Building a toy nucleus for neutrino-nucleus cross sections

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Project

Use a square-well potential to solve the Schrodinger equation in spherical coordinates and generate bound-state single particle energies and wave functions for a ^{12}C nucleus. Use the nuclear radius $R = 1.2 \text{ fm}\, A^{1/3}$, with A the total number of nucleons, as size for the well. The energy eigenvalues are obtained by imposing that the radial part of the wave function $R_{nl}(r)$ disappears at the edge of the nucleus.

Determine the depth of the potential by comparing the energy of the least-bound state with experimental values for the nucleon knockout threshold. What happens to the energies when the nucleus becomes larger ?

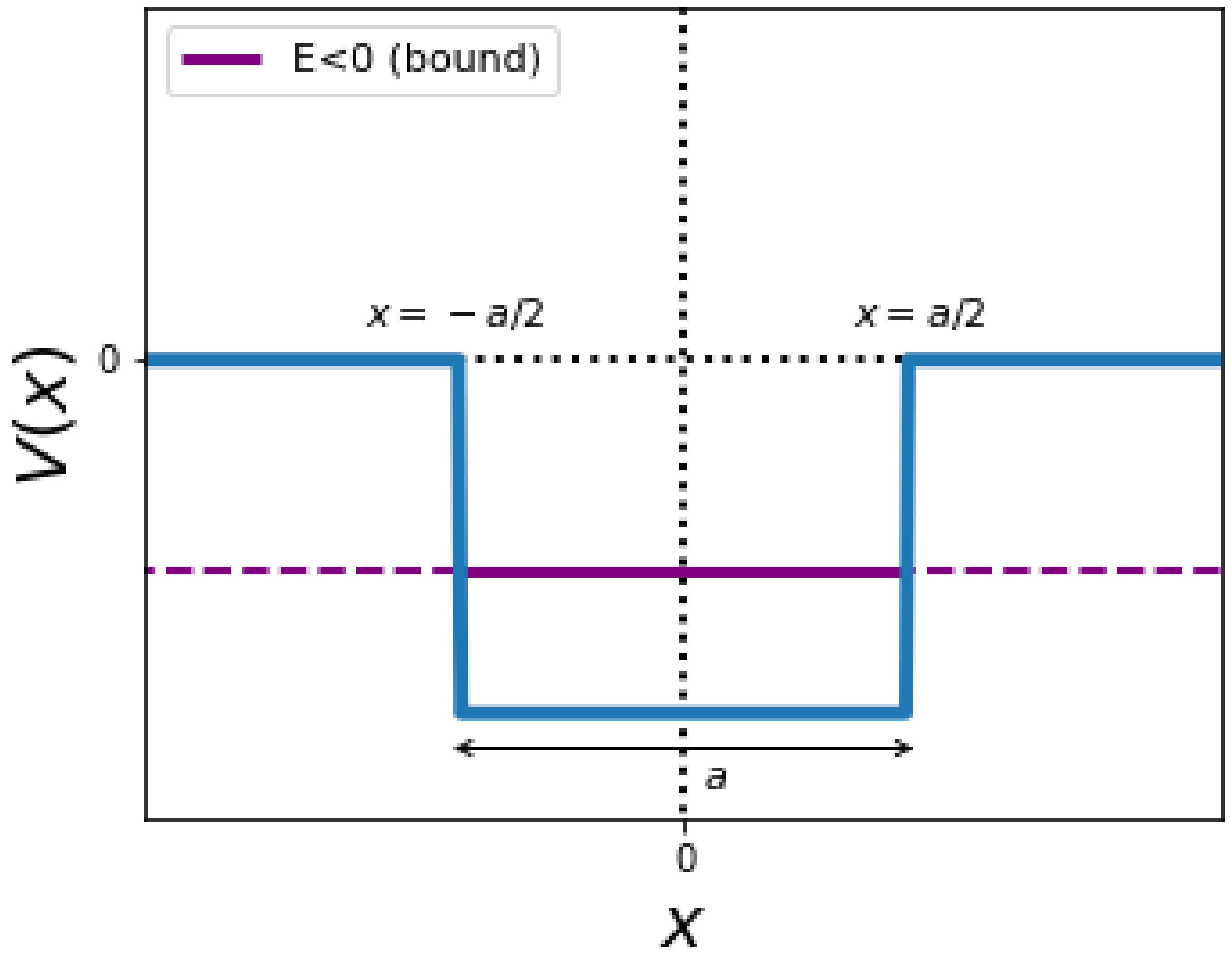
The advantage of the square-well problem is that it can be solved analytically and so provides a lot of insight. You can make the model more realistic by using a Woods-Saxon potential

$$V(r) = -\frac{V_0}{1 + \exp(\frac{r-R_0}{a})}, \tag{1}$$

adding a spin-orbit term (and a Coulomb term for protons !), and solve the equation numerically. Several parametrizations for the Woods-Saxon parameters are available in the literature (see e.g. https://arxiv.org/pdf/0709.3525). What does the nuclear density look like for a carbon and an argon nucleus ?

What are the average binding energies in this model ? Compare to the Fermi model values.

