelian [2312.11291].

4-Loop Abelian (Highest Order) [2506.12253].

1-Loop Standard Model (Foundation).

Electroweak Precision Phenomenology

- The Standard Model is too good but does not describe all phenomena
- Need high-precision tests to probe SM & BSM physics



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BackUp Slides

Chiral Gauge Theories and the γ_5-Problem

- Electroweak interactions act on chiral fermions
  - Left-handed and right-handed fermions interact differently with gauge bosons
  - SM and all its extensions for potential new physics are chiral gauge theories
- γ_5 is manifestly 4-dimensional
- Cannot simultaneously retain the following properties in D dimensions

$$\{\gamma_5, \gamma^\mu\} = 0 \tag{1}$$

$$\text{Tr}(\Gamma_1\Gamma_2) = \text{Tr}(\Gamma_2\Gamma_1) \tag{2}$$

$$\text{Tr}(\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = -4i\varepsilon^{\mu\nu\rho\sigma} \tag{3}$$

- Inconsistent in $D \neq 4 \implies \gamma_5$-problem
- Give up at least one property
  - Breitenlohner-Maison/'t Hooft-Veltman (BMHV) scheme
  - Alternative γ_5-schemes (Naive anticommuting? Reading point? ...)

Alternative γ_5-Schemes

- Limited range of applicability
- Ambiguities at the multi-loop level
- "Kreimer's" scheme / reading point prescription (important alternative) [DK'90-93]
  - Abandoning cyclicity of the trace
  - promises a better behaviour w.r.t. gauge invariance
- However, ... multi-loop properties not fully under control
  - Ambiguities from location of γ_5 [AB,AP'16;MZ'16;CP,AT'19;JD et al.'20,'22;FH'22]
  - Problems with higher order QCD corrections with an external flavour-singlet axial-current [LC'23]

Breitenlohner-Maison/'t Hooft-Veltman (BMHV) Scheme

Abandoning anticommutativity of $\gamma_5 \longrightarrow$ BMHV algebra

- Decomposing the D dimensional space

$$\mathbb{M} = \mathbb{M}_4 \oplus \mathbb{M}_{-2\epsilon}, \qquad \eta_{\mu\nu} = \overline{\eta}_{\mu\nu} + \hat{\eta}_{\mu\nu}, \qquad X^\mu = \overline{X}^\mu + \hat{X}^\mu$$

- BMHV algebra

$$\{\gamma^\mu, \gamma_5\} = \{\hat{\gamma}^\mu, \gamma_5\} = 2\,\hat{\gamma}^\mu\,\gamma_5, \qquad \{\overline{\gamma}^\mu, \gamma_5\} = 0, \qquad [\hat{\gamma}^\mu, \gamma_5] = 0$$

- Breaks gauge invariance in intermediate steps
- Broken symmetry has to be restored $\Longrightarrow$ more complicated renormalization

$$S_{\text{ct}} = S_{\text{sct}} + S_{\text{fct}} = S_{\text{sct,inv}} + S_{\text{sct,non-inv}} + S_{\text{fct}}$$

- Complicated
- But: unitary and proven to be consistent to all orders!

Technical Aspects of the γ-Algebra in the BMHV Scheme

1. Orthogonality of $\mathbb{M}_4$ and $\mathbb{M}_{-2\epsilon} \implies 4D$ – evanescent contractions vanish

$$\overline{\eta}_{\mu\nu}\mathrm{Tr}(\dots\widehat{\gamma}^{\nu}\dots) = 0$$

2. Normal form of γ-traces:

$$\mathrm{Tr}(\widehat{\gamma}^{\nu_1}\dots\widehat{\gamma}^{\nu_m}\overline{\gamma}^{\mu_1}\dots\overline{\gamma}^{\mu_n}\mathbb{\Lambda}), \qquad \mathbb{\Lambda} \in \{\mathbb{1}, \gamma_5, \mathbb{P}_\mathrm{L}, \mathbb{P}_\mathrm{R}\}$$