Square-well Potential for a ^{12}C Nucleus



⋙ The Schrödinger equation in spherical coordinates is:

$$\left(-\frac{\hbar}{2m}\nabla^2 + V(r)\right)\psi(r,\theta,\phi) = E\psi(r,\theta,\phi)$$

⋙ Where the square-well potential is:

$$V(r) = \begin{cases} -V_0 & \text{if } r \leq R_0 \\ 0 & \text{if } r > R_0 \end{cases}$$

⋙ The Wave function can be separated into a radial and angular part:

$$\psi(r,\theta,\phi) = R_{n,l}(r)Y_{l,m}(\theta,\phi)$$

⋙ Introducing our Wave function in the Schrödinger equation:

$$-\frac{\hbar}{2m}\nabla^2 RY = (E - V(r))RY$$

⋙ In this way, we can separate the radial and angular parts. Solving the angular part and keeping in mind that the potential only depends on r, the radial equation is the following:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[\frac{2m(E-V(r))}{\hbar} - \frac{l(l+1)}{r^2}\right]R = 0$$

Square-well Potential for a ^{12}C Nucleus

- For $r \leq R_0, V(r) = -V_0$, the equation is:

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[k^2 - \frac{l(l+1)}{r^2}\right]R = 0, \text{with } k^2 = \frac{2m(E+V_0)}{\hbar}$$

- The independent linear solution to this equation are the spherical Bessel functions:

$$R(r) = Aj_l(kr) + Bn_l(kr)$$

- Since $n_l(kr) \to \infty \text{ when } r \to 0 : B = 0$ (Neumann function):

$$R(r) = A_{jl}(kr)$$

- And taking into consideration the condition of frontier:

$$R(R_0) = Aj_l(kR_0) = 0 \to j_l(kR_0) = 0 \to$$

$$\boxed{kR_0 = \chi_{n,l} \qquad \text{Bessel roots}}$$

Square-well Potential for a ^{12}C Nucleus

⋙ Taking the definition k^2 and the obtained Bessel roots, we can finally obtain the energy eigenvalues:

$$\sqrt{\frac{2m(E+V_0)}{\hbar}} = \frac{\chi_{nl}}{R_0} \rightarrow \boxed{E_{nl} = -V_0 + \frac{\hbar^2 \chi_{nl}}{2mR_0^2}}$$

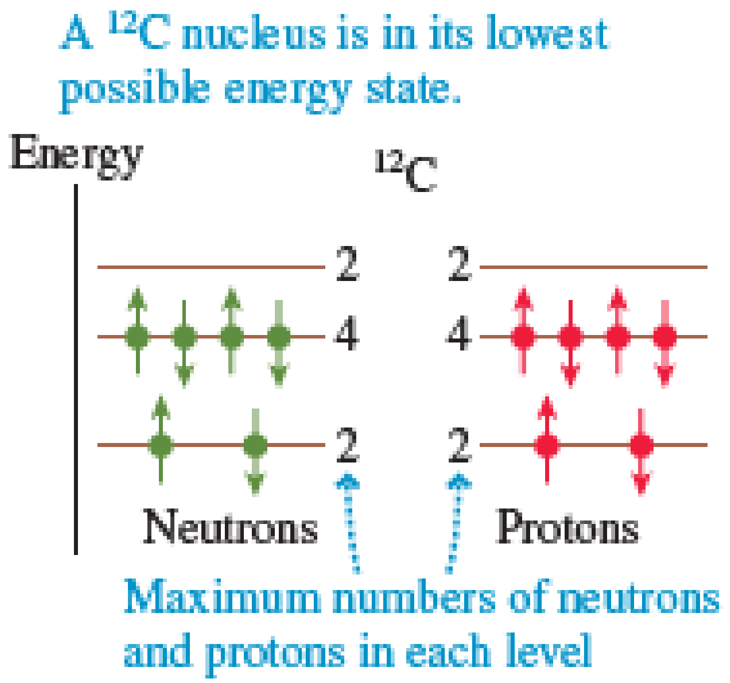
⋙ To compute the depth of the potential, we need to compute the energy of the least-bound state.

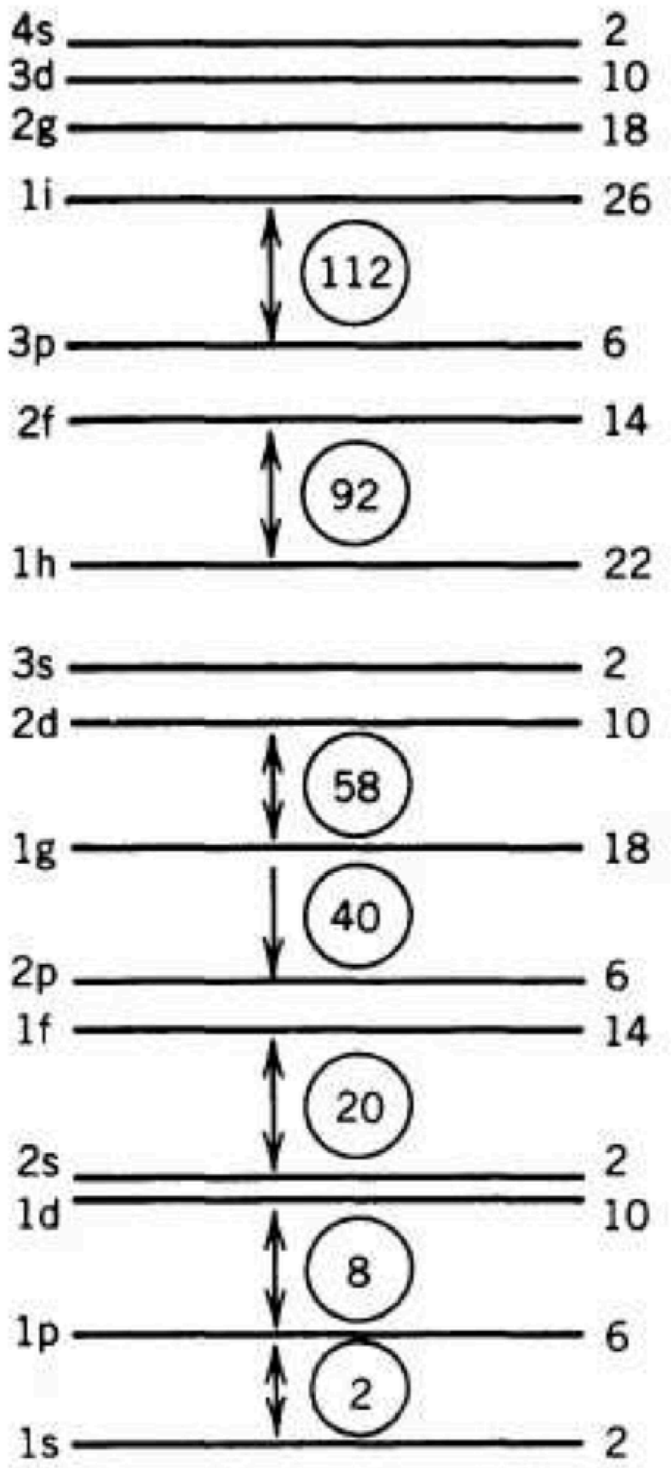
⋙ Since ^{12}C has 6 protons and 6 neutrons, the first level (1s) is completely full and 1p is partially. Therefore, the least-bound state corresponds to this one. According to experimental values:

$$E_{1p} = -16 \text{ MeV}$$

⋙ Using this value in the expression obtained for the energy eigenvalues:

$$-16 \text{ MeV} = -V_0 + \frac{\hbar^2 \chi_{11}}{2mR_0^2} \rightarrow \boxed{V_0 \approx 71 \text{ MeV}}$$





Square-well Potential for a ^{12}C Nucleus

- When the nucleus becomes larger: $A \to \infty \implies R_0 \propto A^{1/3}$ also increases

- From the expression for the eigenvalues, we can see that the kinetic term:

$$E_k \propto \frac{1}{R_0^2} = \frac{1}{A^{2/3}}$$

- As the nucleus increases, the kinetic decreases. If we consider that the depth of the potential remains constant:

$$E_{nl} = -V_0 + E_k$$

- We can see that as the kinetic term decreases, the total energy becomes more negative. This means that the levels become more deeply bound, and the spacing between energy levels shrinks.

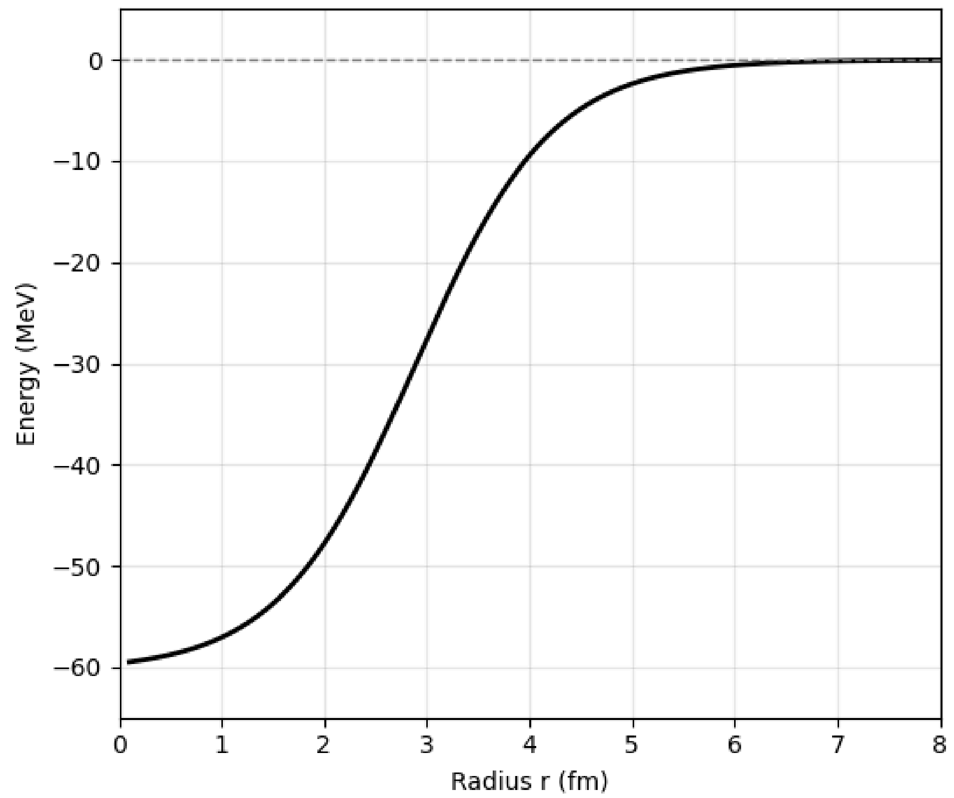
A more realistic model

››› Working harder...