3. γ-trace factorisation:

$$\mathrm{Tr}(\widehat{\gamma}^{\nu_1}\dots\widehat{\gamma}^{\nu_m}\overline{\gamma}^{\mu_1}\dots\overline{\gamma}^{\mu_n}\mathbb{\Lambda}) = \frac{1}{4}\mathrm{Tr}(\widehat{\gamma}^{\nu_1}\dots\widehat{\gamma}^{\nu_m})\mathrm{Tr}(\overline{\gamma}^{\mu_1}\dots\overline{\gamma}^{\mu_n}\mathbb{\Lambda})$$

Quantum Action Principle

Field Transformations

$$\phi_k(x) \longrightarrow \phi_k(x) + \delta\phi_k(x), \qquad S \longrightarrow S + \delta S$$

$$\begin{aligned}
Z[J] &= \int \mathcal{D}\phi\, e^{i\,S[\phi]+i\int \phi_k\cdot J_k} \\
&= \int \mathcal{D}\phi\, e^{i\,S[\phi+\delta\phi]+i\int(\phi_k\cdot J_k+\delta\phi_k\cdot J_k)} && \text{measure invariant} \\
&= \int \mathcal{D}\phi \left(1 + i\,\delta S + i\int \delta\phi_k\cdot J_k\right) e^{i\,S[\phi]+i\int \phi_k\cdot J_k} && \text{1st order expansion}
\end{aligned}$$

$$\Longrightarrow\ 0 = \int \mathcal{D}\phi \left(i\,\delta S + i\int \delta\phi_k\cdot J_k\right) e^{i\,S[\phi]+i\int \phi_k\cdot J_k} = i\,\langle\delta S\rangle + i\int\langle\delta\phi_k\rangle\cdot J_k$$

Variation w.r.t. quantum fields

$$i\langle T(\delta\phi_1)\phi_2\ldots\rangle + i\langle T\phi_1(\delta\phi_2)\ldots\rangle + \ldots = \langle T\phi_1\phi_2\ldots(\delta S)\rangle$$

exactly valid in DReg and provides breaking DReg:[BM'77], DRed:[DS'05], Rev:[HB,AI,PK,MM,DS,MW'23]

Slavnov-Taylor Identity

Symmetry transformations (e.g. BRST transformations)

$$\delta\phi_i = \theta s\phi_i, \qquad \delta S = 0$$

Example: chiral fermions in an Abelian gauge theory

$$s\psi_i(x) = s\psi_{Ri}(x) = -iec(x)\mathcal{Y}_{Rij}\psi_{Rj}(x), \qquad \text{here: } s\psi_{Li}(x) = 0$$

The Slavnov-Taylor operator

$$\mathcal{S}(\Gamma) = \int \mathrm{d}^4x \, \langle s\phi_i \rangle \frac{\delta\Gamma}{\delta\phi_i(x)}$$

Coupling transformations to external sources $\mathcal{L}_{\text{ext}} = K_i s\phi_i$

$$\mathcal{S}(\Gamma) = \int \mathrm{d}^4x \, \frac{\delta\Gamma}{\delta K_i(x)} \frac{\delta\Gamma}{\delta\phi_i(x)}$$

Symmetries — Slavnov-Taylor Identity

- The Slavnov-Taylor identity reflects symmetries in the full quantum theory
- The classical Symmetry can, and may in general, be broken by the Regularisation

$$\mathcal{S}(\Gamma_{\text{reg}}) \neq 0$$

- The Slavnov-Taylor identity needs to be obeyed after renormalisation for consistency
  - physical Hilbert space with positive norm states
  - unitary and gauge independent physical S-matrix
- Require the validity of symmetries as part of the definition of the theory

$$\mathcal{S}(\Gamma_{\text{ren}}) \overset{!}{=} 0$$

- Regularisation induced symmetry breakings need to be restored
  - Symmetries usually valid in DReg
  - However, no gauge-invariant regularisation that preserves chiral symmetry $\Longrightarrow$ γ_5-problem