Central Wood-Saxon Potential

$$f(r, R, a) = \left[1 + \exp\left(\frac{r - R}{a}\right)\right]^{-1}$$

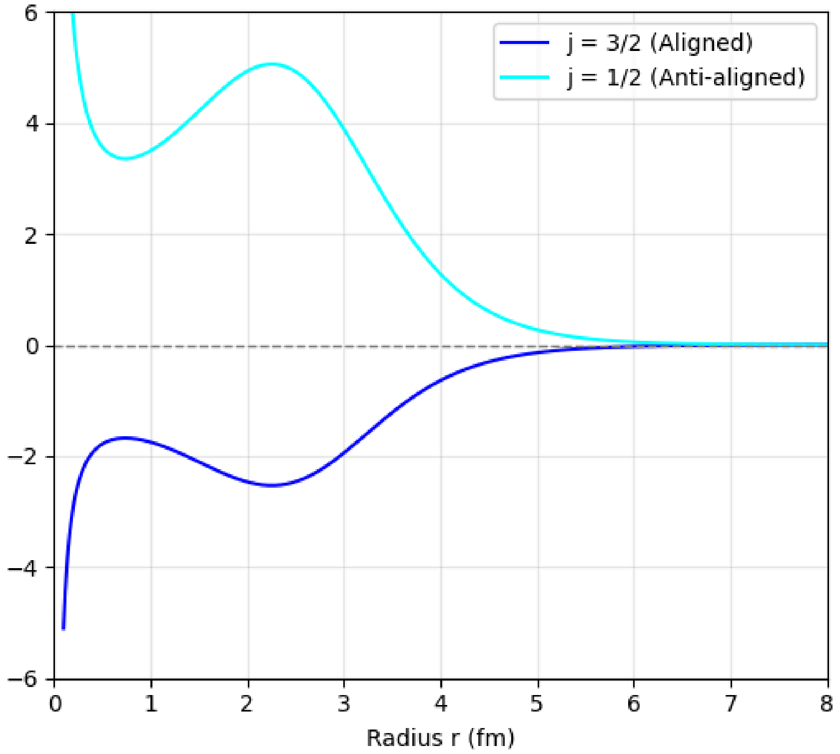
Fermi-function form



$$V(r) = -V f(r, R, a)$$

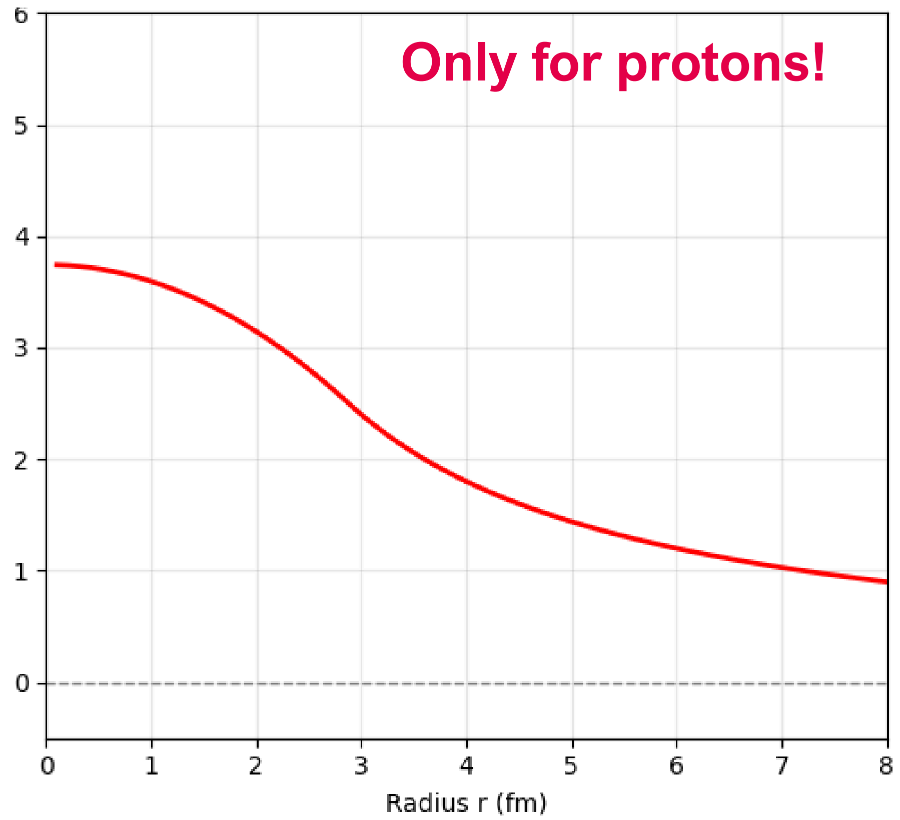
Spin-Orbit Term

$$\frac{1}{2\mu^2 r}\left(\frac{\partial}{\partial r}\tilde{V}(r)\right)\ \mathbf{l}\cdot\mathbf{s}$$



$$\tilde{V}(r) = \tilde{V} f(r, R_{SO}, a_{SO})$$

Coulomb Potential



$$V_c(r) = Z'e^2 \begin{cases} (3R^2 - r^2)/(2R^3), & r \le R, \\ 1/r, & r > R, \end{cases}$$

A more realistic model

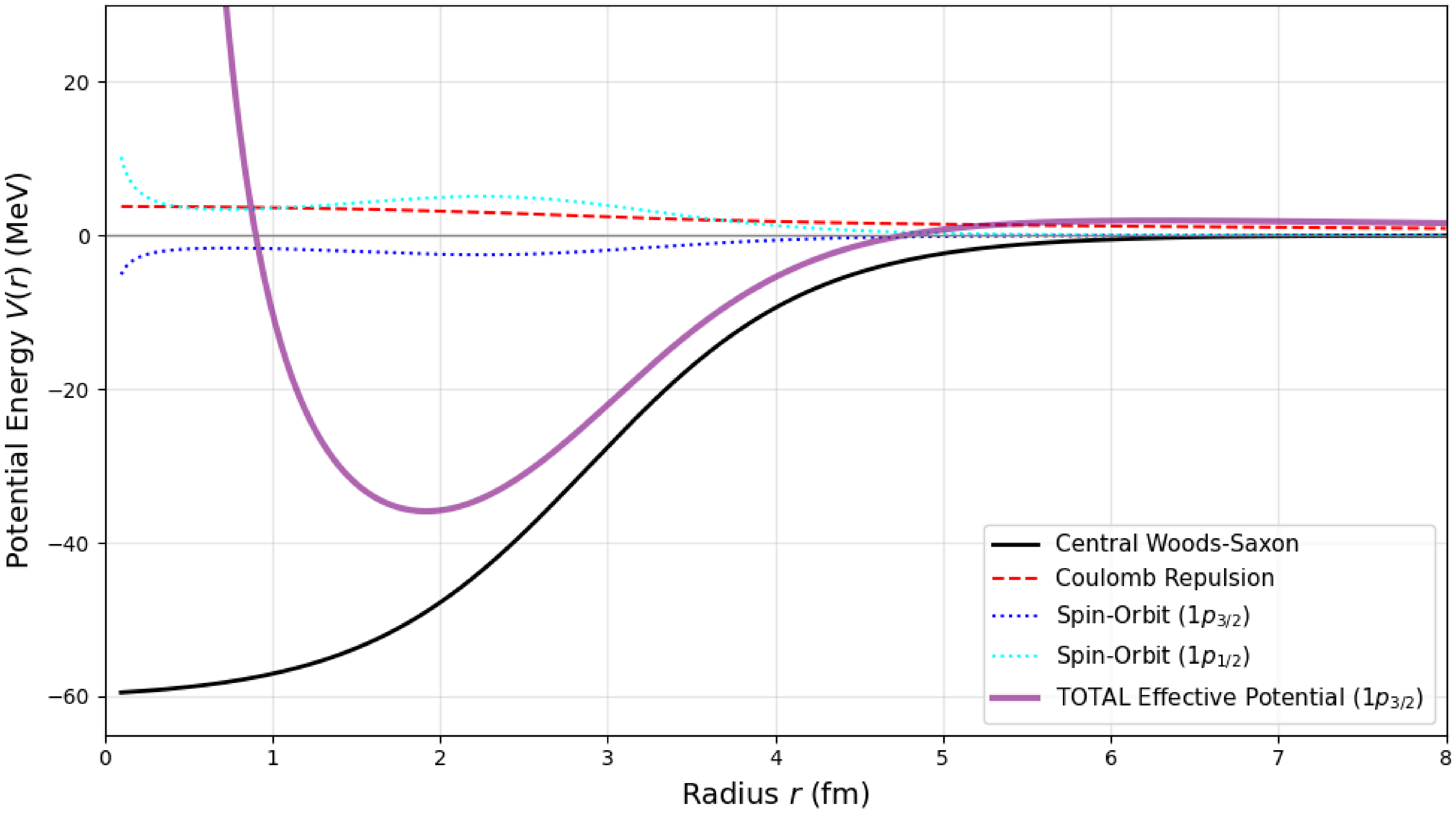
››› Components of the nuclear potential for a proton within ^{12}C-core

$$H = \frac{\mathbf{p}^2}{2\mu} + V(r) + V_c(r) + \frac{1}{2\mu^2 r}\left(\frac{\partial}{\partial r}\tilde{V}(r)\right)\ \mathbf{l}\cdot\mathbf{s}$$

Contribution only for
l = 1 (p-state)

reduced mass

$$\mu = \left(\frac{1}{m_{\pi}^{\nu}} + \frac{1}{M'}\right)^{-1}$$



A more realistic model

››› We follow the **structure of parametrization** from here

$$R = R_C = R_0 A^{1/3}$$

$$R_{SO} = R_{0,SO} A^{1/3}$$

$$a = a_{SO} = const$$

$$\tilde{V} = \lambda V_0$$

$$V = V_0 \left(1 - \frac{4\kappa}{A} \langle \mathbf{t} \cdot \mathbf{T}' \rangle \right)$$

isospin dependence

$$-4\langle \mathbf{t} \cdot \mathbf{T}' \rangle = \begin{cases} 3 & N = Z \\ \pm(N - Z + 1) + 2 & N > Z \\ \pm(N - Z - 1) + 2 & N < Z \end{cases}$$

››› Parameter set values from fits with orbital energy data

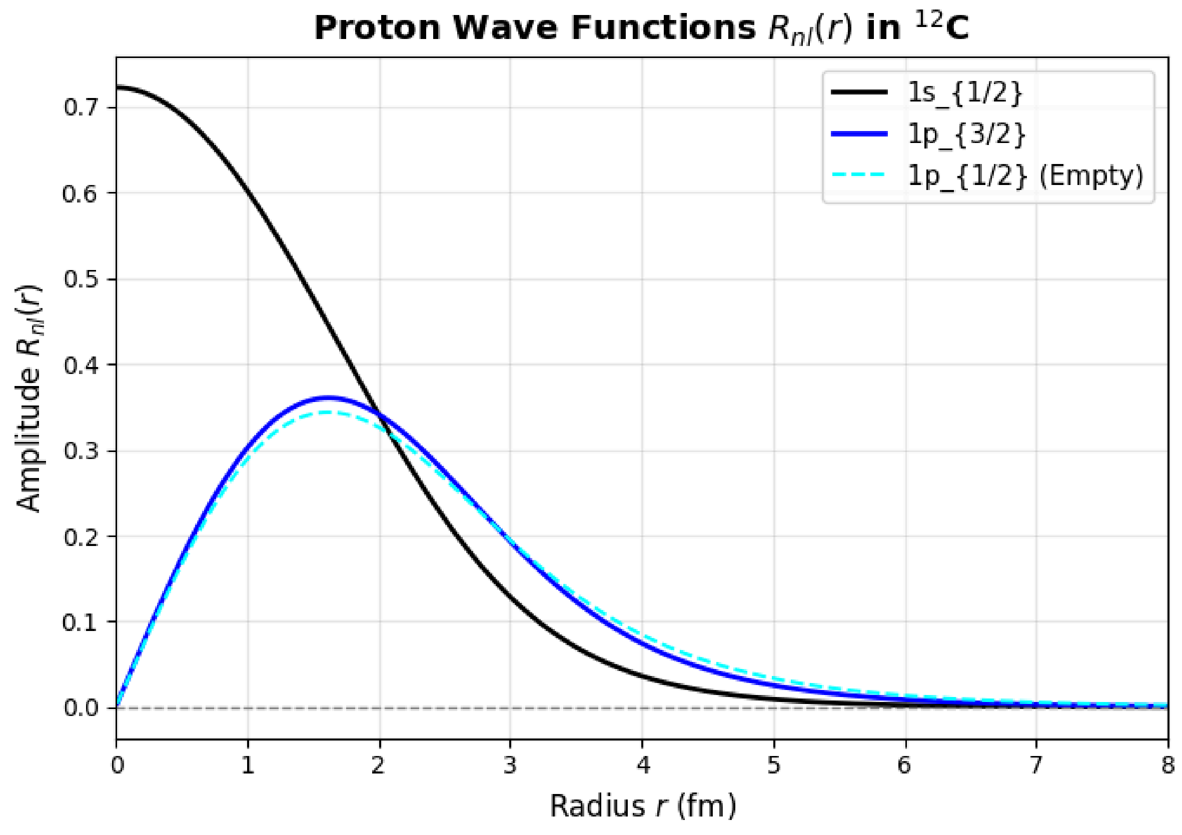
V_0 [MeV]	κ	R_0 [fm]	$a = a_{SO}$ [fm]	λ	$R_{0,SO}$ [fm]
52.06	0.639	1.260	0.662	24.1	1.16