Ward Identities in Abelian chiral Gauge Theories

Abelian gauge theories: Slavnov-Taylor identity $\mathcal{S}(\Gamma) = 0 \implies$ Ward identities

$$p_\nu \, \widetilde{\Gamma}^{\nu\mu}_{AA} = 0,$$
$$p_\sigma \, \widetilde{\Gamma}^{\sigma\rho\nu\mu}_{AAAA} = 0,$$
$$\widetilde{\Gamma}^{ji,\mu}_{\psi\overline{\psi}A}(q=0) = -e \, \mathcal{Y}_{Rjk} \, \frac{\partial}{\partial p_\mu} \widetilde{\Gamma}^{ki}_{\psi\overline{\psi}}$$

Distinguish two cases

1. Regularisation preserves symmetries
   $\implies$ standard multiplicative renormalisation
2. Regularisation breaks symmetries
   $\implies$ include symmetry-restoring counterterms, satisfying $\mathcal{S}(\Gamma^{(n)}_{\text{subren}}) = -bS^n_{\text{ct}}$

Algebraic Renormalisation

- Symmetry restoration $\longrightarrow$ inductive procedure
- Symmetry breaking is restricted in two ways
  - $\Delta =$ local polynomial restricted by power counting
  - Wess-Zumino consistency condition: $b\,\Delta = 0$
- Trivial elements of the BRST Cohomology, i.e. $\exists\,\Delta'$ with $\Delta = b\,\Delta'$
  - Symmetry can be established to all orders
  - Original action can be equipped with a new counterterm

$$S_{\mathrm{fct}} = -\Delta'$$

- Non-trivial elements of the BRST Cohomology
  - Anomaly; the breaking cannot be restored
  - Anomaly cancellation conditions can be derived, e.g.

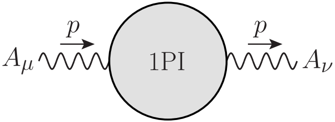
$$\mathrm{Tr}(\mathcal{Y}_R^3) = 0$$

Symmetry Restoration — Explicit Evaluation

Explicit evaluation of all Green functions in $\mathcal{S}_D(\Gamma^{(n)}_{\text{subren}} + S^n_{\text{sct}})$

- Consider gauge boson self energy at 1-loop

$$i\,\widetilde{\Gamma}^{\nu\mu}_{AA}(p)\Big|^{1} =$$



$$\Big|^{1}$$

$$= \frac{ie^2}{16\pi^2}\frac{\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\Bigg\{\frac{2}{\epsilon}\left[\left(\overline{p}^\mu\overline{p}^\nu - \overline{p}^2\overline{\eta}^{\mu\nu}\right) - \frac{1}{2}\widehat{p}^2\overline{\eta}^{\mu\nu}\right]$$

$$+\left[\left(\frac{10}{3} - 2\ln(-p^2)\right)\left(\overline{p}^\mu\overline{p}^\nu - \overline{p}^2\overline{\eta}^{\mu\nu}\right) - \left(\overline{p}^2 + \widehat{p}^2\left(\frac{8}{3} - \ln(-p^2)\right)\right)\overline{\eta}^{\mu\nu}\right]\Bigg\}$$

- Drop finite evanescent contribution, which vanishes in $\text{LIM}_{D\to 4}$
- Regularisation breaks transversality

Symmetry Restoration — Explicit Evaluation

Exhibit the breaking explicitly — probe the gauge boson Ward identity

$$[\mathcal{S}(\Gamma)]^1_{A^\mu c} = i\, p_\nu \widetilde{\Gamma}^{\nu\mu}_{AA}(p)\Big|^1 = \frac{ie^2}{16\pi^2}\frac{\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\left[-\frac{1}{\epsilon}\widehat{p}^2\overline{p}^\mu - \overline{p}^2\overline{p}^\mu\right] \neq 0$$

Need 3 types of counterterms

- singular and symmetric

$$S^1_{\mathrm{sct,inv}} = -\frac{e^2}{16\pi^2\epsilon}\frac{2\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\int d^Dx\left(-\frac{1}{4}\overline{F}^{\mu\nu}\,\overline{F}_{\mu\nu}\right) + \ldots$$

- singular and non-symmetric

$$S^1_{\mathrm{sct,non\text{-}inv}} = -\frac{e^2}{16\pi^2\epsilon}\frac{\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\int d^Dx\,\frac{1}{2}\overline{A}_\mu\widehat{\partial}^2\overline{A}^\mu + \ldots$$

- finite symmetry-restoring

$$S^1_{\mathrm{fct}} = -\frac{e^2}{16\pi^2}\frac{\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\int d^4x\,\frac{1}{2}\overline{A}_\mu\overline{\partial}^2\overline{A}^\mu + \ldots$$

Symmetry Restoration — Explicit Evaluation

Pros

- $+$ obvious and straightforward procedure
- $+$ operates on ordinary Green functions $\Longrightarrow$ convenient

Cons

- $\exists$ in principle ∞ many identities between Green functions
- involves also complicated finite and non-local terms of Green functions
- most finite terms, in particular non-local ones, do not contribute

$\Longrightarrow$ Calculation is unnecessarily complicated; this strategy is not the most efficient

Is there a better and more efficient procedure?

Symmetry Restoration — Quantum Action Principle

- The ultimate symmetry requirement is the Slavnov-Taylor identity

$$\underset{D\to 4}{\mathsf{LIM}}\left(\mathcal{S}_D(\Gamma_{\mathrm{DRen}})\right) = 0$$

- Regularised Quantum Action Principle of Dimensional Regularisation

$$\mathcal{S}_D(\Gamma_{\mathrm{DRen}}) = (\widehat{\Delta} + \Delta_{\mathrm{ct}}) \cdot \Gamma_{\mathrm{DRen}}$$

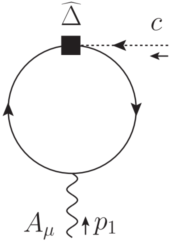
- Possible symmetry breaking can be rewritten as a composite operator insertion

$$\widehat{\Delta} = \mathcal{S}_D(S_0), \qquad \widehat{\Delta} + \Delta_{\mathrm{ct}} = \mathcal{S}_D(S_0 + S_{\mathrm{ct}})$$

Symmetry Restoration — QAP: 1-Loop Example Revisited

Breaking of the gauge boson self energy at 1-loop

$$i\left(\widehat{\Delta}\cdot\Gamma^{(1)}\right)_{A_\mu c} =$$



$$= \frac{e^2}{16\pi^2}\frac{\mathrm{Tr}(\mathcal{Y}_R^2)}{3}\left[-\frac{1}{\epsilon}\widehat{p}^2\overline{p}^\mu - \overline{p}^2\overline{p}^\mu\right]$$

$$= -i\left[\mathcal{S}(\Gamma)\right]^{(1)}_{A^\mu c} = p_\nu \Gamma^{\nu\mu}_{AA}(p)\Big|^{(1)}$$

1-loop breaking of the Slavnov-Taylor identity

$$\left(\widehat{\Delta}\cdot\Gamma\right)^{(1)} = -\frac{1}{16\pi^2}\int d^Dx\, \frac{e^2\,\mathrm{Tr}\left(\mathcal{Y}_R^2\right)}{3}\left[\frac{1}{\epsilon}\,c\,\overline{\partial}_\mu\,\widehat{\partial}^2\,\overline{A}^\mu + c\,\overline{\partial}_\mu\,\overline{\partial}^2\,\overline{A}^\mu\right] + \dots$$

Quantum Action Principle vs. Explicit Evaluation

Simplification is threefold

+ Only UV-divergent part of Green functions contributes
+ Only power-counting divergent Green functions required
+ In general fewer diagrams with Δ insertion

Cons

– Requires special Δ-operator inserted Green functions
– "Indirect" determination of symmetry-restoring counterterms

$\Longrightarrow$ Procedure with the Quantum Action Principle is much more efficient!