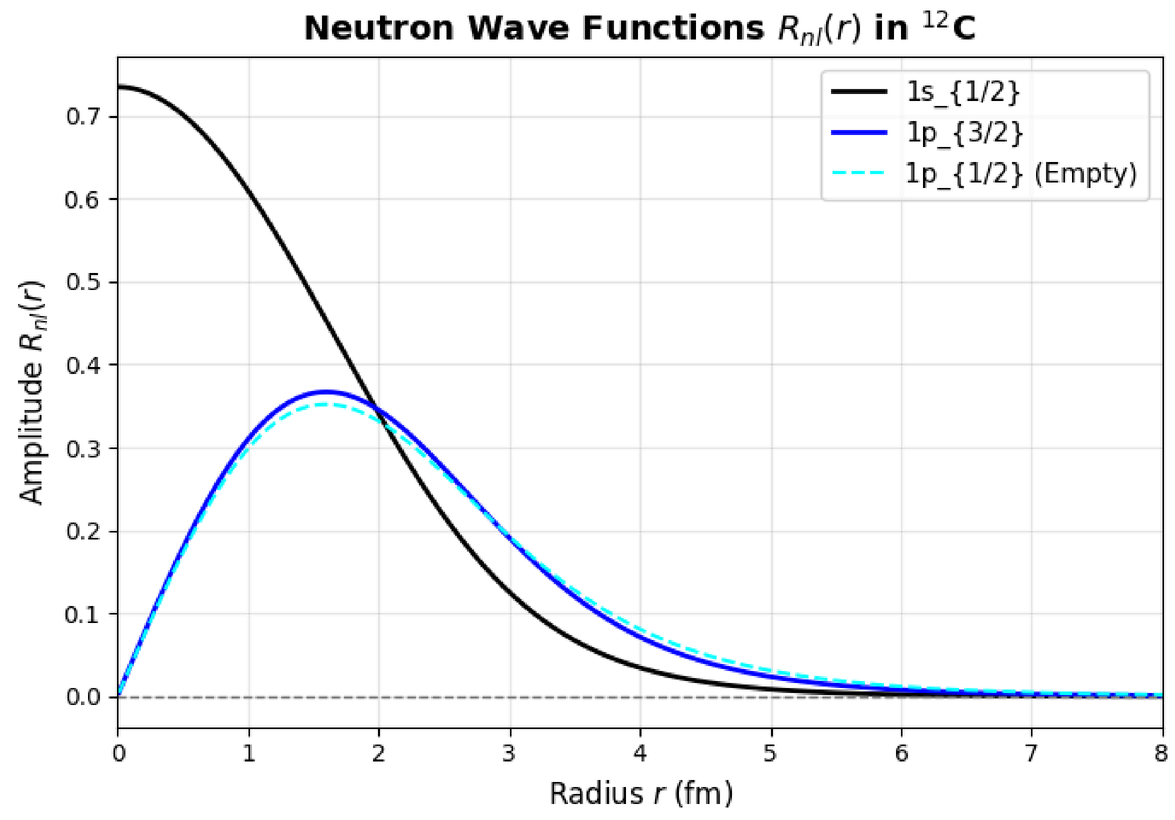
Numerical solution

- 1D substitution as: $u(r) = r \cdot R(r)$
- Creating a grid by discretizing space (radial).
- Finite difference approximation:

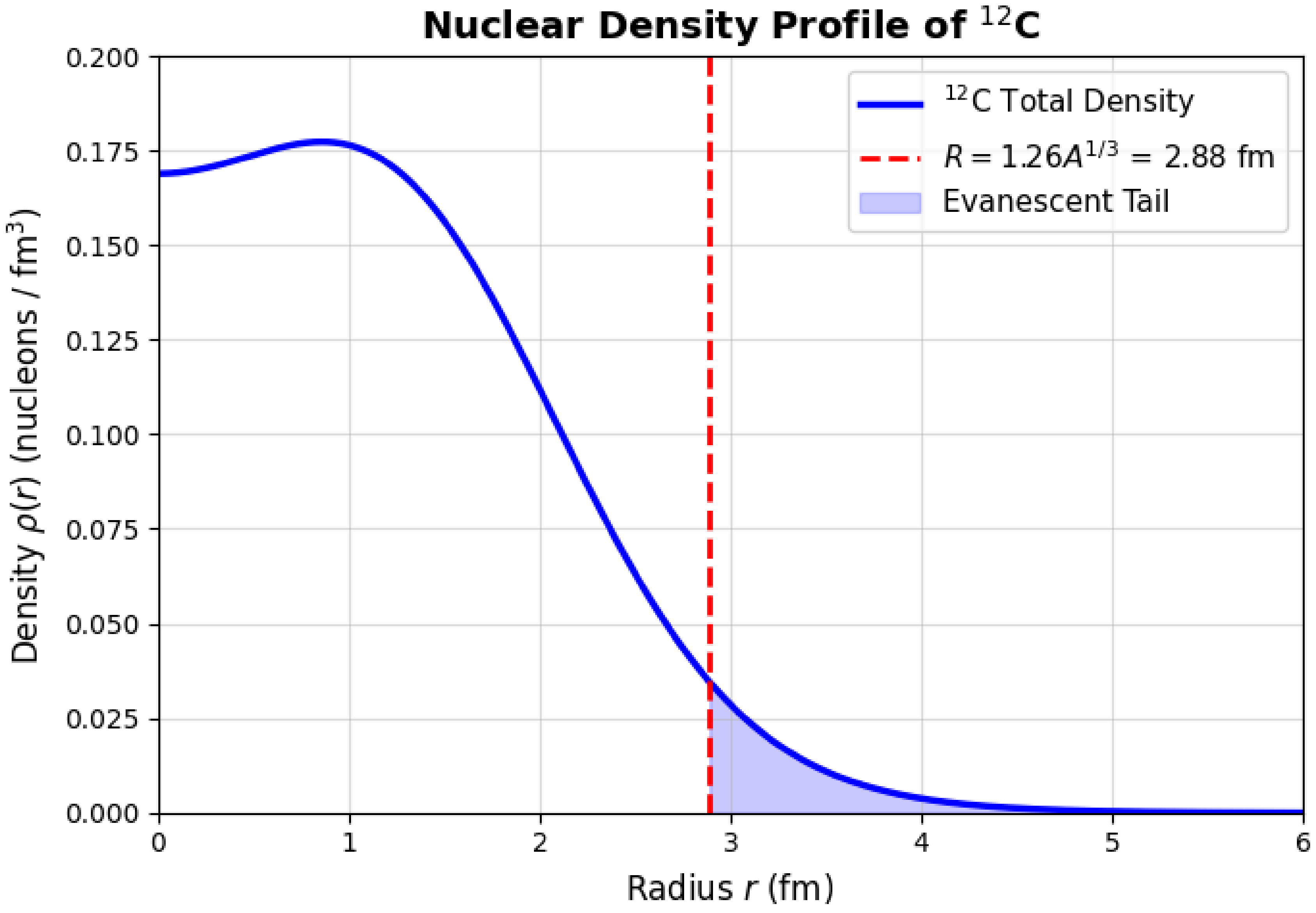
$$\frac{d^2u}{dr^2} \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{(\Delta r)^2}$$

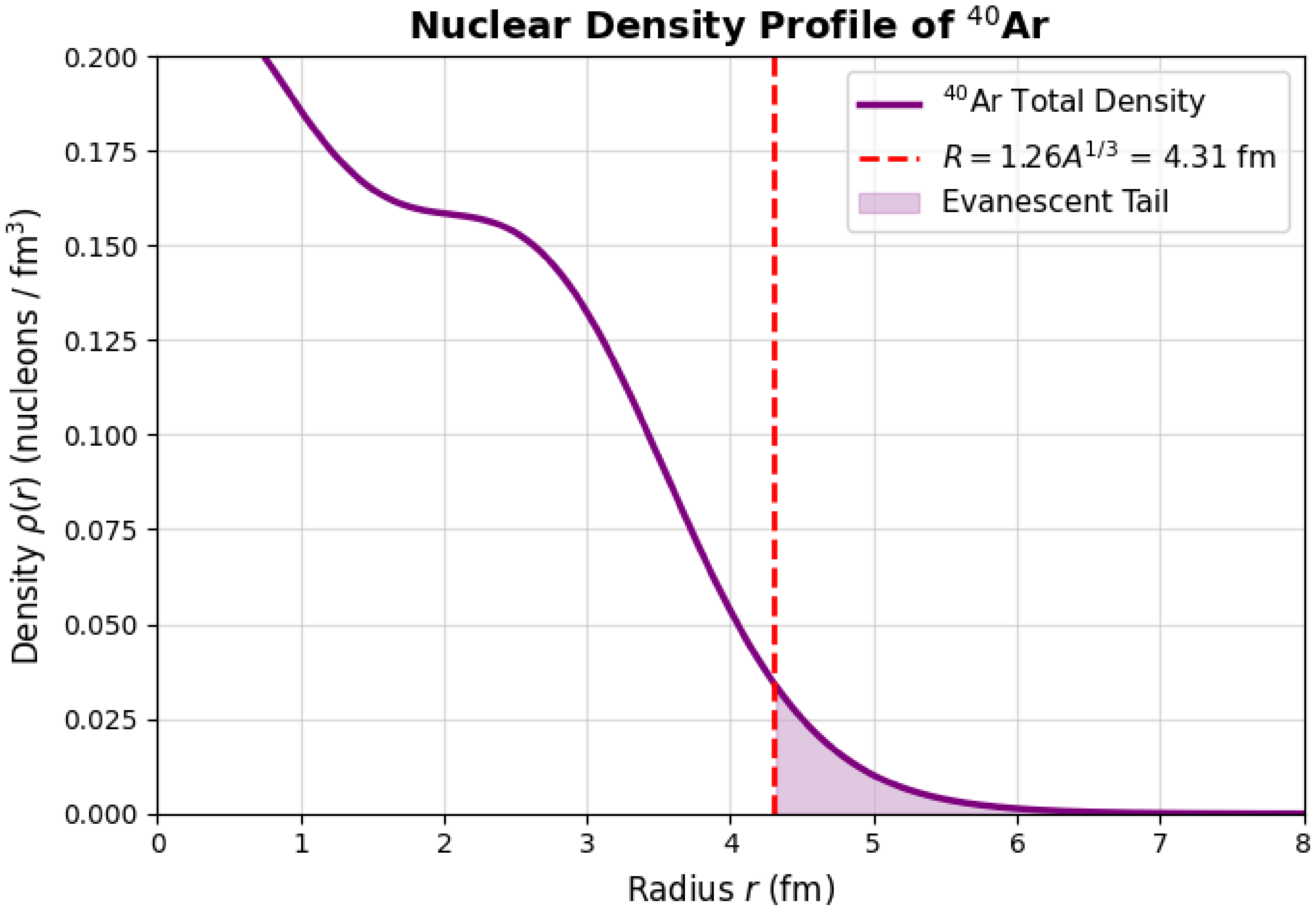
- Building the Hamiltonian matrix.
- Boundary conditions.
- Diagonalization.