All Massive Tadpoles Method

Extracting UV-divergences utilising an infrared rearrangement

$$\frac{1}{(k+p)^2} = \frac{1}{k^2 - M^2} - \frac{p^2 + 2\,k\cdot p + M^2}{k^2 - M^2}\,\frac{1}{(k+p)^2}$$

- Exact decomposition, which can be applied recursively
- Power-counting finite terms can be dropped
- Does not affect the UV-divergences after subtraction of subdivergences

Improved tadpole expansion: Introducing M^2 and performing a Taylor-expansion

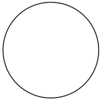
$$\frac{1}{(k+p)^2} \longrightarrow \frac{1}{(k+p)^2 - M^2} = \frac{1}{k^2 - M^2} - \frac{p^2 + 2\,k\cdot p}{(k^2 - M^2)^2} + \frac{(p^2 + 2\,k\cdot p)^2}{(k^2 - M^2)^3} + \ldots$$

- Same result as with the exact decomposition when neglecting numerator terms $\propto M^2$
- Auxiliary counterterms $\propto M^2$ necessary (not part of the theory)

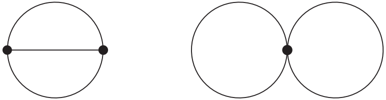
Master Integrals

Single-scale massive vacuum master bubbles

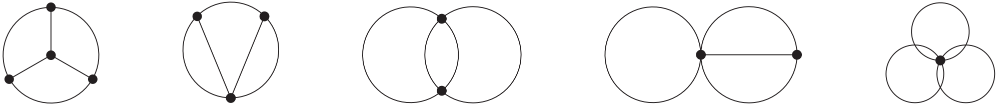
- 1-Loop



- 2-Loop



- 3-Loop



Tensor Reduction

- Reduction of tensor integrals into scalar integrals

$$I^{\mu_1\cdots\mu_r}(p_1,\ldots,p_E) = \sum_{a=1}^{n} C_a(p_1,\ldots,p_E)\, T_a^{\mu_1\cdots\mu_r}(p_1,\ldots,p_E)$$

- Determination of the scalar integrals upon projection, using $P_a^{\mu_1\cdots\mu_r}\, T_{b,\mu_1\cdots\mu_r} = \delta_{ab}$

$$C_a = P_a^{\mu_1\cdots\mu_r}\, I_{\mu_1\cdots\mu_r}$$

- Task: Determination of $\mathcal{B} = \{T_b^{\mu_1\ldots\mu_r}\}_{b=1}^n$ and $\mathcal{B}^* = \{P_b^{\mu_1\ldots\mu_r}\}_{b=1}^n$
- Special case: Vacuum bubble tensor integrals $\Longrightarrow$ $\mathcal{B} = \{\sigma \circ T^{\mu_1\cdots\mu_r} \,|\, \sigma \in S_2^r\}$, $n = |S_2^r|$

$$T_\sigma^{\mu_1\cdots\mu_r} = \eta^{\mu_{\sigma(1)}\mu_{\sigma(2)}} \cdots \eta^{\mu_{\sigma(r-1)}\mu_{\sigma(r)}},$$

$$P_a^{\mu_1\cdots\mu_r} = \sum_{b=1}^{n} A_b^{(a)}\, T_b^{\mu_1\cdots\mu_r}$$

Tensor Reduction — Orbit Partition Approach

- Stabiliser of T_a in S_r: encodes permutation symmetries of T_a

$$H(T_a) = \{h \in S_r \,|\, h \circ T_a = T_a\} \subset S_r$$

- Orbit of T_c under the stabiliser $H(T_a) \subset S_r$: equivalence class

$${}_r\Theta_c^{(a)} = \{h \circ T_c \,|\, h \in H(T_a)\} \subset \mathcal{B},$$

- Orbit Partition Approach:
  - utilise symmetry arguments: $P_a^{\mu_1 \cdots \mu_r}$ shares the same symmetries as $T_a^{\mu_1 \cdots \mu_r}$
  - set of tensors $\mathcal{B}$ can be partitioned into disjoint orbits
  - orbit sums $\implies$ stabilisation under $H(T_a)$

$$P_a^{\mu_1 \cdots \mu_r} = \sum_{c=1}^{m} A_c^{(a)} \left(\sum_{T \in {}_r\Theta_c^{(a)}} T^{\mu_1 \cdots \mu_r} \right)$$

r	2	4	6	8	10	12	14	16
$n = \lvert S_2^r \rvert$	1	3	15	105	945	10395	135135	2027025
$m = p(r/2)$	1	2	3	5	7	11	15	22