Nuclear density and average binding energies





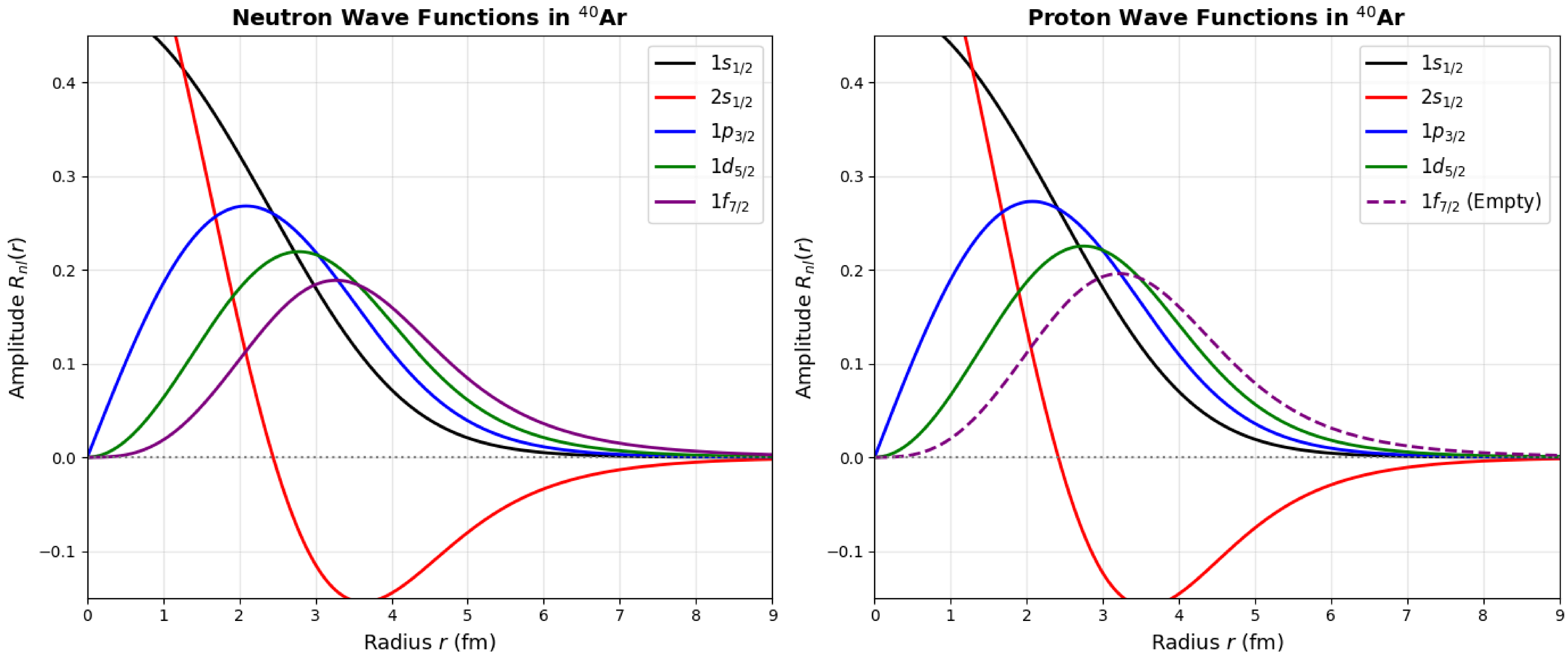
››› Average single-particle energies:

$$-22.57\,\mathrm{MeV} \;\; \text{for} \;\; {}^{12}\mathrm{C} \;\; \text{and} \;\; -20.53\,\mathrm{MeV} \;\; \text{for} \;\; {}^{40}\mathrm{Ar}$$

››› Fermi gas model (theoretical):

$$E_{avg} = -V_0 + \frac{3}{5}E_F \approx -30\,\mathrm{MeV}$$

Thanks


Neutron Wave Functions in ^{40}Ar
Proton Wave Functions in ^{40}Ar
$1s_{1/2}$
$2s_{1/2}$
$1p_{3/2}$
$1d_{5/2}$
$1f_{7/2}$
$1f_{7/2}$ (Empty)
Amplitude $R_{nl}(r)$
Radius r (fm)
0.4
0.3
0.2
0.1
0.0
−0.1
0
1
2
3
4
5
6
7
8
9